Numer. Math. 56, 109–121 (1989)

Numerische Mathematik

A Note on the SSOR and USSOR Iterative Methods Applied to p-Cyclic Matrices

Xiezhang Li and Richard S. Varga
Institute for Computational Mathematics, Kent State University, Kent, OH 44242, USA

Dedicated to the memory of Peter Henrici

Summary. The purpose of this note is threefold: *i)* to derive the new functional equation,

$$[\lambda-(1-\omega)(1-\hat{\omega})]^p=\lambda^k[\lambda\omega+\hat{\omega}-\omega\hat{\omega}]^{|\zeta_L|-k}[\lambda\hat{\omega}+\omega-\omega\hat{\omega}]^{|\zeta_U|-k} \cdot(\omega+\hat{\omega}-\omega\hat{\omega})^{2k}\mu^p,$$

which couples the nonzero eigenvalues of the USSOR iteration matrix $T_{\omega,\hat{\omega}}$ with the eigenvalues μ of the associated block Jacobi matrix B in the p-cyclic case, *ii)* to interpret the exponent k in this equation by means of graph theory, and *iii)* to connect the above equation with known results in the literature.

Subject Classifications: AMS(MOS) 65F10; CR: G.1.

1 Introduction

There have been a number of recent research articles, all concerned with the symmetric successive overrelaxation (SSOR) iterative method and the unsymmetric successive overrelaxation (USSOR) iterative method, applied to p-cyclic matrices. These research articles give generalizations of the following functional equation, derived by Varga et al. [4]:

$$[\lambda-(1-\omega)^2]^p=\lambda[\lambda+1-\omega]^{p-2}(2-\omega)^2\omega^p\mu^p, \tag{1.1}$$

which connects the eigenvalues λ of the associated SSOR matrix S_ω to the eigenvalues μ of a particular weakly cyclic of index p Jacobi matrix B (where $p\geqq 2$). Of course, the functional equation (1.1) strongly resembles in character the related well-known functional equations

$$(\lambda+\omega-1)^2=\lambda\omega^2\mu^2 \tag{1.2}$$

of Young [7, 8], and

$$(\lambda+\omega-1)^p=\lambda^{p-1}\omega^p\mu^p \tag{1.2'}$$

of Varga [5, 6], which similarly connect the eigenvalues λ of an associated successive overrelaxation matrix $\mathscr{L}_\omega$ to the eigenvalues μ of a consistently ordered weakly cyclic of index p Jacobi matrix B (where $p \geqq 2$).

The purpose of this note is threefold. First, we develop the following *new* functional equation (cf. also (2.1) of Theorem 1):

$$[\lambda-(1-\omega)(1-\hat{\omega})]^p=\lambda^k[\lambda\omega+\hat{\omega}-\omega\hat{\omega}]^{|\zeta_L|-k}[\lambda\hat{\omega}+\omega-\omega\hat{\omega}]^{|\zeta_U|-k} \cdot(\omega+\hat{\omega}-\omega\hat{\omega})^{2k}\mu^p, \tag{1.3}$$

which serves to generalize and unify *all* the recent research articles on the SSOR and USSOR iterative methods applied to a block p-cyclic matrix. Second, we give a *graph-theoretic interpretation* of the exponent k in the equation above. As it turns out, a similar analysis applies to a graph-theoretic interpretation for the associated known SOR case. (This is remarked in §2.) Finally, (1.3) and Theorem 1 generalize the recent result of Gong and Cai [1] on the SSOR iterative method for p-cyclic matrices, which has been published only in Chinese. Our final purpose in this note is to connect our new Theorem 1 with known results in the literature, and to bring this result of Gong and Cai [1] to a larger audience.

For the remainder of this section, we give background and notation for our problem. For the iterative solution of the matrix equation

$$A\mathbf{x}=\mathbf{k}, \tag{1.4}$$

where A is a given $n\times n$ complex matrix, assume that the matrix A can be written in block-partitioned form as

$$A=\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,p} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,p} \\ \vdots & & & \vdots \\ A_{p,1} & A_{p,2} & \cdots & A_{p,p} \end{bmatrix}, \tag{1.5}$$

where each diagonal submatrix $A_{i,i}$ is square and nonsingular ($1\leqq i\leqq p$). (We assume throughout that $p\geqq 2$.) With

$$D:=diag[A_{1,1}, A_{2,2}, \ldots, A_{p,p}],$$

the associated block-Jacobi matrix B is defined by

$$B:=I-D^{-1}A, \tag{1.6}$$

which we can write, from the partitioning in (1.5), as

$$B=[B_{i,j}]:=\begin{bmatrix} O & B_{1,2} & \cdots & B_{1,p} \\ B_{2,1} & O & \cdots & B_{2,p} \\ \vdots & & & \vdots \\ B_{p,1} & B_{p,2} & \cdots & O \end{bmatrix}. \tag{1.7}$$

As the block diagonal submatrices of B are by definition all null, we can also express B as the sum

$$B=L+U, \tag{1.8}$$

where L and U are respectively strictly lower and strictly upper triangular matrices.

