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Real vs. Complex Best Rational Approximation

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Dedicated to G. G. Lorentz on the occasion of his 80th Birthday

Abstract. In this article, we survey the recent results on real vs. complex best rational approximation to continuous real-valued functions on the interval $[-1, 1]$.

§1. Introduction

We describe here a phenomenon, of recent research interest, which results when *complex* rational functions are pitted against *real* rational functions (of the same order) in approximating (in the uniform norm) *real* continuous functions on the *real* interval $[-1, +1]$.

For notation, let π_m^r and π_m^c be respectively the sets of polynomials (in the variable z or x) of degree at most m, with real and complex coefficients. For any pair (m, n) of nonnegative integers, $\pi_{m,n}^r$ and $\pi_{m,n}^c$ then denote respectively the sets of rational functions of the form p/q, with p in $\pi_m^r(\pi_m^c)$ and q in $\pi_n^r(\pi_n^c)$. With

$$I := [-1, +1],$$

let $C_r(I)$ be the set of all continuous real-valued functions on I. Then, for any f in $C_r(I)$, we further set

$$E_{m,n}^r(f) := \inf_{g \in \pi_{m,n}^r} \|f - g\|_{L_\infty(I)}; \quad E_{m,n}^c(f) := \inf_{g \in \pi_{m,n}^c} \|f - g\|_{L_\infty(I)}, \tag{1.1}$$

where, for any real- or complex-valued function h defined on I,

$$\|h\|_{L_\infty(I)} := \sup \{|h(x)| : x \in I\}.$$

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The phenomenon to be studied here is this. We claim that, for each pair (m, n) of nonnegative integers with $n \geq 1$, there exists an f in $C_r(I)$ for which

$$\begin{cases} (i) & E^c_{m,n}(f) < E^r_{m,n}(f), \text{ and} \\ (ii) & \text{best uniform approximation to } f \text{ from } \pi^c_{m,n} \text{ on } I \\ & \text{is } \textit{not unique}. \end{cases} \tag{1.2}$$

Since $\pi^r_{m,n}$ is a proper subset of $\pi^c_{m,n}$ for each pair (m, n) of nonnegative integers, it is obvious from (1.1) that $E^c_{m,n}(f) \leq E^r_{m,n}(f)$ for any f in $C_r(I)$. Because an arbitrary complex number consists of two real parameters, then any $r_{m,n}$ in $\pi^r_{m,n}$ can be associated with an $R_{m,n}$ in $\pi^c_{m,n}$ with *twice* the number of real parameters. This alone might heuristically suggest that $E^c_{m,n}(f)$ is roughly at most $E^r_{m,n}(f)/2$ in (1.2*i*) for *certain* f in $C_r(I)$. We shall later see in (1.19) that, except for the case $n = 0$ when $E^c_{m,0}(f) = E^r_{m,0}(f)$ for any f in $C_r(I)$ and any $m \geq 0$, this heuristically deduced inequality (and more) is essentially *correct*.

It is known (cf. Meinardus [5, p. 161]) that any f in $C_r(I)$ admits a *unique* best uniform approximant $r_{m,n}$ from $\pi^r_{m,n}$ on I. On the other hand, Walsh [14, p. 356] has given an example of a continuous complex-valued function, namely $f(z) := z + z^{-1}$ which, on a certain compact crescent-shaped set in the complex plane, does *not* possess a unique best uniform rational approximation from $\pi^c_{1,1}$ on this set. That this phenomenon of *nonuniqueness* can hold even in the case of *real* functions on *real* intervals, as in (1.2*ii*), may come as somewhat of a surprise to the reader.

