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# On a 2-Periodic Lacunary Trigonometric Interpolation Problem

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**Abstract.** In this paper we obtain simple necessary and sufficient conditions for a particular 2-periodic lacunary trigonometric interpolation problem on equidistant nodes in $[0, 2\pi]$ to be regular.

## §1. Introduction

For a given positive integer $n$, define

$$x_k = x_k(n) := k\pi/n \qquad (k = 0, 1, \cdots, 2n-1), \tag{1.1}$$

so that $\{x_k\}_{k=0}^{2n-1}$ is a set of $2n$ equidistant *nodes* in $[0, 2\pi)$. Next, assume that

$$\{m_j\}_{j=1}^{p} \text{ are any } p \text{ distinct positive integers } (p \geq 1). \tag{1.2}$$

We consider here the following 2-*periodic lacunary trigonometric interpolation problem*, denoted by the expression

$$(0 =: m_0, m_1, \cdots, m_p; m_1, m_2, \cdots, m_p), \tag{1.3}$$

on the $2n$ equidistant nodes $\{x_k\}_{k=0}^{2n-1}$ in $[0, 2\pi)$. For arbitrary data consisting of complex numbers $\{\alpha_{j,\nu}\}_{j=0,\nu=0}^{n-1,m_p}$ and $\{\beta_{j,\nu}\}_{j=0,\nu=1}^{n-1,m_p}$, we ask if there is a *unique* trigonometric polynomial of the form

$$t_M(x) = a_0 + \sum_{k=1}^{M} (a_k \cos kx + b_k \sin kx), \tag{1.4}$$

$$t_M(x) = a_0 + \sum_{k=1}^{M-1} (a_k \cos kx + b_k \sin kx) + a_M \cos(Mx + \frac{\varepsilon\pi}{2}), \tag{1.4'}$$

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(where $\varepsilon = 0$ or where $\varepsilon = 1$), such that

$$\begin{cases} t_M^{(m_\nu)}(x_{2j}) = \alpha_{j,\nu} & (j = 0, 1, \cdots, n-1; \quad \nu = 0, 1, \cdots, p) \\ t_M^{(m_\nu)}(x_{2j+1}) = \beta_{j,\nu} & (j = 0, 1, \cdots, n-1; \quad \nu = 1, 2, \cdots, p). \end{cases} \tag{1.5}$$

Note that as the number of nodes in (1.1) is *even*, we see that the interpolation conditions of (1.5) break down into interpolation conditions on the disjoint sets $\{x_{2j}\}_{j=0}^{n-1}$ and $\{x_{2j+1}\}_{j=0}^{n-1}$ of nodes, each set consisting of $n$ nodes. From this, the term 2-*periodic* lacunary trigonometric interpolation is derived. In addition, we see from (1.5) that the first group of integers of (1.3) give the derivative conditions on the set $\{x_{2j}\}_{j=0}^{n-1}$, while the latter group of integers of (1.3) give the derivative conditions on $\{x_{2j+1}\}_{j=0}^{n-1}$.

It is evident that the total number of interpolation conditions in (1.5) is

$$N := n(2p+1). \tag{1.6}$$

Then, as $N$ is odd iff $n$ is odd, the desired trigonometric interpolant $t_M(x)$ in (1.5) when $N$ is odd is necessarily of the form (1.4) (which has an odd number of parameters), and $M = (N-1)/2$ in this case. Continuing, as $N$ is even iff $n$ is even, the desired trigonometric polynomial $t_M(x)$ in (1.5) when $N$ is even is necessarily of the form (1.4′) with $M = N/2$, where $\varepsilon = 0$ or 1 is to be determined. We say that this $(0, m_1, \cdots, m_p; m_1, \cdots, m_p)$ 2-periodic lacunary interpolation problem is *regular* if, for *arbitrary* data, (1.5) admit a *unique* solution, where $t_M(x)$ is of the form (1.4) when $N$ is odd, or of the form (1.4′) when $N$ is even.

The goal of this paper is to derive *simple* (i.e., non-determinantal) necessary and sufficient conditions on $N, n$, the integers $\{m_j\}_{j=1}^p$, and $\varepsilon$ (when (1.4′) is used) for the 2-periodic lacunary trigonometric interpolation problem (1.3) to be *regular*. As we shall see below, this goal is reached.

