Numer. Math. 47, 505–533 (1985)

Numerische Mathematik

A Study of Semiiterative Methods for Nonsymmetric Systems of Linear Equations

Dedicated to Professor Karl Zeller (Universität Tübingen) on the occasion of his sixtieth birthday (December 28, 1984)

M. Eiermann[1], W. Niethammer[1,3], and R.S. Varga[2,3]

[1] Institut für Praktische Mathematik, Universität Karlsruhe, D-7500 Karlsruhe, Federal Republic of Germany

[2] Institute for Computational Mathematics, Kent State University, Kent, Ohio 44242, USA

[3] Research supported by the Air Force Office of Scientific Research

Summary. Given a nonsingular linear system $A\mathbf{x}=\mathbf{b}$, a splitting $A=M-N$ leads to the one-step iteration (1)$\mathbf{x}_m=T\mathbf{x}_{m-1}+\mathbf{c}$ with $T:=M^{-1}N$ and $\mathbf{c}:=M^{-1}\mathbf{b}$. We investigate semiiterative methods (SIM's) with respect to (1), under the assumption that the eigenvalues of T are contained in some compact set Ω of $\mathbb{C}$, with $1\notin\Omega$. There exist SIM's which are optimal with respect to Ω, but, except for some special sets Ω, such optimal methods are not explicitly known in general. Using results about "maximal convergence" of polynomials and "uniformly distributed" nodes from approximation and function theory, we describe here SIM's which are asymptotically optimal with respect to Ω. It is shown that Euler methods, extensively studied by Niethammer-Varga [NV], are special SIM's. Various algorithms for SIM's are also derived here. A $1-1$ correspondence between Euler methods and SIM's, generated by generalized Faber polynomials, is further established here. This correspondence gives that asymptotically optimal Euler methods are quite near the optimal SIM's.

Subject Classifications: AMS(MOS): 65F10, 40G05; CRG1.3.

Contents

1. Introduction

Given a nonsingular system of linear equations $A\mathbf{x}=\mathbf{b}$, where $A\in\mathbb{C}^{n,n}$, and given a splitting $A=M-N$ of A with M nonsingular, this system of equations can be written in the equivalent fixed-point form

$$\mathbf{x}=T\mathbf{x}+\mathbf{c}, \tag{1.1}$$

where $T:=M^{-1}N$ and $\mathbf{c}:=M^{-1}\mathbf{b}$. With (1.1), there is the naturally induced iterative method

$$\mathbf{x}_m=T\mathbf{x}_{m-1}+\mathbf{c}, \quad \mathbf{x}_0=\mathbf{a} \quad (m\geqq 1), \tag{1.2}$$

which is well-known to be convergent, for arbitrary $\mathbf{a}$, to the solution $\mathbf{x}$ of (1.1) iff the spectral radius of T, $\rho(T)$, satisfies $\rho(T)<1$. If the error vectors and the residual vectors for the iterative method of (1.2) are respectively defined by

$$\mathbf{e}_m:=\mathbf{x}-\mathbf{x}_m, \quad \mathbf{r}_m:=\mathbf{c}-(I-T)\mathbf{x}_m, \tag{1.3}$$

then there holds

$$\mathbf{e}_m=T^m\mathbf{e}_0, \quad \mathbf{r}_m=T^m\mathbf{r}_0 \quad (m\geqq 0). \tag{1.3'}$$

Following Varga ([V1] or [V2, p. 132]), a *semiiterative method* (SIM) with respect to the iterative method (1.2) is defined by

$$\mathbf{y}_m:=\sum_{i=0}^{m}\pi_{m,i}\mathbf{x}_i \quad (m\geqq 0), \tag{1.4}$$

where the infinite lower triangular matrix

$$P:=\begin{bmatrix} \pi_{0,0} & & & \\ \pi_{1,0} & \pi_{1,1} & & 0 \\ \pi_{2,0} & \pi_{2,1} & \pi_{2,2} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1.5}$$

satisfies

$$\sum_{i=0}^{m}\pi_{m,i}=1 \quad (m\geqq 0). \tag{1.6}$$

The associated sequence of polynomials, derived from the rows of the infinite triangular matrix P by

$$p_m(z):=\sum_{i=0}^{m}\pi_{m,i}z^i \quad (m\geqq 0), \tag{1.7}$$

evidently satisfies $p_m(1)=1$ from (1.6). For convenience, we assume $\pi_{m,m}\neq 0$, so that $p_m(z)$ can be represented also as

$$p_m(z)=\pi_{m,m}\prod_{i=1}^{m}(z-\xi_i^{(m)}) \quad (m\geqq 0). \tag{1.7'}$$

If we introduce the error vector $\tilde{\mathbf{e}}_m := \mathbf{x} - \mathbf{y}_m$ and the corresponding residual vector $\tilde{\mathbf{r}}_m := \mathbf{c} - (I - T)\mathbf{y}_m$, then there holds

$$\tilde{\mathbf{e}}_m = p_m(T)\mathbf{e}_0, \qquad \tilde{\mathbf{r}}_m = p_m(T)\mathbf{r}_0 \quad (m \geqq 0). \tag{1.8}$$

The problem is to choose the matrix P, or equivalently the polynomials $p_m(z)$, such that the error vectors $\tilde{\mathbf{e}}_m$ decrease rapidly to zero, thereby rendering rapid convergence of the vectors $\mathbf{y}_m$ to $\mathbf{x}$.

In Sect. 2, we introduce the polynomials

$$q_{m-1}(z) := \frac{1 - p_m(z)}{1 - z} \qquad (m \geqq 1). \tag{1.9}$$

Since $p_m(1) = 1$, then $q_{m-1}(z)$ is a polynomial of degree $m-1$. It can be directly seen from (1.9) that $q_{m-1}(z)$ interpolates the function $1/(1-z)$ at the zeros $\xi_i^{(m)}$ of $p_m(z)$ $(i = 1, \ldots, m)$. Writing

$$q_{m-1}(z) = \sum_{i=0}^{m-1} \gamma_{m-1,i} z^i, \tag{1.9'}$$

the coefficients of the polynomials $q_{m-1}(z)$ similarly define an infinite lower triangular matrix Q. Whereas the matrix P of (1.5) induces a linear transformation from the sequence of vectors $\{\mathbf{x}_m\}_{m \geqq 0}$ to the sequence of vectors $\{\mathbf{y}_m\}_{m \geqq 0}$, the matrix Q induces the same transformation from the series, which has the sequence $\{\mathbf{x}_m\}$ as partial sums, to the sequence $\{\mathbf{y}_m\}$ (cf. § 2).

A third infinite lower triangular matrix V induces the corresponding series-to-series transformation. There is a simple algebraic relation between these infinite matrices P, Q and V, i.e., each of these matrices can be used for defining a SIM. *Euler methods* (to be described precisely in § 2) are special SIM's which can be described by such an infinite matrix V_E, which is generated by an *Euler function*. A fourth infinite lower triangular matrix, the *nodal matrix* K_P, consists of the zeros $\xi_i^{(m)}$ of $p_m(z)$ $(m = 1, 2, \ldots; 1 \leqq i \leqq m)$, i.e., with (1.7'),

$$K_P := \begin{bmatrix} \xi_1^{(1)} & & & \\ \xi_1^{(2)} & \xi_2^{(2)} & & \mathbf{0} \\ \xi_1^{(3)} & \xi_2^{(3)} & \xi_3^{(3)} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.10}$$

In Sect. 3, we describe algorithms for computing the sequence $\{\mathbf{y}_m\}_{m \geqq 0}$ of (1.4). Thus, if an Euler function is given as a power series, a recurrence formula is described which uses all preceding iterates $\mathbf{y}_k (k = 0, \ldots, m-1)$ and the coefficients of the power series. If the Euler function has the special form

$$h(\phi) = \frac{\mu_0 \phi}{1 - \mu_1 \phi - \ldots - \mu_k \phi^k}, \tag{1.11}$$

then a k-step stationary iterative procedure can be derived. If the nodal matrix K_P of (1.10) of a SIM is *column-constant* which means that for the nodes there holds $\xi_i^{(m)} = \xi_i$ $(i=1,2,\ldots;m\geqq i)$, then the vectors $\mathbf{y}_m$ can be calculated by a *first-order Richardson method*

$$\mathbf{y}_m = \alpha_m T\mathbf{y}_{m-1} + (1-\alpha_m)\mathbf{y}_{m-1} + \alpha_m \mathbf{c}, \tag{1.12}$$

where $\alpha_m := 1/(1-\xi_m)$.

In Sect. 4, the *asymptotic convergence factor*

$$\kappa(T,P) = \overline{\lim_{m\to\infty}} \left\{ \sup_{\mathbf{e}_0 \neq 0} \left[\frac{\|\tilde{\mathbf{e}}_m\|}{\|\tilde{\mathbf{e}}_0\|} \right]^{1/m} \right\} \tag{1.13}$$

of T, with respect to the SIM *induced by P*, is introduced. It is shown that $\kappa(T,P)$ can be expressed in terms of the sequences $\{p_m(z)\}_{m\geqq 0}$, $\{q_{m-1}(z)\}_{m\geqq 1}$, and $\{v_m(z)\}_{m\geqq 0}$, given by the matrices P, Q and V.

From (1.1), we can see that the coefficient matrix A of the given system is nonsingular iff 1 is not an eigenvalue of T. Thus, if $\sigma(T)$ denotes the set of all eigenvalues of T, our assumption that A is nonsingular yields $1 \notin \sigma(T)$.

In Sect. 5, we assume that a compact set Ω in the complex plane with $\sigma(T) \subset \Omega$ and $1 \notin \Omega$ is known. For a given SIM induced by P, this leads to the definitions of the *asymptotic convergence factor*

$$\kappa(\Omega,P) := \overline{\lim_{m\to\infty}} \{\max_{z\in\Omega} |p_m(z)|\}^{1/m} \tag{1.14}$$

of Ω *with respect to P*, and the *convergence factor* of Ω

$$\kappa(\Omega) := \inf\{\kappa(\Omega,P) \colon P \text{ induces a SIM}\}. \tag{1.15}$$

A SIM, induced by $\tilde{P}$, such that $\kappa(\Omega) = \kappa(\Omega,\tilde{P})$ is called *asymptotically optimal* (AOSIM) *with respect to* Ω. Results from Eiermann and Niethammer [EN], concerning *maximal convergence* of sequences $\{q_{m-1}(z)\}_{m\geqq 1}$, of *uniformly distributed nodes* and of AOSIM's, are reported. We prove that an optimal Euler methods for Ω defined in Niethammer and Varga [NV] is an AOSIM with respect to Ω.

In Sect. 6, *regions of convergence* for special SIM's are described. Thus, after briefly treating cyclic first-order Richardson methods, Euler functions of type (1.11) are considered. It is shown that the case $k=1$ of (1.11) has disks for regions of convergence, while $k=2$ of (1.11) has ellipses for regions of convergence, which are described by the parameters of the corresponding Euler function. Conversely, given sets Ω which are either disks or ellipses, the parameters of the Euler methods which are asymptotically optimal with respect to these regions Ω, are derived.

Finally, in Sect. 7, new results on SIM's generated by generalized Faber polynomials, are derived. From known recurrence relations for the Faber polynomials, recurrence relations for the iterates $\mathbf{y}_m$ of (1.4) are found. In Theorem 21, it is shown that there is a $1-1$ correspondence between Euler methods and SIM's generated by suitably chosen generalized Faber polynomials. Our final result, Theorem 22, gives bounds for the norm of the error iterates $\tilde{\mathbf{e}}_m$ of (1.8) of a SIM in terms of the best uniform approximation error of $1/(1-z)$, by polynomials, on Ω.

From the beginning of the computer age, i.e., from about 1950, many "polynomial" methods for accelerating the convergence of iterative methods for positive definite linear systems have been considered. A discussion of the different approaches is, e.g., given by Varga ([V2, p. 159]) or Householder ([H, p. 115]). Householder points out that, besides the case where the eigenvalues of all matrices involved are real, "no theory has been developed". The aim of our paper is to give a report, partly with new results, on the progress that has been obtained towards such a theory.