From (1.8), the associated unsymmetric successive overrelaxation (USSOR) iteration matrix $T_{\omega,\hat{\omega}}$ is then defined by

$$T_{\omega,\hat{\omega}}:=(I-\hat{\omega}U)^{-1}[(1-\hat{\omega})I+\hat{\omega}L](I-\omega L)^{-1}[(1-\omega)I+\omega U], \tag{1.9}$$

where ω and $\hat{\omega}$ are relaxation parameters. The associated symmetric successive overrelaxation (SSOR) iteration matrix S_ω for (1.8) reduces to the case when $\omega=\hat{\omega}$ in (1.9), i.e.,

$$S_\omega:=T_{\omega,\omega}. \tag{1.10}$$

Our interest here is in the case where the block-Jacobi matrix B of (1.7) has the property that there is a cyclic permutation (a 1-1 onto mapping) of the integers $\{1,2,\dots,p\}$, expressed in cyclic form as $\sigma=(\sigma_1, \sigma_2,\dots,\sigma_p)$, such that

$$B_{\sigma_j,k}\equiv O \qquad \text{for all } k\neq\sigma_{j+1} \qquad (1\leqq j,k\leqq p), \tag{1.11}$$

where $\sigma_{p+1}:=\sigma_1$. It is easily seen that if the block-partitioned matrix B of (1.7) satisfies (1.11), then B is *weakly cyclic of index* p (cf. [6, p. 39]), and, conversely, if the partitioned matrix B is weakly cyclic of index p, then B satisfies (1.11) for a suitable cyclic permutation σ. Thus, we define the block-partitioned matrix B of (1.7) to be a *weakly cyclic matrix generated by the cyclic permutation* $\sigma=(\sigma_1, \sigma_2,\dots,\sigma_p)$ if (1.11) is satisfied. (We do remark that a block-partitioned matrix B, which is weakly cyclic of index p, can, for a different partitioning of B, be weakly cyclic of some index p' with $p'\neq p$.)

Assume that $B=L+U$ of (1.7) is a weakly cyclic of index p matrix generated by a cyclic permutation $\sigma=(\sigma_1,\sigma_2\dots,\sigma_p)$, so that (1.11) is valid. Then, it follows from (1.11) that B^p is a block-diagonal matrix whose σ_j-th diagonal block is given by the product

$$B_{\sigma_j,\sigma_{j+1}}\cdot B_{\sigma_{j+1},\sigma_{j+2}}\cdots B_{\sigma_{j+p-1},\sigma_j} \qquad (1\leqq j\leqq p), \tag{1.12}$$

where $\sigma_i:=\sigma_{i-p}$ if $i>p$. To avoid trivial cases, we further assume that none of the square matrices in (1.12) is a null matrix. This implies that

$$B_{\sigma_j,\sigma_{j+1}}\neq O \qquad (1\leqq j\leqq p). \tag{1.13}$$

Then, with the cyclic permutation $\sigma=(\sigma_1, \sigma_2, \dots, \sigma_p)$, we define its associated disjoint subsets ζ_L and ζ_U of $\{1, 2, \dots, p\}$ as

$$\begin{cases} \zeta_L := \{\sigma_j : \sigma_j > \sigma_{j+1}\}, \\ \zeta_U := \{\sigma_j : \sigma_j < \sigma_{j+1}\}. \end{cases} \tag{1.14}$$

With $|R|$ denoting the *cardinality* of an arbitrary set R, then, by definition, $|\zeta_L|$ and $|\zeta_U|$ are precisely the number of nonzero block submatrices of B which are in L and in U, respectively. Also, as $\zeta_L \cup \zeta_U = \{1, 2, \dots, p\}$ and as $\zeta_L \cap \zeta_U = \emptyset$, then

$$|\zeta_L| + |\zeta_U| = p. \tag{1.15}$$

To determine which entries of the product LU, for the block-partitioning of (1.7), are nonzero, we define the disjoint (and possibly empty) subsets η_L and η_U of ζ_U as

$$\begin{cases} \eta_L := \{\sigma_j : \sigma_{j-1} > \sigma_j, \sigma_{j+1} > \sigma_j, \quad \text{and} \quad \sigma_{j-1} > \sigma_{j+1}\} \\ \eta_U := \{\sigma_j : \sigma_{j-1} > \sigma_j, \sigma_{j+1} > \sigma_j, \quad \text{and} \quad \sigma_{j-1} < \sigma_{j+1}\}, \end{cases} \quad (\text{where } \sigma_0 := \sigma_p). \tag{1.16}$$