## §2. Main Result

For notation, let

$$\begin{cases} e_p := \text{number of even integers in the set} \quad \{m_j\}_{j=1}^p \text{ of (1.2);} \\ o_p := \text{number of odd integers in the set} \quad \{m_j\}_{j=1}^p \text{ of (1.2),} \end{cases} \tag{2.1}$$

so that

$$e_p + o_p = p. \tag{2.2}$$

Our main result is the following

**Theorem 1.** *Let $\{m_j\}_{j=1}^{p}$ be $p$ distinct positive integers and let $\{x_k(n)\}_{k=0}^{2n-1}$ be the $2n$ equidistant nodes in $[0, 2\pi)$ of (1.1).*

1. *If $N := n(2p+1)$ is odd, so that $n$ is also odd, then the 2-periodic trigonometric interpolation problem (1.5), with $t_M(x)$ of the form (1.4), is regular iff $p$ is even and* (cf. (2.1))

$$e_p = o_p = p/2. \tag{2.3}$$

2. *If $N := n(2p+1)$ is even, so that $n$ is also even, then the 2-periodic trigonometric interpolation problem (1.5), with $t_M(x)$ of the form (1.4′), is regular iff $p$ is even, (2.3) is satisfied, and $\varepsilon = 0$ in (1.4′).*

*In particular, the interpolation problem (1.5) is never regular when $p$ is odd.*

**Proof.** (Sketch for $N$ odd): Assume that $N$ is odd, so that $n$ is also odd from (1.6), and we write $n = 2r + 1$. In this case, $M = (N-1)/2 = np + r$, and the desired trigonometric polynomials $t_M(x)$ are of the form (1.4). Using the familiar device for identifying trigonometric polynomials with algebraic polynomials through the transformation $z = e^{ix}$, any trigonometric polynomial $t_M(x)$ of the form (1.4) can be expressed as

$$t_M(x) = z^{-M} q_{2M}(z), \tag{2.4}$$

where $q_{2M}(z)$ is a complex polynomial of degree at most $2M$. On considering null data in (1.5) (i.e., $\alpha_{j,\nu} = 0 = \beta_{j,\nu}$ in (1.5)), $t_M(x)$ can then be expressed as

$$t_M(x) = z^{-M}(z^n - 1) \sum_{\lambda=0}^{2p-1} z^{\lambda n} \sum_{j=0}^{n-1} a_{\lambda,j} z^j \qquad (n = 2r+1), \tag{2.5}$$

where the $2p$ unknowns $\{a_{\lambda,j}\}_{\lambda=0}^{2p-1}$ can be shown to satisfy (for each $j = 0, 1, \cdots, n-1$) the $2p$ homogeneous equations:

$$\begin{aligned}
&\sum_{\lambda=0}^{2p-1} a_{\lambda,j}\{(\alpha_j + \lambda + 1 - p)^{m_\nu} - (\alpha_j + \lambda - p)^{m_\nu}\} = 0 \qquad (\nu = 1, 2, \cdots, p),\\
&\sum_{\lambda=0}^{2p-1} (-1)^\lambda a_{\lambda,j}\{(\alpha_j + \lambda + 1 - p)^{m_\nu} + (\alpha_j + \lambda - p)^{m_\nu}\} = 0 \qquad (\nu = 1, 2, \cdots, p).
\end{aligned} \tag{2.6}$$

where (since $n = 2r + 1$)

$$\alpha_j := (j - r)/(2r+1) \qquad (j = 0, 1, \cdots, 2r). \tag{2.7}$$

Thus, if $\Delta(\alpha_j)$ denotes the determinant of order $2p$ of the coefficients of $\{a_{\lambda,j}\}_{\lambda=0}^{2p-1}$, then $\Delta(\alpha_j) \neq 0$ (for all $j = 0, 1, \cdots, 2r$) iff $t_M(x) \equiv 0$ in (2.4).

In other words, for the interpolation problem (1.5) to be *regular* in the case when $N$ is odd, it is necessary and sufficient that

$$\Delta(\alpha_j) \neq 0 \qquad (\alpha_j := (j-r)/(2r+1); \quad j = 0, 1, \cdots, 2r). \tag{2.8}$$

Since $\alpha_j = 0$ when $j = r$, a close examination of the particular determinant $\Delta(0)$, arising from (2.6) in the case $j = r$, shows that $\Delta(0) \neq 0$ implies that $p$ is even and that (cf. (2.1)) $e_p = o_p = p/2$. In other words, *necessary* conditions that the interpolation problem (1.5) be regular in this case when $N$ is odd are that

$$p \text{ is even, and } e_p = o_p = p/2. \tag{2.9}$$

Conversely, in the case when $N$ is odd, a lengthy proof, using determinantal tools, shows that (2.9) implies that $\Delta(\alpha_j) \neq 0$ for *all* $j = 0, 1, \cdots, 2r$, which, from (2.8), is necessary and sufficient for regularity.

The proof when $N$ is even is similar but more involved, as it requires, from (1.4′), the additional determination of $\varepsilon = 0$ or $\varepsilon = 1$. ∎

## References

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