To give a concrete example exhibiting both parts of (1.2), we first recall (cf. [5, p. 161]) that, for any f in $C_r(I)$ and for any pair (m, n) of nonnegative integers, the unique $r_{m,n} = p/q$ in $\pi^r_{m,n}$ (where p and q are assumed to have no common factors) satisfying

$$E^r_{m,n}(f) = \|f - r_{m,n}\|_{L_\infty(I)}, \tag{1.3}$$

is precisely characterized by the existence of an *alternation set* $\{\xi_j\}_{j=1}^{\ell}$, with $-1 \leq \xi_1 < \xi_2 < \cdots < \xi_\ell \leq 1$, for which (with fixed $\lambda = 1$ or $\lambda = -1$)

$$f(\xi_j) - r_{m,n}(\xi_j) = \lambda(-1)^j E^r_{m,n}(f) \qquad (j = 1, 2, \cdots, \ell), \tag{1.4}$$

and for which

$$\ell \geq 2 + \max\{m + \deg q; n + \deg p\}. \tag{1.5}$$

(Here, we adopt the convention that if $p \equiv 0$, then $\deg p := -\infty$ and $\deg q := 0$, so that $\ell \geq 2 + m$ in (1.5) in this case. We also call ℓ the *length* of the alternation set $\{\xi_j\}_{j=1}^{\ell}$.)

Consider now the particular function $f(x) := x^2$ in $C_r(I)$. With $r_{1,1}(x) := p(x)/q(x) \equiv 1/2$ for all real x, we have that

$$E^r_{1,1}(x^2) = 1/2, \tag{1.6}$$

since all the conditions of (1.4) and (1.5) are fulfilled (with $\ell = 3,\ \lambda = -1,\ m = n = 1,\ \deg p = \deg q = 0,\ \xi_1 := -1,\ \xi_2 := 0$, and $\xi_3 := 1$). On the other hand, consider

$$g_{1,1}(x) := \frac{x + (\sqrt{2} - 1)i}{x + i}, \tag{1.7}$$

which is an element of $\pi^c_{1,1}$. A short calculation shows that

$$\|x^2 - g_{1,1}(x)\|_{L_\infty(I)} = \sqrt{2} - 1 = 0.41421 \cdots, \tag{1.8}$$

so that with (1.1), $E^c_{1,1}(x^2) \leq 0.41421 \cdots$. This implies from (1.6) that

$$E^c_{1,1}(x^2) < E^r_{1,1}(x^2). \tag{1.9}$$

Next, for any pair (m, n) of nonnegative integers and for any f in $C_r(I)$, there always exists (cf. Walsh [14, p. 351]) an $R_{m,n}$ in $\pi^c_{m,n}$ for which $\|f - R_{m,n}\|_{L_\infty(I)} = E^c_{m,n}(f)$. Specifically, there is an $R_{1,1}$ in $\pi^c_{1,1}$ for which $\|x^2 - R_{1,1}\|_{L_\infty(I)} = E^c_{1,1}(x^2)$. But from (1.9), it is clear that $R_{1,1}(x)$ *cannot* be real for all real x. Consequently, as $x^2 - R_{1,1}(x)$ and its complex conjugate have the same uniform norm on I, then

$$E^c_{1,1}(x^2) = \|x^2 - R_{1,1}(x)\|_{L_\infty(I)} = \|x^2 - \overline{R_{1,1}(x)}\|_{L_\infty(I)}. \tag{1.10}$$

Thus, $R_{1,1}$ and $\overline{R_{1,1}}$, which are *distinct* elements in $\pi^c_{1,1}$, are *both* best uniform approximations to x^2 from $\pi^c_{1,1}$ on I, and the function x^2 exhibits both properties of (1.2) in the case $m = n = 1$. We remark that this argument shows in general that if $E^c_{m,n}(f) < E^r_{m,n}(f)$, as in (1.2*i*), then the non-uniqueness in (1.2*ii*) necessarily follows.