2. Different Forms of Semiiterative Methods (SIM's)

The idea of semiiterative methods has been suggested by the classical theory of summability (cf. [V1; V2, p. 132]). In terms of this theory, (1.4) yields a linear transformation from the sequence $\{\mathbf{x}_m\}_{m\geqq 0}$ of (1.2) to the sequence $\{\mathbf{y}_m\}_{m\geqq 0}$ of (1.4), i.e., the infinite matrix P of (1.5) induces a so-called *sequence-to-sequence transformation.* In matrix notation, this transformation can be written in the form

$$\begin{bmatrix}\mathbf{y}_0\\ \mathbf{y}_1\\ \mathbf{y}_2\\ \vdots\end{bmatrix}=P\begin{bmatrix}\mathbf{x}_0\\ \mathbf{x}_1\\ \mathbf{x}_2\\ \vdots\end{bmatrix}. \tag{2.1}$$

Now, with the residual vector $\mathbf{r}_m:=\mathbf{c}-(I-T)\mathbf{x}_m$, we obtain

$$\mathbf{x}_m=T\mathbf{x}_{m-1}+\mathbf{c}=\mathbf{x}_{m-1}+\mathbf{r}_{m-1} \qquad (m\geqq 1),$$

i.e.,

$$\mathbf{x}_m=\mathbf{x}_0+\sum_{i=0}^{m-1}\mathbf{r}_i \qquad (m\geqq 1), \tag{2.2}$$

or, because of (1.3′),

$$\mathbf{x}_m=\mathbf{x}_0+\left(\sum_{i=0}^{m-1}T^i\right)\mathbf{r}_0 \qquad (m\geqq 1). \tag{2.3}$$

Next, the solution vector $\mathbf{x}$ of (1.1) evidently satisfies $\mathbf{x}=\mathbf{x}_0+(I-T)^{-1}\mathbf{r}_0$, since $1\notin\sigma(T)$. Thus, if $\rho(T)<1$, we have

$$\mathbf{x}=\mathbf{x}_0+(I-T)^{-1}\mathbf{r}_0=\mathbf{x}_0+\sum_{i=0}^{\infty}T^i\mathbf{r}_0. \tag{2.4}$$

We see from (2.3) that the sequence $\{\mathbf{x}_m\}_{m\geqq 0}$, generated by the iteration (1.2), is the sequence of partial sums of the series (2.4).

Rather than considering the sequence-to-sequence transformation (2.1) induced by the matrix P, it is useful to consider instead the corresponding *series-to-sequence transformation,* which transforms directly the terms $\{\mathbf{x}_0,\mathbf{r}_0,\mathbf{r}_1,\ldots\}$ of the series (2.2) to the sequence $\{\mathbf{y}_m\}_{m\geqq 0}$, generated by (1.4). First, the connection between the sequence $\{\mathbf{x}_m\}_{m\geqq 0}$ and the terms of the series (2.2) can be expressed in matrix notation simply as

$$\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \end{bmatrix} = S \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{r}_0 \\ \mathbf{r}_1 \\ \vdots \end{bmatrix}, \quad \text{with} \quad S := \begin{bmatrix} 1 & & & \\ 1 & 1 & & 0 \\ 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.5}$$

We see from (2.1) and (2.5) that this series-to-sequence-transformation is induced by the infinite lower triangular matrix Q, where

$$Q := PS =: \begin{bmatrix} 1 & & & \\ 1 & \gamma_{0,0} & & 0 \\ 1 & \gamma_{1,0} & \gamma_{1,1} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.6}$$

Note that the elements of the first column of Q are unity because of (1.6). Note further that, for this *series-to-sequence matrix* Q, the notation of the elements is different from the usual one. From the coefficients $\{\gamma_{m-1,k}\}_{0 \leqq k \leqq m-1}$ of the m-th row of Q, one can define polynomials

$$q_{m-1}(z) := \sum_{i=0}^{m-1} \gamma_{m-1,i} z^i. \tag{2.7}$$

Using (1.3′) and (2.3), together with the definition of Q, a short calculation shows that

$$\mathbf{y}_m = \mathbf{x}_0 + q_{m-1}(T)\mathbf{r}_0. \tag{2.8}$$

Hence from (2.4), we conclude that $\mathbf{y}_m$ defined in (2.8) would be a good approximation for the solution $\mathbf{x}$ if $q_{m-1}(T)$ approximates well $g(T) := (I-T)^{-1}$. A hint as to how to choose these polynomials $q_{m-1}(z)$, can be seen from the next theorem which gives an interesting relation between the polynomial $p_m(z)$ of (1.7) and $q_{m-1}(z)$ of (2.7).

Theorem 1 ([EN]). *a) Let* $P := (\pi_{m,i})_{m \geqq 0, 0 \leqq i \leqq m}$ *be an infinite lower triangular matrix satisfying* $\sum_{i=0}^{m} \pi_{m,i} = 1$ $(m \geqq 0)$. *Then, for* $Q = PS$ *defined by* (2.6), *and for the corresponding sequences of polynomials* $\{p_m(z)\}_{m \geqq 0}$ *of* (1.7) *and* $\{q_{m-1}(z)\}_{m \geqq 1}$ *of* (2.7), *there holds*

$$q_{m-1}(z) = \frac{1 - p_m(z)}{1 - z} \quad (m \geqq 1; q_{-1}(z) := 0), \tag{2.9}$$

and

$$p_m(z) = 1 - (1-z) q_{m-1}(z) \quad (m \geqq 0). \tag{2.10}$$

b) If $\xi_i^{(m)}$ $(i = 1, \ldots, l)$ *is any zero of* $p_m(z)$ *with exact multiplicity* k_i, *then for the "geometric function"* $g(z) = 1/(1-z)$, *there holds*

$$q_{m-1}^{(j)}(\xi_i^{(m)}) = g^{(j)}(\xi_i^{(m)}), \quad (i = 1, \ldots, l; j = 0, \ldots, k_i - 1), \tag{2.11}$$

i.e., $q_{m-1}(z)$ *is the unique Hermite interpolation polynomial which interpolates* $g(z)$ *at the zeros of* $p_m(z)$.

Proof. The infinite triangular matrix S introduced in (2.5) is invertible, its inverse being given explicitly by

$$S^{-1}=\begin{bmatrix} 1 & & & & \\ -1 & 1 & & \mathbf{0} & \\ & -1 & 1 & & \\ \mathbf{0} & & -1 & 1 & \\ & & \vdots & & \ddots \end{bmatrix}. \tag{2.12}$$

Thus from (2.6), we obtain $P=QS^{-1}$. This implies that the elements of the m-th row of P, using (1.5), (2.6) and (2.12), satisfy

$$\begin{aligned} \pi_{m,0} &= \qquad -\gamma_{m-1,0}+1, \\ \pi_{m,k} &= \gamma_{m-1,k-1}-\gamma_{m-1,k} \qquad (k=1,\dots,m-1), \\ \pi_{m,m} &= \gamma_{m-1,m-1}. \end{aligned}$$

Using the definitions of $p_m(z)$ in (1.7) and of $q_{m-1}(z)$ in (2.7), the above relations imply $p_m(z)=zq_{m-1}(z)-q_{m-1}(z)+1=1-(1-z)q_{m-1}(z)$, i.e., (2.10) holds and (2.9) follows from (2.10). The interpolating property in part b) follows by inserting the zeros of $p_m(z)$, according to their multiplicities, in the differentiated identity of (2.10). □

For our purpose of comparing different semiiterative methods existing in the literature, it is useful to introduce a third possibility of representing such matrix transformations from summability theory. The sequence $\{\mathbf{y}_m\}_{m\geqq 0}$ can be interpreted as the sequence of partial sums of the series $\sum_{j=0}^{\infty}\mathbf{z}_j$ with $\mathbf{z}_j:=\mathbf{y}_j-\mathbf{y}_{j-1}$ ($j\geqq 0$, $\mathbf{y}_{-1}:=0$). In a notation similar to that of (2.5), we obtain

$$\begin{bmatrix}\mathbf{z}_0\\ \mathbf{z}_1\\ \mathbf{z}_2\\ \vdots\end{bmatrix}=S^{-1}\begin{bmatrix}\mathbf{y}_0\\ \mathbf{y}_1\\ \mathbf{y}_2\\ \vdots\end{bmatrix}, \tag{2.13}$$

where S^{-1} is the inverse introduced in (2.12). Thus, we can consider the *series-to-series transformation* which transforms the series with terms $\{\mathbf{x}_0,\mathbf{r}_0,\mathbf{r}_1,\dots\}$ to the series with terms $\{\mathbf{z}_0,\mathbf{z}_1,\mathbf{z}_2,\dots\}$. As can be seen from (2.6) and (2.13), this transformation is induced by

$$V:=S^{-1}PS=S^{-1}Q, \tag{2.14}$$

where, from the form of Q in (2.6) and S^{-1} in (2.12), it follows that the infinite lower triangular matrix V can be represented as

$$V=:\begin{bmatrix} 1 & & & \\ 0 & \rho_{1,1} & & \mathbf{0} \\ 0 & \rho_{2,1} & \rho_{2,2} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.15}$$

It should be noted that a matrix transformation, i.e., a semiiterative method, can be given by any one of the three matrices P, Q, and V; the remaining two can always be computed from (2.6) or (2.14) (see also Zeller and Beekmann [ZB, p. 5]). By Theorem 1 then, a fourth infinite lower triangular matrix is given, namely the *nodal matrix* $K_P = (\xi_i^{(m)})_{m \geqq 1, 1 \leqq i \leqq m}$ of (1.10), consisting of the nodes $\xi_i^{(m)}$ of the interpolating polynomials $q_{m-1}(z)$ in (2.11), or equivalently, the zeros of $p_m(z)$ $(m \geqq 1)$. On the other hand, we can start with a nodal matrix K, introduce the polynomials $p_m(z)$ by

$$p_0 = 1, \quad p_m(z) = \prod_{i=1}^{m} \frac{(z - \xi_i^{(m)})}{(1 - \xi_i^{(m)})} \quad (m \geqq 1), \tag{2.16}$$

or construct the polynomials $q_{m-1}(z)$ interpolating $g(z) = 1/(1-z)$ at the nodes $\xi_i^{(m)}$ $(i = 1, \ldots, m)$. The coefficients of $p_m(z)$ and $q_{m-1}(z)$ $(m \geqq 1)$ respectively define the matrices P and Q. Thus, we have different possibilities for defining a SIM.

Euler methods, extensively treated in [NV], are usually introduced as series-to-series transformations, according to (2.14) and (2.15). The corresponding matrix V is generated by an *Euler function* $h(\phi)$ defined as following ([NV, §3]): Let $D_\eta := \{\phi \in \mathbb{C}: |\phi| < \eta\}$, and let $\bar{D}_\eta$ denote its closure. Then, $h(\phi)$ is called an *Euler function* if there exists an open neighborhood $\mathbb{D}$ of $\bar{D}_1$ such that

(i) $h(\phi)$ is meromorphic and univalent in $\mathbb{D}$, and
(ii) $h(0) = 0$, $h(1) = 1$.

Thus, if $h(\phi)$ is an Euler function, there exists a $\nu > 0$ such that $h(\phi)$ is holomorphic in D_ν. Further, there exist power series for all powers of $h(\phi)$, i.e., if $h(\phi) := \sum_{j=1}^{\infty} \rho_{j,1} \phi^j$, $|\phi| < \nu$, then

$$[h(\phi)]^m =: \sum_{j=m}^{\infty} \rho_{j,m} \phi^j, \quad (m = 0, 1, \ldots), \ |\phi| < \nu. \tag{2.17}$$

Now, the coefficients $\{\rho_{j,m}\}_{j \geqq 0; 0 \leqq m \leqq j}$ define an infinite lower triangular matrix $V =: V_E$, which induces a *general Euler method*. As one of our new results, we now give

Theorem 2. *To each Euler method induced by an infinite lower triangular matrix* V_E, *there corresponds a* SIM *induced by*

$$P_E := S V_E S^{-1}, \tag{2.18}$$

where S *and* S^{-1} *are given by* (2.5) *and* (2.12).