Again by definition, $|\eta_L|$ and $|\eta_U|$ are precisely the *number* of nonzero block submatrices of LU which occur in the strictly block-lower and strictly block-upper triangular parts, respectively, of the partitioning for LU. We further set

$$k := \begin{cases} |\eta_L| + |\eta_U| & \text{if } p > 2, \\ 1 & \text{if } p = 2. \end{cases} \tag{1.17}$$

If l is such that $\sigma_l = 1$, then evidently $\sigma_{l-1} > \sigma_l$ and $\sigma_{l+1} > \sigma_l$, so that (cf. (1.16) σ_l is necessarily either an element of η_L or of n_U for $p > 2$. Consequently, (cf. (1.17)), $k \geqq 1$ if $p > 2$. Similarly, if σ_l satisfies $\sigma_{l-1} > \sigma_l$ and $\sigma_{l+1} > \sigma_l$, then neither σ_{l-1} nor σ_{l+1} can be an element of η_L or η_U, so that $k \leqq [[p/2]]$, giving

$$1 \leqq k \leqq [[p/2]], \tag{1.18}$$

where $[[x]]$ denotes the integer part of a real number x. As can be verified, k is precisely the number of nonzero block submatrices of LU. It is further evident that $|\zeta_L| \geqq k$ and $|\zeta_U| \geqq k$.

We finally give in this section a *directed graph* interpretation of the positive integer k of (1.17). Specifically, let $G_\pi[B]$ denote the *directed graph of type* 2 for the block-partitioned matrix B of (1.7), i.e., (cf. [6, p. 121]), we associate with the matrix B of (1.7) a directed graph with p vertices, $V_1, V_2, \dots, V_p$, where an arc from vertex V_i to the vertex V_j is drawn with a *double arrow* only if $B_{i,j} \not\equiv O$ and if $j > i$, while an arc from vertex V_i to the vertex V_j is drawn with a *single arrow* only if $B_{i,j} \not\equiv O$ and if $j < i$. Then, for any simple closed path of length p starting at any vertex V_i and ending at the same vertex V_i (this path consisting of consecutive single- and/or double-arrowed arcs), the positive integer k of (1.17) is *precisely* the number of times (in travelling this closed path) that a double-arrowed arc *follows* a single-arrowed arc. This will be illustrated in three examples in §2.

2 Statement of Main Result and Discussion

With the notations and definitions of §1, our main result is

Theorem 1. *Assume that the block-partitioned matrix A of* (1.5) *is such that all diagonal submatrices $A_{i,i}$ are square and nonsingular* ($1 \leqq i \leqq p$), *and assume that its block-Jacobi matrix B of* (1.7) *is a weakly cyclic matrix of index p, generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$. If $\omega + \hat{\omega} - \omega\hat{\omega} \neq 0$, if λ is an nonzero eigenvalue of the USSOR matrix $T_{\omega,\hat{\omega}}$ of* (1.9), *and if μ satisfies*

$$[\lambda - (1-\omega)(1-\hat{\omega})]^p = \lambda^k [\lambda\omega + \hat{\omega} - \omega\hat{\omega}]^{|\zeta_L| - k} [\lambda\hat{\omega} + \omega - \omega\hat{\omega}]^{|\zeta_U| - k} \cdot (\omega + \hat{\omega} - \omega\hat{\omega})^{2k} \mu^p, \tag{2.1}$$

(where k, $|\zeta_L|$, and $|\zeta_U|$ are defined from σ in §1, *and where the convention $0^0 := 1$ is used in* (2.1)*), then μ is an eigenvalue of B. Conversely, if μ is an eigenvalue of B and if $\hat{\lambda}$ satisfies* (2.1), *then $\hat{\lambda}$ is an eigenvalue of $T_{\omega,\hat{\omega}}$.*

The proof of this theorem will be given in §3. We remark that in the case $\omega = \hat{\omega}$, (2.1) reduces with (1.15) to

$$[\lambda - (1-\omega)^2]^p = \lambda^k [\lambda + 1 - \omega]^{p-2k} (2-\omega)^{2k} \omega^p \mu^p, \tag{2.1'}$$

which was given in Gong and Cai [1, Eq. (1.4)].

To complete this section, we show how this new functional Eq. (2.1) relates to recent results in this area.