A.A. Gonchar first mentioned in 1968 this possibility of nonuniqueness in a footnote of his paper [2]. This possibility was followed up by K.N. Lungu, a student of Gonchar, who gave sufficient conditions in [4] in 1971 for the properties of (1.2) to hold. Independently, Saff and Varga [9,10] made the same discovery in 1977, and obtained more general sufficient conditions for $E^c_{m,n}(f) < E^r_{m,n}(f)$ to hold for an f in $C_r(I)$, as well as a sufficient condition for $E^c_{m,n}(f) = E^r_{m,n}(f)$ to hold for an f in $C_r(I)$. The former sufficient conditions of Saff and Varga were later sharpened by Ruttan [6] who showed that $E^c_{m,n}(f) < E^r_{m,n}(f)$ holds if the best real uniform approximant from $\pi^r_{m,n}$ to f on I attains its maximum error on *no* alternation set (cf. (1.4)) of length greater than $m + n + 1$, and that this lower bound is, in general, *best possible*. For a survey of such results, see [12, Chapter 5].

What we wish to focus on here is the following problem raised in Saff and Varga [10]. For each pair (m, n) of nonnegative integers, determine the nonnegative real number $\gamma_{m,n}$, defined by

$$\gamma_{m,n} := \inf \left\{ E^c_{m,n}(f)/E^r_{m,n}(f) : f \in C_r(I) \backslash \pi^r_{m,n} \right\}. \tag{1.11}$$

In essence, determining the number $\gamma_{m,n}$ amounts to seeing just how much *better* best uniform approximation from $\pi^c_{m,n}$ on I can be, than best uniform approximation from $\pi^r_{m,n}$ on I, for *particular* functions in $C_r(I)\backslash\pi^r_{m,n}$. For example, for the function x^2 in $C_r(I)$, it is known (cf. Bennett, Rudnick and Vaaler [1]) that $E^c_{1,1}(x^2) = (4/27)^{1/2} = 0.38490\cdots$, so that with (1.6),

$$\frac{E^c_{1,1}(x^2)}{E^r_{1,1}(x^2)} = 0.76980\cdots .$$

Consequently, this gives the following upper bound for $\gamma_{1,1}$:

$$\gamma_{1,1} \le 0.76980\cdots . \tag{1.12}$$

To precisely determine $\gamma_{m,0}$ for any nonnegative integer m, we first establish

Proposition 1. ([10]). *Given any f in $C_r(I)$ and given any pair (m,n) of nonnegative integers, then*

$$E^r_{m+n,2n}(f) \le \inf_{g\in\pi^c_{m,n}} \|f - \operatorname{Re} g\|_{L_\infty(I)} \le E^c_{m,n}(f) \le E^r_{m,n}(f). \tag{1.13}$$

Proof: As the final inequality of (1.13) is obvious, consider any p_m/q_n in $\pi^c_{m,n}$. On taking real parts, it is evident that

$$\left| f(x) - \frac{p_m(x)}{q_n(x)} \right| \ge \left| f(x) - \operatorname{Re}\left(\frac{p_m(x)}{q_n(x)}\right) \right| \qquad (x \in I),$$

which establishes the second inequality of (1.13). Since Re (p_m/q_n) is an element of $\pi^r_{m+n,2n}$, the first inequality of (1.13) then follows. ■

On choosing $n = 0$ in (1.13), we see that

$$E^c_{m,0}(f) = E^r_{m,0}(f) \qquad (f \in C_r(I); m = 0, 1, \cdots), \tag{1.14}$$

so that (cf. (1.11))

$$\gamma_{m,0} = 1 \qquad (m = 0, 1, \cdots). \tag{1.15}$$

It turns out that the exact determination of the constants $\gamma_{m,n}$ of (1.11), when $m \ge 0$ and $n \ge 1$, is more delicate than the determination of $\gamma_{m,0}$ in (1.15). Four recent papers have described the behavior of $\gamma_{m,n}$ for $n \ge 1$. First, Trefethen and Gutknecht [11] established in 1983 the rather *remarkable* result that