Proof. Let the Euler method be generated by the Euler function $h(\phi)$. From $[h(\phi)]^0 := 1$, it follows that the first column of the associated infinite lower triangular matrix V_E is (cf. (2.15)) $(1, 0, 0, \ldots)^T$. Then, as (cf. (2.14)) $Q_E := S V_E$, the first column of Q_E is $(1, 1, 1, \ldots)^T$ so that in the notation of (2.6), $\gamma_{j,-1} = 1$ $(j = -1, 0, 1, \ldots)$. Thus, it follows (cf. (2.6) and (2.12)) that the elements $\pi_{m,i}$ of the infinite lower triangular matrix $P_E := Q_E S^{-1}$, satisfy $\pi_{m,i} = \gamma_{m-1,i-1} - \gamma_{m-1,i}$

$(i=0,1,\ldots,m)$, so that

$$\sum_{i=0}^{m} \pi_{m,i} = \sum_{i=0}^{m} (\gamma_{m-1,i-1} - \gamma_{m-1,i}) = 1 - \gamma_{m-1,m} = 1,$$

i.e., (1.6) holds for P_E. □

One can ask what the result is of applying an Euler method to the series (2.2) with the terms $(\mathbf{x}_0, \mathbf{r}_0, \mathbf{r}_1 \ldots)^T$. Formally, this vector is multiplied from the left by V_E which yields the vector $(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2 \ldots)^T$ which represents the terms of the transformed series. Now, things become a bit simpler if we assume, as in [NV], in (1.2) that $\mathbf{x}_0 = \mathbf{c}$, which yields $\mathbf{r}_0 = T\mathbf{c}$ and $\mathbf{r}_j = T^{j+1}\mathbf{c}$ $(j>0)$. Thus, V_E is applied to $(\mathbf{c}, T\mathbf{c}, T^2\mathbf{c}, \ldots)^T$, and we get, with the notation of (2.15), that the terms $\mathbf{z}_j$ of the transformed series are given by

$$\mathbf{z}_0 = \mathbf{c}, \qquad \mathbf{z}_j = \sum_{k=1}^{j} \rho_{j,k}(T^k \mathbf{c}) =: v_j(T)\mathbf{c}, \tag{2.19}$$

so that

$$v_j(T) := \sum_{k=1}^{j} \rho_{j,k} T^k \qquad (j>0). \tag{2.20}$$

The partial sums of the transformed series $\sum_{j=0}^{\infty} \mathbf{z}_j$ are then the iterates $\mathbf{y}_m$, i.e.,

$$\mathbf{y}_m = \sum_{j=0}^{m} v_j(T)\mathbf{c}, \qquad \text{where } v_0(T) := I. \tag{2.21}$$

3. Different Ways of Computing the Iterates of a SIM

If the iterates $\mathbf{y}_m$ of a SIM, as defined in (1.4), were directly computed from (1.4), this would require the storage of all the vectors $\mathbf{x}_m$, as well as the explicit knowledge of the entries of the infinite matrix P of (1.5). The storage of all vectors $\mathbf{x}_m$ alone would naturally lead one to consider alternate ways of computing the vectors $\mathbf{y}_m$. In the symmetric positive definite case treated by Varga (cf. [V2, p. 132]), the well-known recurrence relation for the Chebyshev polynomials yields a corresponding three-term recursive formula for the $\mathbf{y}_m$.

For a given Euler method, we similarly do not use (2.19) which would require the evaluation of the matrix polynomial $v_j(T)$.

Lemma 3 ([NV, Lemma 4]). *If $\rho_{j,1}$ $(j=1,2,\ldots)$ are the coefficients of the power series of an Euler function $h(\phi)$, then the iterates $\mathbf{y}_m$ of (2.21) can be recursively computed by $\mathbf{y}_0 = \mathbf{z}_0 = \mathbf{c}$, $\mathbf{y}_m = \mathbf{y}_{m-1} + \mathbf{z}_m$ with*

$$\mathbf{z}_m := T\left(\sum_{j=0}^{m-1} \rho_{m-j,1} \mathbf{z}_j\right). \tag{3.1}$$

Thus, we don't have to explicitly compute the Euler matrix P_E, but it is necessary to store the vectors $\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1}$.

For our later use (cf. § 7), we shall deduce another recurrence relation for the vectors $\mathbf{y}_m$ of (2.21). For any Euler function $h(\phi)$, we consider the function

$\tilde{h}(\phi) := 1/h(\phi)$ which is obviously meromorphic in a neighborhood of the origin, and has a simple pole at $\phi = 0$. Therefore, we can develop $\tilde{h}(\phi)$ in a Laurent series, of the form

$$\tilde{h}(\phi) = \frac{1}{h(\phi)} = \frac{1}{\mu_0}\left\{\frac{1}{\phi} - \mu_1 - \mu_2\phi - \mu_3\phi^2 - \dots\right\}, \tag{3.2}$$

whose coefficients μ_j can be computed from the Taylor coefficients $\rho_{j,1}$ of $h(\phi)$ (cf. (2.17)), by a simple recursion.

The following result, while new, is similar to Theorem 5 of [NV].

Theorem 4. *Let the Euler function $h(\phi)$ be given. Then, the iterates $\mathbf{y}_m$ of* (2.21) *of the corresponding Euler method obey the following recurrence relation:* $\mathbf{y}_0 = \mathbf{c}$, $\mathbf{y}_1 = \mathbf{c} + \mu_0 T\mathbf{y}_0$,

$$\mathbf{y}_m = \left(1 - \sum_{j=1}^{m-1} \mu_j\right)\mathbf{c} + \mu_0 T\mathbf{y}_{m-1} + \sum_{j=1}^{m-1} \mu_j \mathbf{y}_{m-j} \quad (m \geqq 2) \tag{3.3}$$

where the μ_j's are the Laurent coefficients of $\tilde{h}$ (cf. (3.2)).

Proof. The identity $[h(\phi)]^{m-1} = \tilde{h}(\phi)[h(\phi)]^m$ implies (cf. (2.17) and (3.2)) that

$$\mu_0\phi\left(\sum_{j=m-1}^{\infty} \rho_{j,m-1}\phi^j\right) = \left(1 - \sum_{j=1}^{\infty} \mu_j\phi^j\right)\cdot\left(\sum_{j=m}^{\infty} \rho_{j,m}\phi^j\right).$$

Comparing the coefficients of like powers of ϕ, we obtain

$$\rho_{j,m} = \mu_0\rho_{j-1,m-1} + \sum_{l=1}^{j-m} \mu_l\rho_{j-l,m} \quad (1 \leqq m \leqq j).$$

Further, we have $\rho_{0,0} = 1$, $\rho_{0,m} = 0$ $(m \geqq 1)$ (cf. (2.17)). Using this, we conclude from (2.20) for $j \geqq 1$ that

$$\begin{aligned} v_j(T) &= \sum_{k=1}^{j} \rho_{j,k}T^k = \sum_{k=1}^{j}\left(\mu_0\rho_{j-1,k-1} + \sum_{l=1}^{j-k} \mu_l\rho_{j-l,k}\right)T^k \\ &= \mu_0 T \sum_{k=1}^{j-1} \rho_{j-1,k}T^k + \sum_{l=1}^{j-1} \mu_l \sum_{k=1}^{j-l} \rho_{j-l,k}T^k \\ &= \mu_0 T v_{j-1}(T) + \sum_{l=1}^{j-1} \mu_l v_{j-l}(T). \end{aligned}$$

Now, from (2.21) and $v_0(T)\mathbf{c} = \mathbf{c}$, there follows

$$\begin{aligned} \mathbf{y}_m &= \sum_{j=0}^{m} v_j(T)\mathbf{c} = \mathbf{c} + \sum_{j=1}^{m} v_j(T)\mathbf{c} \\ &= \mathbf{c} + \sum_{j=1}^{m}\left[\mu_0 T v_{j-1}(T) + \sum_{l=1}^{j-1} \mu_l v_{j-l}(T)\right]\mathbf{c} \\ &= \mathbf{c} + \mu_0 T \sum_{j=0}^{m-1} v_j(T)\mathbf{c} + \sum_{l=1}^{m-1} \mu_l \sum_{j=1}^{m-l} v_j(T)\mathbf{c} \\ &= \left(1 - \sum_{l=1}^{m-1} \mu_l\right)\mathbf{c} + \mu_0 T\mathbf{y}_{m-1} + \sum_{l=1}^{m-1} \mu_l \mathbf{y}_{m-l}. \quad \square \end{aligned}$$

An easy consequence of this theorem is the following

Corollary 5 ([NV, Theorem 5]). *Let the Euler function* $h(\phi)$ *be given by*

$$h(\phi)=\frac{\mu_0\phi}{1-\mu_1\phi-\ldots-\mu_k\phi^k}. \tag{3.4}$$

Then, the iterates $\mathbf{y}_m$ *of* (2.21) *of the corresponding Euler method can be resursively computed by* $\mathbf{y}_0=\mathbf{c}$, $\mathbf{y}_1=\mathbf{c}+\mu_0 T\mathbf{y}_0$,

$$\mathbf{y}_m=(1-\mu_1-\ldots-\mu_{m-1})\mathbf{c}+\mu_0 T\mathbf{y}_{m-1}+\mu_1\mathbf{y}_{m-1}+\ldots+\mu_{m-1}\mathbf{y}_1. \tag{3.5}$$

$(2\leqq m\leqq k)$, *and*

$$\mathbf{y}_m=\mu_0(T\mathbf{y}_{m-1}+\mathbf{c})+\mu_1\mathbf{y}_{m-1}+\ldots+\mu_k\mathbf{y}_{m-k} \quad (m>k). \tag{3.6}$$

In deriving (3.6), use has been made of the fact that $h(\phi)$ is an Euler function, so that $h(1)=1$. Thus, from (3.4), it follows that $(1-\mu_1\ldots-\mu_k)=\mu_0$.

Equation (3.6) corresponds to a k-step stationary iterative method. For $k=1$, (3.6) yields to so-called stationary first-order Richardson method. Usually, the parameters $\mu_0=:\alpha$ and $\mu_1=1-\alpha$ are commonly found in the literature. The *nonstationary first-order Richardson method* is defined by

$$\mathbf{y}_m=\alpha_m T\mathbf{y}_{m-1}+(1-\alpha_m)\mathbf{y}_{m-1}+\alpha_m\mathbf{c} \quad (m\geqq 1) \tag{3.7}$$

with variable coefficients $\alpha_m\in\mathbb{C}$, $\alpha_m\neq 0$. If $\tilde{\mathbf{e}}_m:=\mathbf{x}-\mathbf{y}_m$ is the error vector, then from (3.7) it follows that

$$\tilde{\mathbf{e}}_m=p_m(T)\mathbf{e}_0, \quad \text{with} \quad p_m(T)=\prod_{j=1}^{m}(\alpha_j T+(1-\alpha_j)I). \tag{3.8}$$

For the zeros $\xi_i^{(m)}$ of $p_m(z)$, we have

$$\xi_i^{(m)}=\xi_i=1-1/\alpha_i, \quad \alpha_i=1/(1-\xi_i) \quad (i=1,\ldots,m). \tag{3.9}$$

Since $p_m(1)=1$ $(m\geqq 1)$, we see by comparing (3.8) with (1.8) that this first-order Richardson method is a special SIM, induced by the nodal matrix

$$K:=\begin{bmatrix} \xi_1 & & & & \\ \xi_1 & \xi_2 & & \mathbf{0} & \\ \vdots & \vdots & \ddots & & \\ \xi_1 & \xi_2 & \cdots & \cdots\ \xi_m & \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}. \tag{3.10}$$

We call a nodal matrix K of the form (3.10) *column-constant* if the terms of the i-th column beneath the diagonal entry are a fixed number ξ_i $(i=1,2,\ldots)$.

Now, if there is a SIM induced by a column-constant nodal matrix K, then by defining parameters α_i according to (3.9), we see from (3.8) that the corresponding iterates $\mathbf{y}_m$ can be calculated by (3.7). This yields

Theorem 6 ([EN, Theorem 5]). *A* SIM *induced by a column-constant nodal matrix corresponds to a first-order Richardson method* (3.7), *where the parameters are given by* (3.9).

Remark. First-order Richardson methods have been recently considered by Opfer and Schober [OS]. Starting with a system $A\mathbf{x}=\mathbf{b}$, their iteration is defined by

$$\mathbf{y}_m=\mathbf{y}_{m-1}+\alpha_m(\mathbf{b}-A\mathbf{y}_{m-1}), \tag{3.11}$$

with the associated error vectors

$$\mathbf{e}_m=\hat{p}_m(A)\mathbf{e}_0, \quad \text{where } \hat{p}_m(A):=\prod_{j=1}^{m}(I-\alpha_j A), \tag{3.12}$$

and $\hat{p}_m(0)=1$. If $A=I-T$, then the polynomials $\hat{p}_m(z)$ of (3.12) and $p_m(z)$ of (3.8) are related via $\hat{p}_m(z)=p_m(1-z)$.