Example 1. Consider the block-partitioned Jacobi matrix B_1 given by

$$B_1 = \begin{bmatrix} O & O & \ldots & O & B_{1,p} \\ B_{2,1} & O & & O & O \\ O & B_{3,2} & \ddots & & O \\ \vdots & & \ddots & \ddots & \vdots \\ O & O & \ldots & B_{p,p-1} & O \end{bmatrix}, \tag{2.2}$$

where $p \geqq 2$. In this case, B_1 is a weakly cyclic of index p matrix, generated by the cyclic permutation $(1, p, p-1, \ldots, 3, 2)$. From the definitions of §1, we have

$$\begin{aligned} &\zeta_L = \{2, 3, \ldots, p\} \text{ and } |\zeta_L| = p-1; \quad \zeta_U = \{1\} \text{ and } |\zeta_U| = 1, \\ p>2:\ &\eta_L = \emptyset \text{ and } |\eta_L| = 0; \quad \eta_U = \{1\} \text{ and } |\eta_U| = 1; \quad k=1, \\ p=2:\ &\eta_L = \emptyset \text{ and } |\eta_L| = 0; \quad \eta_U = \emptyset \text{ and } |\eta_U| = 0; \quad k=1. \end{aligned}$$

For the case $p = 6$, the block-directed graph of type 2 for the matrix B_1 of (2.2) is shown below in Fig. 1.

In this case, the functional Eq. (2.1) reduces to

$$[\lambda-(1-\omega)(1-\hat{\omega})]^{p}=\lambda[\lambda\omega+\hat{\omega}-\omega\hat{\omega}]^{p-2}(\omega+\hat{\omega}-\omega\hat{\omega})^{2}\,\mu^{p}, \qquad (2.3)$$

which is the functional equation for $T_{\omega,\hat{\omega}}$, derived by Saridakis [3], for the block-Jacobi matrix of (2.2).

Example 2. Consider the block-partitioned Jacobi matrix B_2 given by

$$B_2=\begin{bmatrix} O & B_{1,2} & O & \cdots & O \\ O & O & B_{2,3} & \cdots & O \\ \vdots & & \ddots & \ddots & \vdots \\ O & O & O & & B_{p-1,p} \\ B_{p,1} & O & O & \cdots & O \end{bmatrix}. \qquad (2.4)$$

In this case, B_2 is a weakly cyclic of index p matrix generated by the cyclic permutation $(1, 2, \ldots, p)$, and we have

$$\begin{aligned} &\zeta_L=\{p\} \text{ and } |\zeta_L|=1; \quad \zeta_U=\{1,2,\ldots,p-1\} \text{ and } |\zeta_U|=p-1,\\ p>2:\ &\eta_L=\{1\} \text{ and } |\eta_L|=1; \quad \eta_U=\emptyset \text{ and } |\eta_U|=0; \quad k=1,\\ p=2:\ &\eta_L=\emptyset \text{ and } |\eta_L|=0; \quad \eta_U=\emptyset \text{ and } |\eta_U|=0; \quad k=1. \end{aligned}$$

For the case $p=6$, the block-directed graph of type 2 for the matrix B_2 of (2.4) is shown below in Fig. 2.

In this case, the functional Eq. (2.1) reduces to

$$[\lambda-(1-\omega)(1-\hat{\omega})]^{p}=\lambda[\lambda\hat{\omega}+\omega-\omega\hat{\omega}]^{p-2}(\omega+\hat{\omega}-\omega\hat{\omega})^{2}\,\mu^{p}. \qquad (2.5)$$

For the special case $\omega=\hat{\omega}$, the above functional equation (for S_ω) was obtained in Varga et al. [4]. For general ω and $\hat{\omega}$, (2.5) was also obtained by Saridakis [3].

Example 3. Consider the block-partitioned Jacobi matrix B_3 given by

$$B_3=\begin{bmatrix} O & O & B_{1,3} & O \\ O & O & O & B_{2,4} \\ O & B_{3,2} & O & O \\ B_{4,1} & O & O & O \end{bmatrix}. \qquad (2.6)$$

In this case, B_3 is a weakly cyclic of order 4 matrix generated by the cyclic permutation (1, 3, 2, 4). Thus,

$$\begin{aligned} &\zeta_L=\{3,4\} \text{ and } |\zeta_L|=2; \quad \zeta_U=\{1,2\} \text{ and } |\zeta_U|=2,\\ &\eta_L=\{1\} \text{ and } |\eta_L|=1; \quad \eta_U=\{2\} \text{ and } |\eta_U|=1; \quad k=2, \end{aligned}$$

and the block-directed graph of type 2 for the matrix B_3 of (2.6) is given in Fig. 3.