$$\gamma_{m,n} = 0 \qquad (n \ge m + 3; m = 0, 1, \cdots). \tag{1.16}$$

Next, Levin [3] established in 1986 the complementary result that

$$\gamma_{m,n} = 1/2 \qquad (m+1 \geq n \geq 1). \tag{1.17}$$

Levin's proof of (1.17) consisted of a direct construction to show that $\gamma_{m,n} \leq 1/2$, and an algebraic method to show that $\gamma_{m,n} < 1/2$ was impossible for $m+1 \geq n \geq 1$. The results of (1.16) and (1.17) left unresolved only the constants $\gamma_{m,m+2}$ $(m \geq 0)$. This case was most recently settled by Ruttan and Varga in 1989, where it was shown in [7] that $\gamma_{m,m+2} \leq 1/3$ and in [8] that $\gamma_{m,m+2} < 1/3$ was impossible. Thus,

$$\gamma_{m,m+2} = 1/3 \qquad (m = 0, 1, \cdots). \tag{1.18}$$

The results of (1.15) - (1.18) give all the entries $\{\gamma_{m,n}\}_{m,n\geq 0}$ of Table 1.

n \ m	0	1	2	3	4	5	$\cdots$
0	1	1	1	1	$\cdots$	$\cdots$	$\cdots$
1	1/2	1/2	1/2	1/2	$\cdots$	$\cdots$	$\cdots$
2	1/3	1/2	1/2	1/2	$\cdots$	$\cdots$	$\cdots$
3	0	1/3	1/2	1/2	$\cdots$	$\cdots$	$\cdots$
4	0	0	1/3	1/2	$\cdots$	$\cdots$	$\cdots$
5	0	0	0	1/3	1/2	$\cdots$	$\cdots$
6	0	0	0	0	$\ddots$	$\ddots$	$\cdots$
7	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$

Table 1. $\{\gamma_{m,n}\}_{m,n\geq 0}$

It is interesting that the constants $\{\gamma_{m,n}\}_{m,n\geq 0}$ of (1.11) take on only *four* distinct values: $1, 1/2, 1/3$ and 0, and only *three* distinct values when $n \geq 1 : 1/2, 1/3$, and 0. Specifically, this implies for any ε with $0 < \varepsilon < 1$ that, for each pair (m,n) of integers with $m \geq 0$ and $n \geq 1$, there exists an f in $C_r(I)\backslash\pi^r_{m,n}$ for which

$$\begin{cases} (i) & E^c_{m,n}(f) < (1+\varepsilon)E^r_{m,n}(f)/2 \quad (m+1 \geq n \geq 1); \\ (ii) & E^c_{m,m+2}(f) < (1+\varepsilon)E^r_{m,m+2}(f)/3 \quad (m = 0, 1, \cdots); \\ (iii) & E^c_{m,n}(f) < \varepsilon E^r_{m,n}(f) \quad (n \geq m+3; m = 0, 1, \cdots), \end{cases} \tag{1.19}$$

which is a sharper form of (1.2*i*).

In the next sections, we give a unified treatment from [8] for the determination of the constants $\gamma_{m,n}$ when $m \geq 0$ and $n \geq 1$. We are also interested in those functions f in $C_r(I)\backslash\pi^r_{m,n}$ which satisfy

$$\gamma_{m,n} < \frac{E^c_{m,n}(f)}{E^r_{m,n}(f)} < \gamma_{m,n} + \varepsilon \qquad (m \geq 0; n \geq 1), \tag{1.20}$$

for some given $\varepsilon > 0$. For such an f satisfying (1.20), if g in $\pi^c_{m,n}$ and if h in $\pi^r_{m,n}$ are such that

$$\|f - g\|_{L_\infty(I)} = E^c_{m,n}(f), \text{ and } \|f - h\|_{L_\infty(I)} = E^r_{m,n}(f),$$

we are likewise intrigued by the behavior of the associated errors:

$$f(x) - g(x) \text{ and } f(x) - h(x) \qquad (x \in I).$$

This will generate a number of interesting graphs.