If the polynomials $p_m(z)$ of (1.7) satisfy a $(k+1)$-term recurrence relation of the type

$$p_m(z)=\mu_{m,0}zp_{m-1}(z)+\sum_{j=1}^{k}\mu_{m,j}p_{m-j}(z), \quad (m\geqq k;\ \mu_{m,j}\in\mathbb{C}, 0\leqq j\leqq k), \tag{3.13}$$

then the iterates $\mathbf{y}_m$ of (1.4) also can be computed recursively. We shall demonstrate this for the case $k=2$:

Theorem 7. *Assume that the polynomials* $p_m(z)$ *of* (1.7) *obey the following three-term recurrence relation*

$$p_m(z)=(\mu_{m,0}z+\mu_{m,1})p_{m-1}(z)+\mu_{m,2}p_{m-2}(z) \tag{3.14}$$

$\left(m\geqq 2;\ \mu_{m,j}\in\mathbb{C}, 0\leqq j\leqq 2;\ \sum_{j=0}^{2}\mu_{m,j}=1\right)$, *with* $p_0(z)\equiv 1$, $p_1(z)=\pi_{1,0}+\pi_{1,1}z$ (cf. (1.7)). *Then, for the corresponding iterates* $\mathbf{y}_m$, *there holds:* $\mathbf{y}_0=\mathbf{a}$, $\mathbf{y}_1=\mathbf{y}_0+\pi_{1,1}\tilde{\mathbf{r}}_0$ *and*

$$\mathbf{y}_m=\mathbf{y}_{m-1}+\mu_{m,0}\tilde{\mathbf{r}}_{m-1}-\mu_{m,2}(\mathbf{y}_{m-1}-\mathbf{y}_{m-2}) \quad (m\geqq 2), \tag{3.15}$$

where $\tilde{\mathbf{r}}_m=\mathbf{c}-(I-T)\mathbf{y}_m$ *is the associated residual vector.*

Proof. From (2.8) and (2.10), we easily conclude that the iterates $\mathbf{y}_m$ can be represented in the form

$$\mathbf{y}_m=q_{m-1}(T)\mathbf{c}+p_m(T)\mathbf{a} \quad (m\geqq 0), \tag{3.16}$$

where the polynomials $q_m(z)$ are given by (2.7) and (2.9) (where $q_{-1}(z):=0$). The recurrence (3.14) then yields

$$q_{m-1}(z)=\mu_{m,0}(1+zq_{m-2}(z))+\mu_{m,1}q_{m-2}(z)+\mu_{m,2}q_{m-3}(z) \quad (m\geqq 2). \tag{3.17}$$

Inserting (3.14) and (3.17) in (3.16), we obtain the desired recursion (3.15). □

As an example, we consider the SIM which is generated by the translated and scaled Chebyshev polynomials

$$p_m(z)=T_m\left(\frac{2z-(\alpha+\beta)}{\beta-\alpha}\right)\Big/ T_m\left(\frac{2-(\alpha+\beta)}{\beta-\alpha}\right) \quad (m\geqq 0),$$

(cf. [V2, Chapter 5] in the real case) where here we assume α and β are complex numbers such that $z=1$ is not contained in the line segment joining α

and β, and where $T_m(z)$ denotes the "ordinary" Chebyshev polynomial of degree m, of the first kind, i.e.,

$$T_0(z)=1; \quad T_1(z)=z; \quad T_m(z)=2zT_{m-1}(z)-T_{m-2}(z) \quad (m\geqq 2).$$

With the abbreviations $\gamma:=\frac{1}{2}(\beta-\alpha)$ and $\delta:=1-\frac{1}{2}(\alpha+\beta)$, we obtain for the polynomials $p_m(z)$:

$$\begin{aligned} &p_0(z)=1; \quad p_1(z)=(z-1+\delta)/\delta \\ &p_m(z)=(z-1+\delta)\frac{2}{\gamma}\frac{T_{m-1}(\delta/\gamma)}{T_m(\delta/\gamma)}p_{m-1}(z)-\frac{T_{m-2}(\delta/\gamma)}{T_m(\delta/\gamma)}p_{m-2}(z) \quad (m\geqq 2). \end{aligned} \tag{3.18}$$

Thus, the polynomials $p_m(z)$ satisfy a recursion of the form (3.14) with the coefficients

$$\mu_{m,0}=\frac{2}{\gamma}\frac{T_{m-1}(\delta/\gamma)}{T_m(\delta/\gamma)}; \quad \mu_{m,1}=(-1+\delta)\mu_{m,0}; \quad \mu_{m,2}=1-\delta\mu_{m,0}.$$

With $\mathbf{z}_m:=\mathbf{y}_m-\mathbf{y}_{m-1}$, we see that (3.15) now can be written in the form

$$\begin{aligned} &\mathbf{y}_0=\mathbf{a}; \quad \mathbf{y}_m=\mathbf{y}_{m-1}+\mathbf{z}_m; \\ &\mathbf{z}_1=\frac{1}{\delta}\tilde{\mathbf{r}}_0; \quad \mathbf{z}_m=\alpha_m\tilde{\mathbf{r}}_{m-1}+\beta_m\mathbf{z}_{m-1}, \end{aligned} \tag{3.19}$$

where the coefficients α_m, β_m $(m\geqq 2)$ are given by

$$\begin{aligned} &\alpha_2=\mu_{2,0}=\frac{2\delta}{2\delta^2-\gamma^2}; \quad \beta_2=-\mu_{2,2}=\delta\alpha_2-1; \\ &\alpha_m=\mu_{m,0}=\left[\delta-\left(\frac{\gamma}{2}\right)^2\alpha_{m-1}\right]^{-1}; \quad \beta_m=-\mu_{m,2}=\delta\alpha_m-1 \quad (m\geqq 3). \end{aligned} \tag{3.20}$$

The formulas (3.19) and (3.20) are known as the "Chebyshev iteration" which, for example, is treated by Manteuffel [M]. Since Manteuffel starts with the system $A\mathbf{x}=\mathbf{b}$ and considers the iteration

$$\mathbf{x}_m=\mathbf{x}_{m-1}+\sum_{i=1}^{m-1}\hat{\gamma}_{m,i}\mathbf{r}_i \quad \text{with} \quad \mathbf{r}_i:=\mathbf{b}-A\mathbf{x}_i, \tag{3.21}$$

his notation slightly differs from the one used here. But using the transformation $z\to 1-z$ (cf. the Remark following Theorem 6), it is obvious that (3.21) can be considered as a SIM (in series-to-series form).

To summarize this section, there are three practical ways of computing the iterates $\mathbf{y}_m$ of a SIM:

1. Recurrence formulas for the polynomials $p_m(z)$ in (1.7) yield recurrence formulas for the $\mathbf{y}_m$.

2. If the SIM is an Euler method induced by an Euler function of the form (3.4), then the $\mathbf{y}_m$ can be computed by (3.5) and (3.6).

3. If the nodal matrix of the SIM is column-constant, then the computation can be done via (3.7).

4. The Asymptotic Convergence Factor of a SIM

We have seen that a SIM can be induced by any one of the four infinite lower triangular matrices P, Q, V, and K. The m-th rows of P, Q and V define polynomials $p_m(z)$, $q_{m-1}(z)$, and $v_m(z)$. How should these polynomials be chosen in our problem? There is a trivial solution, namely, if all eigenvalues of T are chosen as nodes, i.e., as zeros of $p_m(z)$, then $p_m(T)=0$ by the Theorem of Cayley-Hamilton (cf. [V2, p. 135]). But usually, $\sigma(T)$ is not known. Thus, it is more realistic to assume that a compact set $\Omega\subset\mathbb{C}$ is known such that $\sigma(T)\subset\Omega$. In this section, we give criteria for the convergence of the $\{\mathbf{y}_m\}_{m\geqq 0}$ and for a certain asymptotic decrease of the error which involve the eigenvalues of T, whereas in the next section, criteria are given which only use the knowledge of a set Ω such that $\sigma(T)\subset\Omega$.

From (1.8), we see that for arbitrary $\mathbf{y}_0$, the sequence $\{\mathbf{y}_m\}_{m\geqq 0}$ of (1.4) converges to the solution of (1.1) iff the sequence $\{p_m(T)\}_{m\geqq 0}$ of matrix polynomials converges to the zero matrix. Comparing (2.4) and (2.8) shows that the sequence $\{q_{m-1}(T)\}_{m\geqq 1}$ should converge to $(I-T)^{-1}$. The following lemma gives a well-known criterion for this convergence.

Lemma 8 (Gantmacher [G, p. 102]). *Let* $m(z):=\prod_{i=1}^{k}(z-\lambda_i)^{n_i}$, *(where* $\lambda_i\neq\lambda_j$ *for* $i\neq j$*) be the minimal polynomial for* T*. Then, for arbitrary* $\mathbf{y}_0$, *the sequence* $\{\mathbf{y}_m\}_{m\geqq 0}$ *of* (1.4) *converges to the solution of* (1.1) *iff one of the following two conditions holds:*

$$\lim_{m\to\infty} p_m^{(j)}(\lambda_i)=0 \quad \text{for } 1\leqq i\leqq k,\ 0\leqq j\leqq n_i-1, \tag{4.1}$$

$$\lim_{m\to\infty} q_m^{(j)}(\lambda_i)=\left[\frac{d^j}{dx^j}\left(\frac{1}{1-x}\right)\right]_{x=\lambda_i}, \quad \text{for } 1\leqq i\leqq k,\ 0\leqq j\leqq n_i-1. \tag{4.2}$$

If T is diagonalizable and $\sigma(T)$ denotes the spectrum of T, then both criteria reduce to

$$\lim_{m\to\infty} p_m(\lambda)=0, \qquad \text{for all } \lambda\in\sigma(T), \tag{4.1'}$$

and

$$\lim_{m\to\infty} q_m(\lambda)=\frac{1}{1-\lambda}, \qquad \text{for all } \lambda\in\sigma(T). \tag{4.2'}$$

If G is an open set in $\mathbb{C}$ such that $1\notin G$ and such that $\sigma(T)\subset G$, then the following conditions are sufficient:

$$\{p_m(z)\}_{m\geqq 0} \text{ converges to } 0, \text{ uniformly on every compact subset of } G; \tag{4.3}$$

$$\{q_{m-1}(z)\}_{m\geqq 1} \text{ converges to } 1/(1-z), \text{ uniformly on every compact subset of } G. \tag{4.4}$$

Now, let P induce a SIM such that $\{\mathbf{y}_m\}_{m\geqq 0}$ converges to the solution $\mathbf{x}$ of (1.1). The error (cf. (1.8)),

$$\tilde{\mathbf{e}}_m=p_m(T)\mathbf{e}_0, \tag{4.5}$$

depends on P, T and $\mathbf{e}_0$. Thus, an appropriate measure for the asymptotic decrease of the error $\tilde{\mathbf{e}}_m$ is the *asymptotic convergence factor*

$$\kappa(T,P) := \overline{\lim_{m\to\infty}} \left\{ \sup_{\mathbf{e}_0 \neq 0} \left[\frac{\|\tilde{\mathbf{e}}_m\|}{\|\mathbf{e}_0\|} \right]^{1/m} \right\}. \tag{4.6}$$

We remark that for iterative methods, $\kappa(T,P)$ is sometimes denoted as "root-convergence factor" (cf. Ortega and Reinboldt [OR, p. 288]) and is independent of the norm chosen. The following lemma is easily derived, using the Jordan normal form of T.

Lemma 9. *Let* $m(z) = \prod_{i=1}^{k} (z-\lambda_i)^{n_i}$ *(where* $\lambda_i \neq \lambda_j$ *for* $i \neq j$*) be the minimal polynomial of a given matrix T. For any P, there holds*

$$\kappa(T,P) = \overline{\lim_{m\to\infty}} \left\{ \max_{1 \leq i \leq k} \max_{0 \leq l \leq n_i - 1} \left| \frac{d^l}{d\lambda^l} p_m(\lambda_i) \right|^{1/m} \right\}. \tag{4.7}$$

If T is diagonalizable, then (4.7) *has the simpler form*

$$\kappa(T,P) = \overline{\lim_{m\to\infty}} \{ \max_{\lambda \in \sigma(T)} |p_m(\lambda)|^{1/m} \}. \tag{4.7'}$$

The next lemma shows how $\kappa(T,P)$ is determined by the matrices Q and V through their corresponding sequences of polynomials $\{q_{m-1}(z)\}_{m \geq 1}$ and $\{v_m(z)\}_{m \geq 0}$.

Lemma 10. *a) For given diagonalizable matrices T and* $Q = PS$ (cf. (2.6)), *there holds*

$$\kappa(T,P) = \overline{\lim_{m\to\infty}} \{ \max_{\lambda \in \sigma(T)} |q_{m-1}(\lambda) - 1/(1-\lambda)|^{1/(m-1)} \}. \tag{4.8}$$

b) If $0 < \kappa(T,P) < 1$, *then for* $V = S^{-1}PS$, *as defined in* (2.14), *there holds*

$$\kappa(T,P) = \overline{\lim_{m\to\infty}} \{ \max_{\lambda \in \sigma(T)} |v_m(\lambda)|^{1/m} \}. \tag{4.9}$$

Proof. From (2.9), $q_{m-1}(z) - 1/(1-z) = -p_m(z)/(1-z)$, so that part a) follows from (4.7′). For part b), since $\kappa(T,P) < 1$ by hypothesis, the series (cf. (2.21)) $\sum_{j=0}^{\infty} v_j(T)\mathbf{c}$ converges to the solution $\mathbf{x}$. Thus, for the error vector, there then holds

$$\tilde{\mathbf{e}}_m = \mathbf{x} - \mathbf{y}_m = p_m(T)\mathbf{c} = \sum_{j=m+1}^{\infty} v_j(T)\mathbf{c}.$$

By Lemma 8, we have $p_m(\lambda) = \sum_{j=m+1}^{\infty} v_j(\lambda)$ for all $\lambda \in \sigma(T)$. From Theorem 1 in Wimp [Wi, p. 6], it follows that $\overline{\lim_{m\to\infty}} |p_m(\lambda)|^{1/m} = \overline{\lim_{m\to\infty}} |v_m(\lambda)|^{1/m}$. Thus, (4.9) is a consequence of (4.7′). □

5. Construction of SIM's

One of our basic aims is to show the connection between the results on SIM's in [EN] and the results on Euler methods in [NV]. For this, we have to recall the main theorems in [EN].