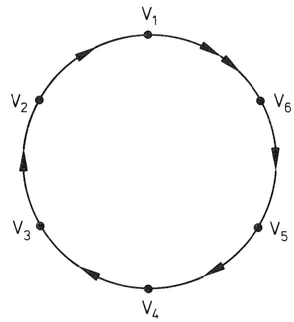

Fig. 1. $G_\pi(B_1)$

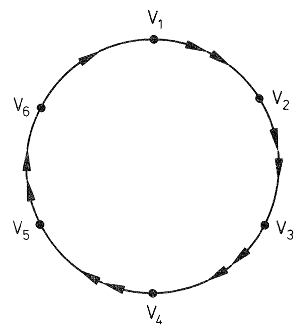

Fig. 2. $G_\pi(B_2)$

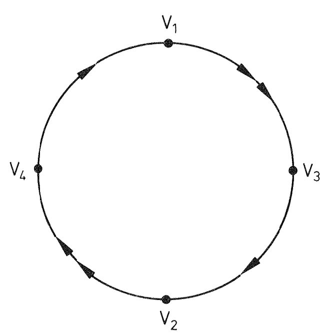

Fig. 3. $G_\pi(B_3)$

In this case, the functional Eq. (2.1) reduces to

$$[\lambda-(1-\omega)(1-\hat{\omega})]^4=\lambda^2(\omega+\hat{\omega}-\omega\hat{\omega})^4\mu^4. \tag{2.7}$$

For the special case $\omega=\hat{\omega}$, the above functional equation was obtained in Varga, et al. [4, Eq. (2.36)], and, again for $\omega=\hat{\omega}$, was given as an example in Gong and Cai [1, Eq. (1.6)].

As mentioned in §1, we can also apply the above graph-theoretic ideas to the analysis of the SOR (successive overrelaxation) iterative method. Specifically, associated with the block-Jacobi matrix B of (1.6)–(1.8) for the matrix problem (1.4), is the well-known SOR iteration matrix $\mathcal{L}_\omega$, defined by

$$\mathcal{L}_\omega := (I-\omega L)^{-1}[(1-\omega)I+\omega U]. \tag{2.8}$$

If B is a weakly cyclic matrix of index p, generated by the cyclic permutation $(\sigma_1, \sigma_2, \ldots, \sigma_p)$, then the functional equation (analogous to (1.2), (1.2′), and (2.1)), which couples the eigenvalues μ of B to the eigenvalues λ of $\mathcal{L}_\omega$, is known (cf. Nickel and Fox [2] and [6, p. 109, Exercise 2]) to be

$$(\lambda+\omega-1)^p = \lambda^\tau \omega^p \mu^p. \tag{2.9}$$

It turns out (as is easily seen) that the exponent τ in (2.9) is *precisely* $|\zeta_L|$, and $|\zeta_L|$ is, from our discussions in §1, exactly the number of nonzero lower triangular block submatrices of B. Equivalently, in terms of the associated directed graph $G_\pi(B)$ of type 2 described in §1, τ is *precisely* the number of single-arrowed arcs in any simple closed path of length p starting at any vertex V_i and ending at the same vertex V_i.

3 Proof of the Theorem

It can be verified from (1.9) that

$$\lambda I - T_{\omega,\hat{\omega}} = (I-\hat{\omega}U)^{-1}(I-\omega L)^{-1}(\gamma I - \alpha L - \beta U - \delta LU), \tag{3.1}$$

where

$$\begin{cases} \gamma := \lambda-(1-\omega)(1-\hat{\omega}), \\ \alpha := \lambda\omega+\hat{\omega}-\omega\hat{\omega}, \\ \beta := \lambda\hat{\omega}+\omega-\omega\hat{\omega}, \\ \delta := (1-\lambda)\,\omega\hat{\omega}. \end{cases} \tag{3.2}$$

Hence, λ is an eigenvalue of $T_{\omega,\hat{\omega}}$ if and only if

$$\det\{\gamma I - \alpha L - \beta U - \delta LU\} = 0. \tag{3.3}$$

Before we prove Theorem 1, we first establish Lemmas 2, 3, and 4. For notation, we introduce two $p \times p$ block-partitioned matrices $H_L := [H_{i,h}]$ and $H_U := [\tilde{H}_{h,j}]$, associated with the block-partitioned matrix B of (1.7), where

$$H_{i,h} := \begin{cases} B_{i,h} & \text{if } h \in \eta_L, \\ O & \text{otherwise}, \end{cases} \tag{3.4}$$

and

$$\tilde{H}_{h,j} := \begin{cases} B_{h,j} & \text{if } h \in \eta_U, \\ O & \text{otherwise}. \end{cases} \tag{3.5}$$

For example, in the case of B_3 of (2.6), we have

$$H_L=\begin{bmatrix} O & O & O & O \\ O & O & O & O \\ O & O & O & O \\ B_{4,1} & O & O & O \end{bmatrix}, \quad H_U=\begin{bmatrix} O & O & O & O \\ O & O & O & B_{2,4} \\ O & O & O & O \\ O & O & O & O \end{bmatrix}.$$

Lemma 2. *Let $B=L+U$ of (1.7) be a weakly cyclic of index p matrix, generated by a cyclic permutation $\sigma=(\sigma_1, \sigma_2, \ldots, \sigma_p)$. Then, for arbitrary complex numbers α, β, δ and δ with $\alpha \neq 0$ and $\beta \neq 0$,*

$$\begin{aligned} &\det\{\gamma I-\alpha L-\beta U-\delta LU\} \\ &=\det\left\{\gamma I-\left(\frac{\alpha\beta+\gamma\delta}{\beta}\right)H_L-\alpha(L-H_L)-\left(\frac{\alpha\beta+\gamma\delta}{\alpha}\right)H_U-\beta(U-H_U)\right\}, \end{aligned} \tag{3.6}$$

where the matrices H_L and H_U are defined in (3.4) *and* (3.5).