§2. Upper Bounds for $\gamma_{m,n}$

As a means for obtaining general upper bounds for $\gamma_{m,n}$ for $n \geq 1$, we next establish

Proposition 2. ([8]). *Given any pair (m,n) of nonnegative integers with $n \geq 1$, assume that*

$$g \text{ is in } \pi^c_{m,n} \backslash \pi^r_{m,n}, \text{ with Re } g \text{ in } C_r(I), \tag{2.1}$$

and assume that S in $C_r(I)$ is such that there are $L \geq m+2$ distinct points $\{x_j\}_{j=1}^L$ with $-1 \leq x_1 < x_2 < \cdots < x_L \leq 1$, for which (with fixed $\lambda = 1$ or $\lambda = -1$)

$$\lambda(-1)^j \left[S(x_j) + \text{Re } g(x_j)\right] > 0 \qquad (j = 1, 2, \cdots, L). \tag{2.2}$$

Then,

$$\gamma_{m,n} \leq \|S - i \text{ Im } g\|_{L_\infty(I)}/M, \tag{2.3}$$

where

$$M := \min_{1 \leq j \leq L} |S(x_j) + \text{Re } g(x_j)|. \tag{2.4}$$

Proof: Set $f := S + \text{Re } g$, so that f is an element of $C_r(I)$. If the best approximation to f in $\pi^r_{m,n}$ were the identically zero function, then the convention in (1.5) requires the existence of an alternation set in I of length $\ell \geq m+2$. Now, the hypothesis of (2.2) gives that f oscillates in sign in $L \geq m+2$ distinct points in I, and from this, using a result of de la Vallée-Poussin (cf. [5, p. 162]), one has the following lower bound for $E^r_{m,n}(f)$:

$$E^r_{m,n}(f) \geq M,$$

where M is defined in (2.4). On the other hand, as

$$E^c_{m,n}(f) \le \|f - g\|_{L_\infty(I)} = \|S - i\ \mathrm{Im}\ g\|_{L_\infty(I)},$$

we have from the definition of $\gamma_{m,n}$ in (1.11) that (2.3) is valid. ■

It turns out that Trefethen and Gutknecht [11], Levin [3], and Ruttan and Varga [7] each, in essence, applied a variant of Proposition 2, with appropriate choices of $g(x)$ and $S(x)$, to determine upper bounds for $\gamma_{m,n}$. Their constructions of particular complex rational functions, which lead to sharp upper bounds for $\gamma_{m,n}$, are described in the next paragraphs.

We begin with the clever construction of a complex rational function, by Trefethen and Gutknecht [11], for establishing that $\gamma_{m,m+3} = 0$ $(m = 0, 1, \cdots)$. For any fixed nonnegative integer m and for any ε with $0 < \varepsilon < 1$ (and with $0 < \varepsilon < 1/(2m-1)$ if $m \ge 1$), consider the following complex rational function in $\pi^c_{m,m+3} \backslash \pi^r_{m,m+3}$, defined by

$$h_{m,\varepsilon}(x) := \frac{\varepsilon \prod_{j=1}^{m} [-1 + (2j-1)\varepsilon - x]}{[x + 1 + \varepsilon]^{m+1}(i\sqrt{\varepsilon} - x)(1 + \varepsilon - x)}, \tag{2.5}$$

(where, as usual, $\prod_{j=1}^{m} := 1$ if $m = 0$). It follows from (2.5) that

$$\mathrm{Re}\ h_{m,\varepsilon}(x) = \frac{-\varepsilon x \prod_{j=1}^{m} [-1 + (2j-1)\varepsilon - x]}{[x + 1 + \varepsilon]^{m+1}(\varepsilon + x^2)(1 + \varepsilon - x)} \in \pi^r_{m+1,m+4}, \tag{2.6}$$

and

$$\mathrm{Im}\ h_{m,\varepsilon}(x) = \frac{-\varepsilon\sqrt{\varepsilon} \prod_{j=1}^{m} [-1 + (2j-1)\varepsilon - x]}{[x + 1 + \varepsilon]^{m+1}(\varepsilon + x^2)(1 + \varepsilon - x)} \in \pi^r_{m,m+4}. \tag{2.7}$$