Let us assume that we know a compact set Ω with $\sigma(T)\subset\Omega$. Since the matrix A of the given system is nonsingular iff 1 is not an eigenvalue of T, we may assume $1\notin\Omega$. We want to find SIM's induced by P such that $\kappa(T,P)$ is as small as possible, i.e., we seek a SIM with a maximal asymptotic rate of convergence. An even better aim would be to maximize the *average rate of convergence* ([V2, p. 134]) which leads to the minimization problem

$$\min\{\max_{z\in\Omega}|p_m(z)|:\ p_m\in\pi_m \text{ and } p_m(1)=1\},\tag{5.1}$$

where π_m is the set of all polynomials of degree m. A classical result (see, e.g., Smirnov-Lebedev [SL, p. 367]) states that (5.1) has a solution $p_m^*(z)$ for each m. These polynomials $p_m^*(z)$ yield an infinite lower triangular matrix P^*, but only for very special sets Ω is P^* explicitly known. For example, if Ω is an interval on the real line, appropriately normalized Chebyshev polynomials yield the solution of (5.1) ([V2, p. 135]). For more general domains Ω, we confine oursolves to the minimization of the asymptotic rate of convergence.

Given a SIM induced by P, let us introduce, in analogy to (4.7′), the quantity

$$\kappa(\Omega,P):=\overline{\lim_{m\to\infty}}\{\max_{z\in\Omega}|p_m(z)|^{1/m}\}.\tag{5.2}$$

Thus, if $\sigma(T)\subset\Omega$ and if T is diagonalizable, we have from (4.7′) that $\kappa(T,P)\leqq\kappa(\Omega,P)$. Further, let us define the class of infinite lower triangular matrices

$$\mathbb{P}:=\left\{P=(\pi_{i,j})_{i\geqq 0,\,0\leqq j\leqq i}:\ \pi_{i,j}\in\mathbb{C},\ \sum_{j=0}^{i}\pi_{i,j}=1\ \ (i\geqq 0)\right\}.\tag{5.3}$$

Then, each $P\in\mathbb{P}$ induces a SIM. This leads to the *asymptotic convergence factor of* Ω defined by

$$\kappa(\Omega):=\inf_{P\in\mathbb{P}}\kappa(\Omega,P).\tag{5.4}$$

A SIM induced by $\tilde{P}$ such that $\kappa(\Omega,\tilde{P})=\kappa(\Omega)$ is called an *asymptotically optimal* SIM *with respect to* Ω (AOSIM). The matrix P^*, mentioned in connection with problem (5.1), clearly induces an AOSIM with respect to Ω. Thus, we wish to construct AOSIM's which are different from that induced by P^*, but which can be described more explicitly.

For this, on setting $\bar{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$, we introduce the class

$$\begin{aligned}\mathbb{M}:=\{\Omega\subset\mathbb{C}:\ &\Omega \text{ is compact, } 1\in\bar{\mathbb{C}}\backslash\Omega,\ \bar{\mathbb{C}}\backslash\Omega \text{ is simply connected}\\ &\text{and } \Omega \text{ contains more than one point}\}.\end{aligned}\tag{5.5}$$

Some comments on the case where $\bar{\mathbb{C}}\backslash\Omega$ is connected (but not simply connected) will be given at the end of this section.

By the Riemann Mapping Theorem, there exists, for each $\Omega \in \mathbb{M}$, a conformal mapping

$$\psi: \bar{\mathbb{C}} \setminus \{w: |w| \leqq 1\} \to \bar{\mathbb{C}} \setminus \Omega, \tag{5.6}$$

with

$$\psi(\infty) = \infty, \qquad \psi'(\infty) =: \gamma(\Omega) > 0. \tag{5.7}$$

The constant $\gamma(\Omega)$ is known as the *capacity* of Ω (cf. Walsh [W, p. 74]). There exists a unique complex number $\hat{w}$ with

$$\psi(\hat{w}) = 1 \quad \text{and} \quad \hat{\eta} := |\hat{w}| > 1. \tag{5.8}$$

Now, we know from Theorem 1 that, to each $P \in \mathbb{P}$, there corresponds an infinite matrix $Q = PS$ defining a sequence of polynomials $\{q_{m-1}(z)\}_{m \geqq 1}$. Thus, $\kappa(\Omega)$ can be introduced either by (5.4), or by

$$\kappa(\Omega) = \inf_{\substack{Q = PS \\ P \in \mathbb{P}}} [\overline{\lim_{m \to \infty}} \{\max_{z \in \Omega} |1/(1-z) - q_{m-1}(z)|^{1/(m-1)}\}]. \tag{5.9}$$

Now, we can apply the following result of Walsh.

Theorem 11 ([W, Chap. 4, Theorem 7]). *For $\Omega \in \mathbb{M}$ and for $\hat{\eta}$ defined by* (5.8), *there holds*

$$\kappa(\Omega) = 1/\hat{\eta}. \tag{5.10}$$

Corollary 12 ([EN, Theorem 4]). *Given $\Omega \in \mathbb{M}$, then any* SIM *generated by P (or Q or V) is an* AOSIM *with respect to Ω iff $\kappa(\Omega, P) = 1/\hat{\eta}$.*

Any sequence of polynomials $\{q_{m-1}(z)\}_{m \geqq 1}$ which yields the infimum in (5.9) is said to converge *maximally* to $g(z) := 1/(1-z)$ in Ω (cf. [W, p. 80]), i.e., any sequence $\{q_{m-1}(z)\}_{m \geqq 1}$ which converges maximally to $(1-z)^{-1}$ on Ω, induces an AOSIM with respect to Ω.

Since maximal convergence persists after differentiation (cf. [W, Chap. 4, Theorem 9]), we obtain from Lemma 9 the

Corollary 13 ([EN, Corollary 5]). *For $\Omega \in \mathbb{M}$, let P generate an* AOSIM *with respect to Ω. Then, there holds*

$$\frac{1}{\hat{\eta}} = \kappa(\Omega, P) = \sup_{T: \sigma(T) \subseteq \Omega} \kappa(T, P) = \sup_{T: \sigma(T) \subseteq \Omega} \left\{ \overline{\lim_{m \to \infty}} \sup_{\mathbf{e}_0 \neq \mathbf{0}} \left(\frac{\|\tilde{\mathbf{e}}_m\|}{\|\mathbf{e}_0\|} \right)^{1/m} \right\}.$$

In particular, for any operator T whose spectrum is contained in Ω, even if T is not diagonalizable, there holds

$$\overline{\lim_{m \to \infty}} \left(\frac{\|\tilde{\mathbf{e}}_m\|}{\|\mathbf{e}_0\|} \right)^{1/m} \leqq \frac{1}{\hat{\eta}}.$$

But maximal convergence can also be described by the behavior of the nodes of the nodal matrix K_P introduced in Sect. 2. The nodes $\xi_i^{(m)}$ $(m \geqq 1; 1 \leqq i \leqq m)$ of K_P are called *uniformly distributed* on $\Omega \in \mathbb{M}$ if

(i) no accumulation point of $\{\xi_i^{(m)}\}$ lies in $\bar{\mathbb{C}} \setminus \Omega$, and if

(ii) $\overline{\lim_{m \to \infty}} \left\{ \max_{z \in \Omega} \prod_{i=1}^{m} |z - \xi_i^{(m)}|^{1/m} \right\} = \gamma(\Omega)$,

where $\gamma(\Omega)$ is the capacity of Ω (cf. (5.7)). Then, there holds

Corollary 14 ([EN, Theorem 6]). *Let* $K_P=(\xi_i^{(m)})_{m\geqq 1, 1\leqq i\leqq m}$ *be a given nodal matrix such that the nodes* $\xi_i^{(m)}$ *are uniformly distributed on* $\Omega\in\mathbb{M}$. *Then, the corresponding* P *induces an* AOSIM *with respect to* Ω.

There are many different node sets which are known to be uniformly distributed on Ω, e.g., the Fejér nodes, the Fekete nodes and the Leja nodes (see Gaier [Ga] or [W]). Usually, the corresponding nodal matrix is not column constant. Thus, there is no simple method for computing the iterates as indicated in Theorem 6. But, it is possible to choose a special subsequence of these nodes which is also uniformly distributed, but which yields a column constant nodal matrix. We will show this with the Fejér nodes, but this principle works with any set of uniformly distributed nodes.

Let $\Omega\in\mathbb{M}$, and let us assume that the conformal mapping ψ defined in (5.6) has a continuous extension to the boundary of the unit disk. This holds, for example, if the boundary of Ω is a Jordan curve. Then, the Fejér nodes are defined by

$$\xi_j^{(m)}:=\psi\left(\exp\left[2\pi i\left(\frac{j-1}{m}\right)\right]\right) \quad (m\geqq 1, 1\leqq j\leqq m) \tag{5.11}$$

Now, if $j>1$, there are uniquely determined integers k and l with $2^k<j\leqq 2^{k+1}$ and $j=2^k+l$ $(1\leqq l\leqq 2^k)$. We now define $\zeta_1=1$, and $\zeta_j:=\exp(2\pi i(2l-1)/2^{k+1})$ $(j>1)$. This means that ζ_j is a 2^{k+1}-th root of unity, and we define

$$\xi_j:=\psi(\zeta_j) \quad (j\geqq 1). \tag{5.12}$$

Then, $\{\xi_j\}_{j\geqq 1}$ is a set of uniformly distributed nodes which yields an associated column constant nodal matrix.

What can be said about the subclass

$$\mathbb{P}_E:=\{P_E:=SV_ES^{-1}:\ V_E \text{ is generated by an Euler function}\} \tag{5.13}$$

of $\mathbb{P}$, where each element of $\mathbb{P}_E$ induced an Euler method? If we introduce the quantity

$$\kappa_E(\Omega):=\inf_{P_E\in\mathbb{P}_E}\kappa(\Omega, P_E) \tag{5.14}$$

for $\Omega\in\mathbb{M}$, then an Euler method induced by P_E is called *optimal with respect to* Ω if $\kappa(\Omega, P_E)=\kappa_E(\Omega)$ (cf. [NV]). Since $\mathbb{P}_E\subset\mathbb{P}$, it is clear from (5.4) and (5.14) that $\kappa_E(\Omega)\geqq\kappa(\Omega)$. But, we have the following new result.

Theorem 15. *For each* $\Omega\in\mathbb{M}$, *there holds* $\kappa_E(\Omega)=\kappa(\Omega)$, i.e., *an Euler method optimal with respect to* Ω *is an* AOSIM *with respect to* Ω.

Proof. From Theorem 8 in [NV], we know that there exists an optimal Euler method with respect to Ω. Let this method be generated by the Euler function $h(z)$. Then, from the proof of Theorem 8 in [NV], it follows that there exists a $\tilde\eta>1$ such that $1/h(z)$ conformally maps the disk $D_{\tilde\eta}$ with radius $\tilde\eta$ onto $\bar{\mathbb{C}}\setminus\Omega$ and

$$\kappa_E(\Omega)=1/\tilde\eta. \tag{5.15}$$

Then,

$$\tilde\psi(\omega):=1/h(\tilde\eta/w) \tag{5.16}$$

maps $\mathbb{C}\backslash\{w: |w|\leqq 1\}$ onto $\mathbb{C}\backslash\Omega$ with $\tilde{\psi}(\tilde{\eta})=1$. But, $\tilde{\psi}$ coincides with the mapping ψ claimed in (5.6) and (5.7), up to a rotation $w\to e^{i\theta}w$, i.e., $\psi(e^{-i\theta}\tilde{\eta})=1$. From (5.8), Theorem 11 and (5.15), the assertion follows. □

Remark. It should be noted that the class $\mathbb{P}_E$ is much smaller than the class $\mathbb{P}$. This is clear from the fact that each element P_E from $\mathbb{P}_E$ is determined by the single column $[\rho_{1,1}, \rho_{2,1}, \rho_{3,1}, \ldots]^T$ (see (2.17)), i.e., there is only one parameter free in each row of P_E, whereas for a $P\in\mathbb{P}$, besides the condition (1.5), all elements of the m-th row can be arbitrarily chosen.