Proof. With (1.8), set

$$E:=\gamma I-\alpha L-\beta U-\delta LU. \tag{3.7}$$

As we shall see, eliminating from the matrix E (by means of elementary block-row and block-column transformations applied to the matrix E) those nonzero submatrices of $LU:=[C_{i,j}]$, will directly give the desired result of (3.6).

It follows from the definition of η_L and η_U (cf. (1.16)) that for each $C_{i,j} \neq 0$ with $i>j$ (in the lower triangular part of LU), there exists a unique h in η_L such that $C_{i,j}=B_{i,h}B_{h,j}$. Focusing on the six associated submatrices in E (namely, $E_{h,h}$, $E_{h,j}$, $E_{h,i}$ and $E_{i,h}$, $E_{i,j}$, $E_{i,i}$) we have from the form of E that

$$E=\begin{bmatrix} & \vdots & & \vdots & & \vdots & \\ \cdots & \gamma I_{h,h} & \cdots & -\beta B_{h,j} & \cdots & O & \cdots \\ & \vdots & & & & \vdots & \\ & \vdots & & & & \vdots & \\ & \vdots & & & & \vdots & \\ \cdots & -\alpha B_{i,h} & \cdots & -\delta B_{i,h}B_{h,j} & \cdots & \gamma I_{i,i} & \cdots \\ & \vdots & & \vdots & & \vdots & \end{bmatrix}. \tag{3.8}$$

Because $\beta \neq 0$ by assumption, consider the lower block-triangular matrix Q, defined by

$$Q:=\begin{bmatrix} I_{1,1} & O & \cdots & O & O \\ O & I_{h,h} & & O & O \\ \vdots & & \ddots & & \vdots \\ O & -\frac{\delta}{\beta}B_{i,h} & & I_{i,i} & O \\ O & O & \cdots & O & I_{p,p} \end{bmatrix}, \tag{3.9}$$

where Q has a sole nonzero block, i.e., $-\delta B_{i,h}/\beta$, in its strictly lower block-triangular part. Then, it is easily seen that the matrix product QE (corresponding to an elementary block-row transformation of E) satisfies

$$\det(QE)=\det E; \quad (QE)_{i,j}\equiv O; \quad (QE)_{i,h}=-\frac{\gamma\delta}{\beta}H_{i,h}-\alpha L_{i,h},$$

so that the submatrix $C_{i,j}$ has been reduced to zero in this step. In this fashion, *all* nonzero submatrices $C_{i,j}$ (with $i>j$) can be eliminated by such blow-row elementary transformations, and the resulting lower triangular part of the transformed matrix E is $-(\gamma\delta/\beta)\, H_L-\alpha L$. Similarly, for all nonzero submatrices $C_{i,j}$ in the upper triangular part ($i<j$) of LU, we apply corresponding block-column elementary transformations to E. Then, the resulting upper triangular part in E becomes $-(\gamma\delta/\alpha)\, H_U-\beta U$. As such elementary transformations leave the associated determinants univariant, the lemma is proved. □

Lemma 3. *Let $B=L+U$ of* (1.7) *be a weakly cyclic of index p matrix generated by a cyclic permutation $\sigma=(\sigma_1, \sigma_2, \ldots, \sigma_p)$. Then, for arbitrary complex numbers γ, a, b, c, and d,*

$$\begin{aligned}&\det\{\gamma I-aH_L-b(L-H_L)-cH_U-d(U-H_U)\}\\ &=\det\{\gamma I-t^{1/p}B\},\end{aligned} \tag{3.10}$$

where $t:=a^{|\eta_L|}\, b^{|\zeta_L|-|\eta_L|}\, c^{|\eta_U|}\, d^{|\zeta_U|-|\eta_U|}$, where the matrices H_L and H_U are defined in (3.4) *and* (3.5), *and where the convention $0^0=1$ is used in the definition of t.*