It is evident from (2.6) that Re $h_{m,\varepsilon}(x)$ has $m+1$ distinct zeros in $(-1, 0]$, with m closely packed zeros to the right of -1, plus an additional zero at the origin. Next, define the $m+2$ distinct points, namely $\{x_j(\varepsilon) := -1 + 2j\varepsilon\}_{j=0}^{m}$ and $x_{m+1}(\varepsilon) := 1$, which satisfy

$$-1 = x_0(\varepsilon) < x_1(\varepsilon) < \cdots < x_{m+1}(\varepsilon) = 1.$$

These $m+2$ points $\{x_j(\varepsilon)\}_{j=0}^{m+1}$ *interlace* the $m+1$ zeros of Re $h_{m,\varepsilon}(x)$, and it can be verified that Re $h_{m,\varepsilon}(x)$ *oscillates* in sign in these points. Moreover, the pole of order $m+1$ at $-1-\varepsilon$ and the pole of order 1 at $1+\varepsilon$ of Re $h_{m,\varepsilon}(x)$ contribute in making these oscillations roughly of the same modulus, i.e., there is a constant c, dependent on m but independent of ε, such that (cf. [11])

$$(-1)^j \mathrm{Re}\ h_{m,\varepsilon}(x_j(\varepsilon)) \ge c\|\mathrm{Im}\ h_{m,\varepsilon}\|_{L_\infty(I)}/\sqrt{\varepsilon} \quad (j = 0, 1, \cdots, m+1). \tag{2.8}$$

With the rational function $h_{m,\varepsilon}(x)$ of (2.5) and with Proposition 2, we have

Theorem 3. (Trefethen and Gutknecht [11]). *For any nonnegative integer m and for any integer $n \geq m+3$,*

$$\gamma_{m,n} = 0 \qquad (n \geq m+3;\ m = 0, 1, \cdots). \tag{2.9}$$

Proof: We first show that $\gamma_{m,m+3} = 0$ for every nonnegative integer m. With $n := m+3$, set $S :\equiv 0, L := m+2$, and $g := h_{m,\varepsilon} \in \pi^c_{m,m+3}$, and apply Proposition 2. The discussion above shows that (2.2) of Proposition 2 is valid, and from (2.8) we see (cf. (2.4)) that

$$M := \min_{0 \leq j \leq m+1} |\text{Re } h_{m,\varepsilon}(x_j(\varepsilon))| \geq c \|\text{Im } h_{m,\varepsilon}\|_{L_\infty(I)}/\sqrt{\varepsilon}.$$

It thus follows from (2.3) of Proposition 2 that

$$\gamma_{m,m+3} \leq \sqrt{\varepsilon}/c. \tag{2.10}$$

But since c is independent of ε in (2.10) and since ε can be taken arbitrarily small, then

$$\gamma_{m,m+3} = 0. \tag{2.11}$$

Moreover, as $\pi^c_{m,n} \supset \pi^c_{m,m+3}$ for every $n \geq m+3$, the same function $h_{m,\varepsilon}$ can be used to deduce that

$$\gamma_{m,n} = 0 \quad (n \geq m+3; m = 0, 1, \cdots),$$

the desired result of (2.9). ■

In the above construction, $f(x) :=$ Re $h_{m,\varepsilon}(x)$ of (2.6) is the function in $C_r(I)$ which is simultaneously approximated by the identically zero function of $\pi^r_{m,m+3}$ and by $h_{m,\varepsilon}(x)$ of $\pi^c_{m,m+3}$. Because of the oscillations of f in the $m+2$ distinct points of I of (2.8), the identically zero function in $\pi^r_{m,m+3}$ is a near best approximation to f from $\pi^r_{m,m+3}$ on I, and

$$E^r_{m,m+3}f) \doteq \|f\|_{L_\infty(I)} = \|\text{Re } h_{m,\varepsilon}\|_{L_\infty(I)}.$$