An important consequence of Theorem 15 is that all results proven in [NV] for $\kappa_E(\Omega)$ hold for $\kappa(\Omega)$ as well. One example is the following "Comparison Theorem".

Theorem 16 ([NV]). *Let the sets Ω_1 and Ω_2 in $\mathbb{M}$ satisfy $\Omega_1\subsetneqq\Omega_2$. Then,*

$$\kappa(\Omega_1)<\kappa(\Omega_2). \tag{5.17}$$

Further, for each $\Omega\in\mathbb{M}$, we have introduced two characteristic numbers, namely the asymptotic convergence factor $\kappa(\Omega)$ and the capacity $\gamma(\Omega)$. The next lemma gives a relation between these two numbers.

Lemma 17. *For each $\Omega\in\mathbb{M}$, let the optimal Euler method, optimal with respect to Ω, be generated by the Euler function*

$$h(\phi)=\rho_{1,1}\phi+\rho_{2,1}\phi^2+\ldots. \tag{5.18}$$

Then, there holds

$$|\rho_{1,1}|\cdot\gamma(\Omega)=\kappa(\Omega). \tag{5.19}$$

Proof. The mapping ψ in (5.6) has the form (cf. [Ga, p. 64])

$$\psi(w)=\gamma(\Omega)w+c_0+c_1/w+\ldots. \tag{5.20}$$

From (5.16) and (5.18), we see that

$$\begin{aligned}\tilde{\psi}(w)&=1/h(\tilde{\eta}/w)=1/(\rho_{1,1}(\tilde{\eta}/w)+\rho_{2,1}(\tilde{\eta}/w)^2+\ldots)\\ &=\frac{w}{\rho_{1,1}\cdot\tilde{\eta}}\left[\frac{1}{1+\rho'_{2,1}(\tilde{\eta}/w)+\ldots}\right],\end{aligned} \tag{5.21}$$

where $1/\tilde{\eta}=\kappa(\Omega)$ from Theorem 11. Since $\psi(w)=\tilde{\psi}(e^{i\theta}w)$ for some θ with $0\leqq\theta<2\pi$, as we have seen in the proof of Theorem 15, we see, by comparing (5.20) and (5.21), that $0<\gamma(\Omega)=\kappa(\Omega)\cdot 1/|\rho_{1,1}|$. □

Given $\Omega\in\mathbb{M}$, we have two possibilities for constructing an AOSIM with respect to Ω. The first uses a set of uniformly distributed nodes. A special subsequence of these nodes, as shown for the Fejér nodes in (5.11) and (5.12), yields a column constant nodal matrix. By Theorem 6, the corresponding iterative procedure reduces to a first-order Richardson method. The second possibility is to construct an Euler method optimal with respect to Ω. The corresponding iterates $\{\mathbf{y}_m\}_{m\geqq 0}$ of (2.21) can then be calculated either by Lemma 3 or by Corollary 5.

One main difference between SIM's induced by $P \in \mathbb{P}$, and the subclass of Euler methods induced by $P_E \in \mathbb{P}_E \subset \mathbb{P}$, appears when compact sets Ω are considered such that $\bar{\mathbb{C}} \setminus \Omega$ is connected, but not simply connected. Then, $\Omega \notin \mathbb{M}$ and an Euler method optimal with respect to Ω does not exist. But, the concept of uniformly distributed modes holds even in this case. For example, it can be shown that if $\bar{\mathbb{C}} \setminus \Omega$ is connected and possesses a Green's function $\tilde{G}(\phi)$ with a pole at infinity, then the convergence factor $\kappa(\Omega)$ is equal to $\exp(-\tilde{G}(1))$ (cf. [W, Chap. 4]).

6. Regions of Convergence

In certain cases, it may be difficult to find an AOSIM with respect to some given set Ω in $\mathbb{C}$. In such cases, it is useful to be able to find a SIM, induced by an infinite triangular matrix P, which leads to a relatively simple iteration and for which $K(\Omega, P)$ is only slightly larger than $K(\Omega)$. This requires knowing examples of SIM's, induced by P, and particular regions Ω' for which $\kappa(\Omega') = \kappa(\Omega', P)$. From this point of view, Euler methods are especially well-suited, as we shall see.

Given a SIM, induced by an infinite triangular matrix P, it is convenient to set, in analogy with (5.2),

$$\kappa(z, P) := \overline{\lim_{m \to \infty}} |p_m(z)|^{1/m}. \tag{6.1}$$

Then, we introduce the *region of convergence* $S(P)$ of the SIM induced by P, defined by

$$S(P) := \{z \in \mathbb{C} \colon \kappa(z, P) < 1\}, \tag{6.2}$$

and a subset $S_\eta(P)$ of $S(P)$, defined by

$$S_\eta(P) := \{z \in \mathbb{C} \colon \kappa(z, P) \leqq 1/\eta\} \qquad (\eta > 1). \tag{6.3}$$

Without further assumptions on P, the region $S_\eta(P)$ may be empty for, say $\eta > \hat{\eta}$. Further, it may be difficult to determine these regions. But from the corresponding definitions, it follows that $\kappa(\Omega) \leqq 1/\eta$ if $\Omega \subset S_\eta(P)$. Thus, let us consider the following examples.

1. *Cyclic first-order Richardson method.* Here, the parameters α_j of (3.7) are used in a cyclic manner:

$$\alpha_{kl+j} := \alpha_j \qquad (j = 1, 2, \ldots, k; l = 1, 2, \ldots). \tag{6.4}$$

Thus, for the corresponding polynomials $p_m(z)$ of (3.8), there holds

$$p_{kl}(T) = (p_k(T))^l, \tag{6.5}$$

If P induces the corresponding SIM, we obtain from (6.1) that $\kappa(z, P) = |p_k(z)|^{1/k}$, and that

$$S_\eta(P) := \{z \in \mathbb{C} \colon |p_k(z)| \leqq 1/\eta^k\}, \tag{6.6}$$

i.e., $S_\eta(P)$ is the closed interior of a lemniscate of the polynomial $p_k(z)$. For $k = 1$, we get disks, which are considered by Opfer and Schober [OS], and which appear again in connection with Euler methods. For $k>1$, there may be difficulties which result from the fact that determining a lemniscate means determining the *pre-image* of a disk with radius $1/\eta^k$ for a *nonlinear* mapping $p_k(z)$ $(k>1)$. Examples for $k=2$ are treated in [E] and [EN].

2. *Euler methods.* With D_η denoting an open disk with center zero and radius η, let us introduce for an Euler function $h(z)$ (defined in Sect. 2) its *maximal extension*

$$\hat{\eta}(h) := \sup\{\eta\colon h(z) \text{ is meromorphic and univalent in } D_\eta\}. \tag{6.7}$$

Theorem 18 ([NV, Theorem 1 and Corollary 2]). *Let $h(z)$ be an Euler function, and let V_E and $P_E := SV_ES^{-1}$ induce the corresponding Euler method according to Theorem 2. Then, with $\tilde{h}(z) := 1/h(z)$,*

$$S(P_E) = \bar{\mathbb{C}} \setminus \tilde{h}(\bar{D}_1), \tag{6.8}$$

and

$$S_\eta(P_E) = \bar{\mathbb{C}} \setminus \tilde{h}(D_\eta), \quad \text{for } 1<\eta \leqq \hat{\eta}(h). \tag{6.9}$$

A direct consequence of Theorem 18 is that the boundaries of $S(P_E)$ and $S_\eta(P_E)$ for $1<\eta\leqq\hat{\eta}(h)$ are respectively the *images* of the unit circle, and the circle with radius η under the mapping $\tilde{h}(z)=1/h(z)$. From Corollary 10 in [NV], we have

Theorem 19. *Each Euler method generated by the Euler function $h(z)$ and induced by P_E, is an* AOSIM *with respect to $S_\eta(P_E)$ for $1<\eta\leqq\hat{\eta}(h)$. Furthermore,*

$$\kappa(S_\eta(P_E)) = 1/\eta. \tag{6.10}$$

Thus, if $\Omega \in \mathbb{M}$ with $S_{\eta_1}(P_E) \subsetneqq \Omega \subsetneqq S_{\eta_2}(P_E)$, we obtain from Theorem 16 that

$$1/\eta_1 < \kappa(\Omega) < 1/\eta_2.$$

We now give two simple, but important, examples which result from Euler functions of the form (3.4).

For $k=1$, we have $\tilde{h}(\phi) = (1-(1-\alpha)/\phi)/(\alpha\phi)$ where $\alpha := \mu_0$. From (3.4), we see that the Euler method generated by this Euler function $h(\phi)$ corresponds to the stationary first-order Richardson method. By Theorem 18, we see that $S(P_E)$ is the open disk with center $m := 1-1/\alpha$ and radius $1/|\alpha|$. Similarly, $S_\eta(P_E)$ is the closed concentric disk with radius $1/(\eta|\alpha|)$, $1<\eta<\infty$. Conversely, if $\Omega\in\mathbb{M}$ is a disk with center m and radius ρ, then by Theorem 19, the stationary first-order Richardson method with $\alpha := 1/(1-m)$ is an AOSIM with respect to Ω. It can be seen that $\kappa(\Omega)$ is then given as the quotient of the radius ρ and the distance of m from 1, i.e.,

$$\kappa(\Omega) = \rho/|1-m| = \rho|\alpha|. \tag{6.11}$$

For $k=2$, we use that

$$\tilde{h}(\phi) = \frac{1}{\mu_0}\left(\frac{1}{\phi} - \mu_1 - \mu_2\phi\right) \tag{6.12}$$

is a mapping of Joukowski type. One finds (cf. [NV, (7.3) and (7.7)]) that $h(\phi)$ is univalent in a neighborhood of the unit disk iff $|\mu_2|<1$. Further, we have

$\hat{\eta}(h)=1/|\mu_2|$. From Sect. 7 of [NV], we further have that $S(h)$ is the interior of an ellipse E with $1\in E$, and with foci α, β given by

$$\alpha, \beta=(-\mu_1 \pm 2\sqrt{-\mu_2})/\mu_0. \tag{6.13}$$

For $1<\eta<\hat{\eta}$, the region $S_\eta(h)$ is the closed interior of a confocal ellipse E_η within E and $S_\eta(h)$ is the interval between the foci α and β. If, for a given E_η, the value of η has to be determined, then if z is an arbitrary point on E_η, one then solves the quadratic equation $z=\tilde{h}(\phi)$ for ϕ. There is then a solution with $1<|\phi|\leqq\hat{\eta}$, and this yields $\eta=|\phi|$.

Conversely, let us assume that Ω is the closed interior of an ellipse. Suppose we wish to find parameters μ_0, μ_1, μ_2 such that for the corresponding $h(z)$, there holds $\Omega=S_\eta(h)$ for some η with $1<\eta<\hat{\eta}(h)$. Let α and β be the foci of Ω, and let

$$\theta^{\pm}:=\frac{(\sqrt{1-\alpha}\pm\sqrt{1-\beta})^2}{\beta-\alpha}. \tag{6.14}$$

It follows from (6.14) that $\theta^{+}=1/\theta^{-}$. Let θ be that value with $|\theta|>1$. Then, with $\gamma:=(\alpha-\beta)/2$, $\delta:=(\alpha+\beta)/2$, we obtain (cf. [NV, formula (7.3)])

$$\mu_0=\frac{2}{\gamma\theta}, \quad \mu_1=\frac{-2\delta}{\gamma\theta}, \quad \mu_2=\frac{-1}{\theta^2}. \tag{6.15}$$

The value of η can be determined as mentioned above. It should be noted that, for all confocal ellipses E' such that $1\notin\bar{E}'$, we obtain from (6.15) the *same* iteration parameters μ_0, μ_1, μ_2. Only the value of η will be different in this case.

7. Faber Polynomials and Asymptotically Optimal SIM's

As we have already mentioned in Sect. 5, each polynomial sequence $\{q_{m-1}(z)\}_{m\geqq 1}$, converging maximally to $g(z):=1/(1-z)$ on a compact set $\Omega\in\mathbb{M}$ (cf. (5.5)), gives raise to an AOSIM with respect to Ω. In this section, we shall consider another class of polynomials, the so-called *Faber polynomials* which play an important role in approximation theory (cf. Faber [F1, F2]). We shall investigate how these polynomials can be used to solve linear systems iteratively.