Proof. Assume first that $abcd\neq 0$. We define the $p\times p$ block-partitioned matrix $M(a, b, c, d)$ by

$$M(a,b,c,d):=t^{-1/p}\{aH_L+b(L-H_L)+cH_U+d(U-H_U)\}. \tag{3.11}$$

On comparing the matrix $M:=M(a, b, c, d)$ with the matrix B, it is easily seen that the matrix M has exactly the same partitioning structure as the matrix B, except for scalar multipliers of its nonzero submatrices. Thus, M is a weakly cyclic of index p matrix, generated by the same permutation $\sigma=(\sigma_1, \sigma_2, \ldots, \sigma_p)$, and M^p and B^p are both block-diagonal matrices having the same diagonal submatrices, except for scalar multipliers. Since there are $|\eta_L|$ and $|\eta_U|$ nonzero submatrices in the matrices H_L and H_U, respectively, then there are $|\zeta_L|-|\eta_L|$ and $|\zeta_U|-|\eta_U|$ nonzero submatrices in matrices $L-H_L$ and $U-H_U$, respectively. Recalling from (1.15) that $|\zeta_L|+|\zeta_U|=p$, it follows from the definition of t and by direct computation that the scalar multiplier of each diagonal submatrix in M^p is

$$t^{-1}\, a^{|\eta_L|}\, b^{|\zeta_L|-|\eta_L|}\, c^{|\eta_U|}\, d^{|\zeta_U|-|\eta_U|}=1.$$

Thus,

$$[M(a,b,c,d)]^p=B^p, \tag{3.12}$$

and the eigenvalues of matrix $M(a, b, c, d)$ are *independent* of a, b, c, and d. Note that as $M(1,1,1,1)=B$, we have

$$\det\{\gamma I - t^{1/p} M(a,b,c,d)\} = \det\{\gamma I \text{-} t^{1/p} B\}, \tag{3.13}$$

which is the desired result of (3.10) when $abcd \neq 0$.

The remaining case, $abcd=0$, similarly follows by continuity since both sides of (3.10) are *continuous* functions of the parameters a, b, c, and d. For example, if, as in Example 1, $|\zeta_U| = 1 = |\eta_U|$, then $d^{|\zeta_U| - |\eta_U|} \equiv 1$ for all $d \neq 0$. Thus, on letting $d \to 0$, $d^{|\zeta_U| - |\eta_U|}$, arising as a factor of t in (3.10), has the value unity (which explains our use of the convention $0^0 := 1$). □

By applying Lemma 2 and Lemma 3, we can establish the following result, Lemma 4, which gives a general determinantal invariance associated with weakly cyclic of index p matrices.

Lemma 4. *Let $B = L + U$ of* (1.7) *be a weakly cyclic of index p matrix, generated by a cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$. Then, for arbitrary complex numbers α, β, γ, and δ,*

$$\det\{\gamma I - \alpha L - \beta U - \delta LU\} = \det\{\gamma I - [\alpha^{|\zeta_L| - k} \beta^{|\zeta_U| - k} (\alpha\beta + \gamma\delta)^k]^{1/p} B\}, \tag{3.14}$$

where $|\zeta_L|$, $|\zeta_U|$ and k are as defined in §1, *and where the convention $0^0 := 1$ is used in* (3.14).

Proof. For $\alpha \neq 0$ and $\beta \neq 0$, Lemma 4 is the straightforward consequence of (1.17) and Lemmas 2 and 3. As in the proof of Lemma 3, continuity considerations then allow us to extend (3.14) to cases when $\alpha = 0$ and $\beta = 0$, provided that the convention $0^0 := 1$ is used. □

This brings us to the

Proof of Theorem 1. If $\phi(\lambda) := \det(\lambda I - T_{\omega,\hat{\omega}})$, then $\phi(\lambda) = \det\{\gamma I - \alpha L - \beta U - \delta LU]$ from (3.1). Thus, from (3.14) of Lemma 4, we further have

$$\phi(\lambda) = \det\{\gamma I - [\alpha^{|\zeta_L| - k} \beta^{|\zeta_U| - k} (\alpha\beta + \gamma\delta)^k]^{1/p} B\}. \tag{3.15}$$

As remarked at the very beginning of this section, λ is an eigenvalue of $T_{\omega,\hat{\omega}}$ if and only if $\phi(\lambda) = 0$, i.e. (cf. (3.15)), if and only if

$$\det\{\gamma I - [\alpha^{|\zeta_L| - k} \beta^{|\zeta_U| - k} (\alpha\beta + \gamma\delta)^k]^{1/p} B\} = 0. \tag{3.16}$$

Now, the proof follows the procedure of the proof of (1.2′) (cf. [6, Th. 4.3]). First, from the definitions of (3.2), there follows

$$\alpha\beta + \gamma\delta = \lambda(\omega + \hat{\omega} - \omega\hat{\omega})^2. \tag{3.17}$$

Since B is weakly cyclic of index p, it follows from (3.15), (3.2), and Romanovsky's Theorem (cf. [6, p. 40]) that

$$\phi(\lambda)=\gamma^m \prod_{i=1}^{r} \{\gamma^p-\alpha^{|\zeta_L|-k}\beta^{|\zeta_U|-k}(\alpha\beta+\gamma\delta)^k\mu_i^p\}$$

$$=[(\lambda-(1-\omega)(1-\hat{\omega})]^m \prod_{i=1}^{r}\{[\lambda-(1-\omega)(1-\hat{\omega})]^p$$

$$-\lambda^k(\omega+\hat{\omega}-\omega\hat{\omega})^{2k}(\lambda\omega+\hat{\omega}-\omega\hat{\omega})^{|\zeta_L|-k}(\lambda\hat{\omega}+\omega-\omega\hat{\omega})^{|\zeta_U|-k}\mu_i^p\}, \tag{3.18}$$

where the μ_i are nonzero eigenvalue of B if $r \geqq 1$ and where m is a nonnegative integer. To establish the second part of this theorem, let μ be an eigenvalue of B and let $\hat{\lambda}$ satisfy (2.1). Then, one of the factors of $\phi(\hat{\lambda})$ of (3.18) vanishes, proving that $\hat{\lambda}$ is an eigenvalue of $T_{\omega,\hat{\omega}}$, the desired second part of Theorem 1. To establish the first part of Theorem 1, let $\omega+\hat{\omega}-\omega\hat{\omega}\neq 0$ and let λ be an nonzero eigenvalue of $T_{\omega,\hat{\omega}}$. It follows that at least one factor of (3.18) vanishes. It is convenient to note that (2.1), from (3.2) and (3.17), can be expressded as

$$\gamma=\lambda^k\alpha^{|\zeta_L|-k}\beta^{|\zeta_U|-k}(\omega+\hat{\omega}-\omega\hat{\omega})^{2k}\mu^p. \tag{3.19}$$

If $\mu\neq 0$ and μ satisfies (3.19), then, assuming in addition that $\alpha\beta\neq 0$, we must have that $\gamma=\lambda-(1-\omega)(1-\hat{\omega})\neq 0$. Thus, (2.1) is valid for some nonzero μ_i where $1\leqq i\leqq r$. Combining this with (2.1), we have that $\mu^p=\mu_i^p$. Taking pth roots, then

$$\mu=\mu_i e^{2\pi i s/p}, \tag{3.20}$$

where s is a nonnegative integer satisfying $0\leqq s<p$. But, from the weakly cyclic of index p nature of the matrix B, it is evident that μ is also an eigenvalue of B, which is the desired first part of Theorem 1. To conclude the proof, if $\omega+\hat{\omega}-\omega\hat{\omega}\neq 0$, if λ is a nonzero eigenvalue of $T_{\omega,\hat{\omega}}$, and if $\mu=0$ satisfies (3.19), then we must show that $\mu=0$ is an eigenvalue of B. But with these hypotheses and $\alpha\beta\neq 0$, it is evident from (3.19) that $\gamma=0$. In this case, (3.16) reduces to

$$\det\{-[\alpha^{|\zeta_L|-k}\beta^{|\zeta_U|-k}\lambda^k(\omega+\hat{\omega}-\omega\hat{\omega})^{2k}]^{1/p}B\}=0. \tag{3.21}$$

But, as the multiplicative factor of B in (3.21) is nonzero, then $\det B=0$. Hence, $\mu=0$ is an eigenvalue of B which is again the desired first part of Theorem 1, under the added assumption that $\alpha\beta\neq 0$. To establish the first part of Theorem 1 when $\alpha\beta=0$ is similar but tedious, and this is omitted. □

Acknowledgements. The research of the authors was supported in part by the Air Force Office of Scientific Research under AFOSR-85-0245.

References

1. Gong, L., Cai, D.Y.: Relationship between eigenvalues of Jacobi and SSOR iterative matrix with p-weak cyclic matrix. J. Comput. Math. of Colleges and Universities **1**, 79–84 (1985) (in Chinese)

2. Nickols, N.K., Fox, L.: Generalized consistent ordering and the optimum successive over-relaxation factor. Numer. Math. **13**, 425–433 (1969)
3. Saridakis, Y.G.: On the analysis of the unsymmetric successive overrelaxation method when applied to *p*-cyclic matrices. Numer. Math. **49**, 461–473 (1986)
4. Varga, R.S., Niethammer, W., Cai, D.Y.: *p*-cyclic matrices and the symmetric successive overrelaxation method. Linear Algebra Appl. **58**, 425–439 (1984)
5. Varga, R.S.: *p*-Cyclic matrices: a generalization of the Young-Frankel successive overrelaxation scheme. Pacific J. Math. **9**, 617–628 (1959)
6. Varga, R.S.: Matrix Iterative Analysis. Englewood Cliffs, N.J.: Prentice Hall 1962
7. Young, D.M., Jr.: Iterative methods for solving partial difference equations of elliptic type. Trans. Amer. Math. Soc. **76**, 92–111 (1954)
8. Young, D.M.: Iterative Solution of Large Linear Systems, New York: Academic Press 1971

Received June 10, 1988/November 17, 1988