First, we need additional terminology. For $\Omega\in\mathbb{M}$, let $\gamma(\Omega)$ denote the capacity of Ω (cf. (5.7)), and let $\hat{\psi}(w)$ denote the conformal mapping of $\bar{\mathbb{C}}\setminus\{w: |w|\leqq\gamma(\Omega)\}$ onto $\bar{\mathbb{C}}\setminus\Omega$, with $\hat{\psi}(\infty)=\infty$ and $\hat{\psi}'(\infty)=1$. (We remark that $\hat{\psi}$ is connected with the mapping ψ, introduced in (5.6), through

$$\hat{\psi}(w)=\psi(w/\gamma(\Omega)), \text{ for all } |w|>\gamma(\Omega)).$$

Now, $\hat{\psi}(w)$ has an expansion of the form

$$\hat{\psi}(w)=w+\sum_{k=0}^{\infty}\alpha_k w^{-k} \quad (|w|>\gamma(\Omega)). \tag{7.1}$$

Let $\hat{\phi}$ be the inverse mapping of $\hat{\psi}$. Further, for $\tau > \gamma(\Omega)$, let the image of the circle $\{w\colon |w| = \tau\}$ under $\hat{\psi}$ be denoted by $\Gamma_\tau := \{z\colon |\hat{\psi}(z)| = \tau\}$, and let the compact set with boundary Γ_τ by denoted by Ω_τ. Moreover, let

$$\chi(w) = \sum_{k=0}^{\infty} \gamma_k w^{-k} \tag{7.2}$$

be any function regular in $|w| > \gamma(\Omega)$ with $\chi(\infty) = \gamma_0 \neq 0$. We assume that χ has no zeros in the domain $\{w\colon |w| > \gamma(\Omega)\}$.

Now, for any nonnegative integer n, we can expand $\chi(\hat{\phi}(z)) \cdot [\hat{\phi}(z)]^n$ into a Laurent series of the form

$$\chi(\hat{\phi}(z)) \cdot [\hat{\phi}(z)]^n = \gamma_0 z^n + \sum_{k=-\infty}^{n-1} \beta_{n,k} z^k. \tag{7.3}$$

Its principal (i.e., polynomial) part is defined as

$$F_n(z;\chi) := \gamma_0 z^n + \sum_{k=0}^{n-1} \beta_{n,k} z^k \qquad (n \geqq 0). \tag{7.4}$$

Clearly, $F_n(z;\chi)$ is a polynomial of exact degree n, and it is called the n-th *generalized Faber polynomial for the set Ω with respect to the weight function χ.* (If $\chi \equiv 1$, then F_n is known as the n-th "ordinary" Faber polynomial for Ω.)

One of the reasons for the importance of these polynomials in approximation theory is the following. For $\tau > \gamma(\Omega)$, any function which is regular in the interior domain of Γ_τ can be expanded into a series of generalized Faber polynomials (cf. Smirnov and Lebedev [SL, §2.2.4]). For the special case $g(z) = 1/(1-z)$, we have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \sigma_n(\chi) F_n(z;\chi), \tag{7.5}$$

for all z from the interior domain of $\Gamma_{\gamma(\Omega)\hat{\eta}}$ (where $\hat{\eta} := |\hat{\phi}(1)|/\gamma(\Omega)$ (cf. (5.8)), where the coefficients are explicitly given by

$$\sigma_n(\chi) = \frac{\hat{\phi}'(1)}{\chi(\hat{\phi}(1))\,\hat{\phi}(1)} \left[\frac{1}{\hat{\phi}(1)}\right]^n \qquad (n \geqq 0). \tag{7.6}$$

The convergence in (7.5) is uniform on every compact subset of the interior domain of $\Gamma_{\gamma(\Omega)\hat{\eta}}$ (cf. [SL, §2.2.3]).

Now, we are in a position to introduce SIM's that are generated by generalized Faber polynomials. We approximate $g(z) = 1/(1-z)$ by its truncated Faber series

$$q_{m-1}(z;\chi) := \sum_{n=0}^{m-1} \sigma_n(\chi) F_n(z;\chi) \qquad (m \geqq 1), \tag{7.7}$$

and, in view of (2.8), we consider the vector sequence

$$\mathbf{y}_0 = \mathbf{c}; \qquad \mathbf{y}_m = \mathbf{c} + q_{m-1}(T;\chi)\mathbf{r}_0 = \mathbf{c} + \left[\sum_{n=0}^{m-1} \sigma_n(\chi) F_n(T;\chi)\right] \mathbf{r}_0, \tag{7.8}$$

which, of course, represents a SIM with respect to the basic iteration

$$\mathbf{x}_0=\mathbf{c}, \qquad \mathbf{x}_m=T\mathbf{x}_{m-1}+\mathbf{c} \qquad (m\geqq 1).$$

In our first new theorem of this section, we shall show that the iterates $\mathbf{y}_m$ of (7.8) can be computed recursively. To avoid unnecessary complications, we confine ourselves to the special weight function $\chi(w)=1/\hat{\psi}'(\omega)$, which obviously satisfies the conditions following (7.2).

Theorem 20. *For the iterates*

$$\mathbf{y}_m=\mathbf{c}+q_{m-1}\left(T;\frac{1}{\hat{\psi}'}\right)\mathbf{r}_0=\mathbf{c}+\left[\sum_{n=0}^{m-1}\sigma_n\left(\frac{1}{\hat{\psi}'}\right)F_n\left(T;\frac{1}{\hat{\psi}'}\right)\right]\mathbf{r}_0 \tag{7.9}$$

(cf. (2.8), (7.8)), *there holds*

$$\mathbf{y}_0=\mathbf{c}; \qquad \mathbf{y}_1=\mathbf{c}+\mu_0 T\mathbf{y}_0,$$

and (7.10)

$$\mathbf{y}_m=\left(1-\sum_{j=1}^{m-1}\mu_j\right)+\mu_0 T\mathbf{y}_{m-1}+\sum_{j=1}^{m-1}\mu_j\mathbf{y}_{m-j} \qquad (m\geqq 2),$$

where the coefficients μ_k are given by

$$\mu_0=1/\hat{\phi}(1),$$

and (7.11)

$$\mu_k=-\alpha_{k-1}/[\hat{\phi}(1)]^k \qquad (k\geqq 1).$$

(*The α_k's are the Laurent coefficients of $\hat{\psi}$* (cf. (7.1)).

Proof. It is well-known (cf. [SL, §2.2.1(1)]) that a generating function of the generalized Faber polynomials is given by

$$\chi(w)\frac{\hat{\psi}'(w)}{\hat{\psi}(w)-z}=\sum_{n=0}^{\infty}F_n(z;\chi)\frac{1}{w^{n+1}}.$$

From this, one easily derives (cf. Suetin [S2, §1]) the recurrence relations

$$F_0(z;\chi)=\gamma_0^*; \qquad F_1(z;\chi)=(z-\alpha_0)F_0(z;\chi)+\gamma_1^*;$$

$$F_{m+1}(z;\chi)=(z-\alpha_0)F_m(z;\chi)-\sum_{n=1}^{m}\alpha_n F_{m-n}(z;\chi)+\gamma_{m+1}^* \qquad (m\geqq 1),$$

where the α_n's have the same meaning as above, and where the γ_n^*'s are the Laurent coefficients of $\chi(w)\cdot\hat{\psi}'(w)$, i.e.,

$$\chi(w)\hat{\psi}'(w)=\sum_{n=0}^{\infty}\gamma_n^* w^{-n} \qquad (|w|>\gamma(\Omega)).$$

For the special choice $\chi(w)=1/\hat{\psi}'(w)$, we obtain $\gamma_0^*=1$ and $\gamma_n^*=0$ $(n\geqq 1)$. On the other hand, we conclude from (7.6) that

$$\sigma_n\left(\frac{1}{\hat{\psi}'}\right)=\frac{\hat{\psi}'(\hat{\phi}(1))\hat{\phi}'(1)}{\hat{\phi}(1)}\left[\frac{1}{\hat{\phi}(1)}\right]^n=(\hat{\psi}\circ\hat{\phi})'(1)\left[\frac{1}{\hat{\phi}(1)}\right]^{n+1}=\left[\frac{1}{\hat{\phi}(1)}\right]^{n+1}.$$

Now, for the polynomials $q_m\left(z;\frac{1}{\hat{\psi}'}\right)$ of (7.7), there holds

$$q_0\left(z;\frac{1}{\hat{\psi}'}\right)=\mu_0, \quad \text{and,} \quad \text{for } m\geqq 1,$$

$$q_m\left(z;\frac{1}{\hat{\psi}'}\right)=\mu_0+\mu_0 z q_{m-1}\left(z;\frac{1}{\hat{\psi}'}\right)+\sum_{j=1}^{m}\mu_j q_{m-j}\left(z;\frac{1}{\hat{\psi}'}\right).$$

(The coefficients μ_k are defined by (7.11)). With the above relations, (7.10) follows by direct computation. □

Comparing (3.3) and (7.10), we see that both formulas are identical. This observation leads to our second (new) result in this section.

Theorem 21. *The iterates* $\mathbf{y}_m$ *of* (7.9)–(7.10) *are just the vectors* $\mathbf{y}_m$ *of* (2.21) *resulting from an Euler method with the Euler function*

$$h(w):=1/\hat{\psi}\left(\frac{\hat{\phi}(1)}{w}\right). \tag{7.12}$$

Conversely, if an Euler-function $h(z)$ *is given, then*

$$\hat{\psi}(w):=\frac{1}{h}\left(\frac{1}{h'(0)w}\right) \tag{7.13}$$

is a conformal mapping of the type (7.1) *which maps the exterior of a disk onto the exterior of a compact set* $\Omega\in\mathbb{M}$. *Constructing the generalized Faber polynomials* $F_m\left(z;\frac{1}{\hat{\psi}'}\right)$ *for* Ω *and generating the vector sequence* $\{\mathbf{y}_m\}_{m\geqq 0}$ *according to* (7.9) *and* (7.10), *we obtain the iterates of the Euler method which is defined by the given function h.*

Proof. The mapping $h(z)$ of (7.12) is meromorphic and univalent for $|w|<|\hat{\phi}(1)|/\gamma(\Omega)(>1)$, and satisfies the conditions $h(0)=0$, $h(1)=1$. Thus, $h(z)$ is an Euler-function (cf. Sect. 2). The function $\tilde{h}(w):=1/h(w)=\hat{\psi}\left(\frac{\hat{\phi}(1)}{w}\right)$ has the expansion

$$\tilde{h}(w)=\frac{\hat{\phi}(1)}{w}+\alpha_0+\alpha_1\frac{w}{\hat{\phi}(1)}+\alpha_2\left[\frac{w}{\hat{\phi}(1)}\right]^2+\dots$$

(cf. (7.1)), or, with the definitions of (7.11),

$$\tilde{h}(w)=\frac{1}{\mu_0}\left(\frac{1}{w}-\mu_1-\mu_2 w-\mu_3 w^2-\dots\right).$$

Our first assertion now follows from Theorems 4 and 20, whereas the second assertion can be derived in an analogous way. □

This one-to-one correspondence between Euler methods and SIM's generated by the Faber polynomials $F_m\left(z;\frac{1}{\hat{\psi}'}\right)$ allows us to obtain results on Euler methods by using well-known properties of Faber polynomials. For instance, it is known that Faber expansions converge maximally (see Sect. 5). A consequence of this fact is that the corresponding SIM is asymptotically optimal with respect to the set Ω.

But, new results for Euler methods can also be derived by this correspondence. We demonstrate this for the special case $\Omega=[\alpha,\beta]\in\mathbb{M}$ (i.e., $\alpha,\beta\in\mathbb{C}$, $1\notin[\alpha,\beta]$) which has practical importance. With $\delta:=\frac{1}{2}(\alpha+\beta)$ and $\gamma:=\frac{1}{2}(\beta-\alpha)$, the conformal mapping $\hat{\psi}$ (cf. (7.1)) of $\{w: |w|>\frac{1}{2}|\gamma|\}$ onto $\bar{\mathbb{C}}\setminus[\alpha,\beta]$ is given by

$$\hat{\psi}(w)=w+\delta+\frac{\gamma^2}{4}\frac{1}{w},$$

and the associated Faber polynomials $F_m\left(z;\frac{1}{\hat{\psi}'}\right)$ can be computed recursively (cf. the proof of Theorem 20) from

$$F_0\left(z;\frac{1}{\hat{\psi}'}\right)=1;\qquad F_1\left(z;\frac{1}{\hat{\psi}'}\right)=(z-\delta),$$

$$F_{m+1}\left(z;\frac{1}{\hat{\psi}'}\right)=(z-\delta)F_m\left(z;\frac{1}{\hat{\psi}'}\right)-\frac{\gamma^2}{4}F_{m-1}\left(z;\frac{1}{\hat{\psi}'}\right)\qquad(m\geqq 1).$$

First, we show that the above Faber polynomial $F_m\left(z;\frac{1}{\hat{\psi}'}\right)$ can be expressed in terms of the Chebyshev polynomial $U_m(z)$, of the *second* kind. As is well-known (cf. Todd [T, Chap. 2]),

$$U_m(z):=\frac{\sin[(m+1)\arccos z]}{\sin[\arccos z]}\qquad(m\geqq 0),$$

and these polynomials satisfy

$$U_0(z)=1,\qquad U_1(z)=2z,$$

$$U_{m+1}(z)=2z\,U_m(z)-U_{m-1}(z)\quad(m\geqq 1).$$

The translated polynomials $U_m\left(\frac{1}{\gamma}z-\frac{\delta}{\gamma}\right)$ obviously have the leading coefficient $\left(\frac{2}{\gamma}\right)^m$ $(m\geqq 0)$, and, for the scaled polynomials $\hat{U}_m(z):=\left(\frac{\gamma}{2}\right)^m U_m\left(\frac{1}{\gamma}z-\frac{\delta}{\gamma}\right)$, we obtain

$$\hat{U}_0(z)=1;\qquad \hat{U}_1(z)=(z-\delta);$$

$$\hat{U}_{m+1}(z)=(z-\delta)\,\hat{U}_m(z)-\frac{\gamma^2}{4}\,\hat{U}_{m-1}(z)\quad(m\geqq 1).$$

It is evident that

$$F_m\left(z;\frac{1}{\hat{\psi}'}\right)=\left(\frac{\gamma}{2}\right)^m\hat{U}_m\left(\frac{1}{\gamma}z-\frac{\delta}{\gamma}\right).$$

The Faber coefficients $\sigma_m\left(\frac{1}{\hat{\psi}}\right)$ (cf. (7.6)) are then given by

$$\sigma_m\left(\frac{1}{\hat{\psi}'}\right)=\kappa^{-(m+1)},$$

where κ is the zero of $x^2+(\delta-1)x+\frac{\gamma^2}{4}=0$ whose absolute value is greater than $\frac{1}{2}|\gamma|$.

Setting $\theta:=\kappa^{-1}\frac{\gamma}{2}$ (so that $|\theta|<1$), we see that the polynomials $q_{m-1}(z;1/\hat{\psi}')$ of (7.7) satisfy

$$q_{m-1}\left(z;\frac{1}{\hat{\psi}'}\right)=\sum_{n=0}^{m-1}\sigma_n\left(\frac{1}{\hat{\psi}'}\right)F_n\left(z;\frac{1}{\hat{\psi}'}\right)=\frac{2\theta}{\gamma}\sum_{n=0}^{m-1}\theta^n U_n\left(\frac{1}{\gamma}z-\frac{\delta}{\gamma}\right),$$

and thus

$$\max_{z\in[\alpha,\beta]}\left|\frac{1}{1-z}-q_{m-1}\left(z;\frac{1}{\hat{\psi}'}\right)\right|=\max_{z\in[\alpha,\beta]}\left|\sum_{n=m}^{\infty}\sigma_n\left(\frac{1}{\hat{\psi}'}\right)F_n\left(z;\frac{1}{\hat{\psi}}\right)\right|$$
$$\leqq 2\frac{|\theta|}{|\gamma|}\cdot\sum_{n=m}^{\infty}|\theta|^n\max_{z\in[\alpha,\beta]}\left|U_n\left(\frac{1}{\gamma}z-\frac{\delta}{\gamma}\right)\right|.$$

Since $\max_{z\in[\alpha,\beta]}\left|U_n\left(\frac{1}{\gamma}z-\frac{\delta}{\gamma}\right)\right|=U_n\left(\frac{1}{\gamma}\beta-\frac{\delta}{\gamma}\right)=U_n(1)=n+1$ (cf. [T, Chap. 2]), we have

$$\max_{z\in[\alpha,\beta]}\left|\frac{1}{1-z}-q_{m-1}\left(z;\frac{1}{\hat{\psi}'}\right)\right|\leqq 2\frac{|\theta|}{|\gamma|}\sum_{n=m}^{\infty}|\theta|^n\cdot(n+1)=\frac{2}{|\gamma|}\cdot\frac{1+m(1-|\theta|)}{(1-|\theta|)^2}|\theta|^{m+1}.$$

Now, let T be any matrix whose eigenvalues are contained in Ω, and assume that the vectors $\mathbf{y}_m$ are generated by the Euler method which is optimal for Ω (cf. § 5). If the spectral norm of T is just its spectral radius (as is the case when T is a normal matrix), then there holds (cf. (1.8) and (2.10))

$$\|\tilde{\mathbf{e}}_m\|_2=\|\mathbf{x}-\mathbf{y}_m\|_2\leqq\max_{z\in[\alpha,\beta]}\left|1-(1-z)q_{m-1}\left(z;\frac{1}{\hat{\psi}'}\right)\right|\|\mathbf{e}_0\|_2$$
$$\leqq\max\{|1-\delta\pm\gamma|\}\max_{z\in[\alpha,\beta]}\left|\frac{1}{1-z}-q_{m-1}\left(z;\frac{1}{\hat{\psi}'}\right)\right|\|\mathbf{e}_0\|_2$$
$$\leqq\max\{|1-\delta\pm\gamma|\}\frac{2}{|\gamma|}\frac{1+m(1-|\theta|)}{(1-|\theta|)^2}|\theta|^{m+1}\|\mathbf{e}_0\|_2.$$

From the above inequality, we remark that it follows from (4.6) that

$$\kappa(T,P)\leqq|\theta|.$$

In other words, from the geometry of the interval $[\alpha,\beta]$, it is possible to derive upper bounds for the norms of the iteration vectors $\tilde{\mathbf{e}}_m$ for each $m\geqq 0$, which depend *only* on the constants α and β.

Sharper results can be derived if one requires the boundary $\partial\Omega$ of Ω to be smooth in the following sense. Let $\theta(s)$, $0 \leqq s \leqq 1$, be a parametric representation of $\partial\Omega$ with respect to the arc length s. We say $\partial\Omega \in C(p, \alpha)$ (cf. [S1, § 1]) if θ is p-times continuously differentiable, and $\theta^{(p)}$ satisfies the Lipschitz condition $|\theta^{(p)}(s) - \theta^{(p)}(\bar{s})| \leqq \lambda |s - \bar{s}|^{\alpha}$, $0 \leqq s, \bar{s} \leqq 1$, with some constant λ. Our final new result in this section is

Theorem 22. *Given the set $\Omega \in \mathbb{M}$, let $\psi(w)$ be the associated conformal mapping* (cf. (7.1)), *and let $\chi(w)$ be a given weight function* (cf. (7.2)). *Then, the* SIM *induced by the corresponding generalized Faber polynomials $F_n(z;\chi)$ is asymptotically optimal with respect to Ω (and also with respect to Ω_τ for any $\gamma(\Omega) < \tau < |\hat{\phi}(1)|$). In addition, assume that $\partial\Omega \in C(1, \varepsilon)$ for some $\varepsilon > 0$, and that the weight function χ satisfies a Lipschitz condition $|\chi(w) - \chi(\bar{w})| \leqq \lambda |w - \bar{w}|^{\alpha}$, $w, \bar{w} \in \bar{\mathbb{C}} \setminus \Omega$, with $\alpha > \frac{1}{2}$. Suppose that the matrix T is diagonalizable and has all its eigenvalues in Ω. Then, the following estimation of the error $\tilde{\mathbf{e}}_m$ of the m-th iterate is valid:*

$$\|\tilde{\mathbf{e}}_m\| \leqq L \cdot \ln(m-1) E_{m-1}(g, \Omega) \|\mathbf{e}_0\|, \quad (m \geqq 3). \tag{7.14}$$

Here, $\|\cdot\|$ is any vector norm, L is a constant independent of m, and $E_m(g, \Omega)$ denotes the error of the best uniform approximation by polynomials of degree m to $g(z) = 1/(1-z)$ on Ω.

Proof. Let P be the transformation matrix (in sequence-to-sequence form, cf. Sect. 2) which induces our SIM. We have to show that

$$\kappa(\Omega, P) = \gamma(\Omega)/|\hat{\phi}(1)| (= 1/\hat{\eta} \text{ from (5.8)}),$$

and that

$$\kappa(\Omega_\tau, P) = \tau/|\hat{\phi}(1)|, \quad \text{for all } \gamma(\Omega) < \tau < |\hat{\phi}(1)|.$$

Since $\kappa(\Omega, P)$ can be written as

$$\kappa(\Omega, P) = \overline{\lim_{m\to\infty}} \left(\sup_{z\in\Omega} |1/(1-z) - q_{m-1}(z)|^{1/(m-1)}\right),$$

where the polynomials $q_{m-1}(z)$ are given by (7.7), our assertion follows directly from the following result on Faber expansions. It is known (cf. [SL, § 2.2.3, § 2.2.4]) that

$$\overline{\lim_{m\to\infty}} \left(\sup_{z\in\Omega} |1/(1-z) - q_{m-1}(z)|^{1/(m-1)}\right) = \gamma(\Omega)/|\hat{\phi}(1)| \text{ holds.}$$

(The analogous statement is valid for Ω_τ.)

The upper bound (7.14) is also a direct consequence of a theorem (cf. [S1, Theorem 5]) on the deviation of the Faber expansion of an analytic function from its polynomial best-approximation. □

Remark. On choosing the special weight function $\chi(w) = 1/\psi'(w)$, the upper bound (7.14) is also valid for Euler methods (cf. Theorem 21). Thus, for any $\Omega \in \mathbb{M}$ fulfilling the conditions of Theorem 21, we can find an Euler method which is not only an AOSIM with respect to Ω but also differs from the best possible SIM for Ω only by a multiplicative factor which grows only like $\ln(m)$!

References

[E] Eiermann, M.: Numerische analytische Fortsetzung durch Interpolationsverfahren. Dissertation, U. Karlsruhe 1982

[EN] Eiermann, M., Niethammer, W.: On the construction of semiiterative methods. SIAM J. Numer. Anal. **30**, 1153–1160 (1983)

[F1] Faber, G.: Über polynomische Entwicklungen. Math. Ann. **57**, 398–408 (1903)

[F2] Faber, G.: Über Tschebyscheffsche Polynome. J. Reine Angew. Math. **150**, 79–106 (1920)

[Ga] Gaier, D.: Vorlesungen über Approximation im Komplexen. Basel-Boston-Stuttgart: Birkhäuser 1980

[G] Gantmacher, F.R.: Matrizenrechnung, I. Berlin: Deutscher Verlag der Wissenschaften 1965

[H] Householder, A.S.: The Theory of Matrices in Numerical Analysis. New York-Toronto-London: Blaisdell 1964

[M] Manteuffel, T.A.: The Tschebychev iteration for nonsymmetric linear systems. Numer. Math. **28**, 307–327 (1977)

[NV] Niethammer, W., Varga, R.S.: The analysis of k-step iterative methods for linear systems from summability theory. Numer. Math. **41**, 177–206 (1983)

[OS] Opfer, G., Schober, G.: Richardson's iteration for nonsymmetric matrices. Linear Algebra Appl. **58**, 343–367 (1984)

[OR] Ortega, J.M., Reinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables. New York-London: Academic Press 1970

[SL] Smirnov, V.L., Lebedev, N.A.: Functions of a Complex Variable: Constructive Theory. Cambridge, MA: MIT Press 1968

[S1] Suetin, P.K.: Fundamental properties of Faber polynomials. Russ. Math. Surv. **19**, 121–149 (1964)

[S2] Suetin, P.K.: Series in Faber polynomials and several generalizations. J. Sov. Math. **5**, 502–551 (1976)

[T] Todd, J.: Special polynomials in numerical analysis. In: On Numerical Approximation (R.E. Langer, ed.), pp. 423–446. Madison: University of Wisconsin Press 1959

[V1] Varga, R.S.: A comparison of the successive overrelaxation method and semi-iterative methods using Chebyschev polynomials. J. Soc. Indust. Appl. Math. **5**, 39–46 (1957)

[V2] Varga, R.S.: Matrix Iterative Analysis. Englewood Cliffs, NJ: Prentice Hall 1962

[W] Walsh, J.L.: Interpolation and Approximation by Rational Functions in the Complex Domain. Providence, RI: AMS Colloquium Publications 1956

[Wi] Wimp, J.: Sequence Transformations and their Applications. New York-London: Academic Press 1981

[ZB] Zeller, K., Beekmann, W.: Theorie der Limitierungsverfahren. Berlin-Heidelberg-New York: Springer 1978

Received November 12, 1984/June 24, 1985