Thus, in Figure 1 we graph the function $f(x) :=$ Re $h_{5,\varepsilon}(x)$ for x in I, with $\varepsilon = 0.1$, to show its 7 oscillations in I. As some of these oscillations are very tiny, there is a 15-fold magnification (in y) given in Figure 2, of the dotted rectangular portion of Figure 1, which show these tiny oscillations more clearly. Next, the complex error in this case is, from our definitions, just $f(x) - g(x) = -i\ Im\ h_{5,\varepsilon}(x)$, and its path, as x increases from -1 to 1, in the complex plane is *uninteresting* (and omitted), since this path is just motions confined to a symmetric segment of the imaginary axis.

Concerning the result (2.9) of Theorem 3, it was pointed out by E.B. Saff that the *existence* of arbitrarily small numbers $\gamma_{m,n}$ was already implied in 1934 by a result of Walsh [13, Theorem IV], although this connection of Walsh's result to the numbers $\gamma_{m,n}$ was not previously noticed. Specifically, Walsh showed, for each fixed nonnegative integer m, that the functions $\bigcup_{n=0}^{\infty}\pi_{m,n}^{c}$ are *dense* in $C_c(I)$, the space of all continuous complex-valued functions on I. Thus, on choosing any f in $C_r(I)$, a subset of $C_c(I)$, this density implies that

$$\lim_{n\to\infty} E_{m,n}^{c}(f) = 0 \qquad (m = 0, 1, \cdots). \tag{2.12}$$

On the other hand, let $T_j(x)$ $(j \geq 0)$ denote the j-th Chebyshev polynomial (of the first kind). On specifically choosing $f(x) := T_{m+1}(x)$ in $C_r(I)$ and $r_{m,n} \equiv 0$ in $\pi_{m,n}^{r}$, then $f - 0$ has an alternation set in I of length exactly $m+2$, and it follows from the known fact that $\|T_{m+1}\|_{L_\infty(I)} = 1$ that (cf. (1.4))

$$E_{m,n}^{r}(f) = 1 \qquad (n = 0, 1, \cdots). \tag{2.13}$$

Obviously, combining (2.12) and (2.13) then gives

$$\lim_{n\to\infty} \gamma_{m,n} = 0 \qquad (m = 0, 1, \cdots). \tag{2.14}$$

We see however that (2.9) of Theorem 3 is a much more precise form of (2.14).

We next present the construction of Levin [3] for obtaining upper bounds for $\gamma_{n+2k,n}$ where $n \geq 1$ and $k \geq 1$. (We show later in §3 that these upper bounds are *sharp*.)

For $\varepsilon > 0$ sufficiently small, consider the complex rational function defined by

$$h_{2k,n,\varepsilon}(x) := T_{2k}(x)\left(\frac{x - i\varepsilon}{x + i\varepsilon}\right)^{n} \in \pi_{2k+n,n}^{c} \backslash \pi_{2k+n,n}^{r} \quad (n \geq 1), \tag{2.15}$$

where $T_{2k}(x)$ again denotes the $2k$-th Chebyshev polynomial (of the first kind) for any positive integer k. Then,

$$\text{Re } h_{2k,n,\varepsilon}(x) = T_{2k}(x) \text{ Re}\left\{\left(\frac{x - i\varepsilon}{x + i\varepsilon}\right)^{n}\right\} \in \pi_{2n+2k,2n}^{r}. \tag{2.16}$$

For each real x, write $x + i\varepsilon = \rho e^{i\theta}$, so that $[(x - i\varepsilon)/(x + i\varepsilon)]^n = e^{-2ni\theta}$. This gives that

$$\text{Re } \{[(x - i\varepsilon)/(x + i\varepsilon)]^n\} = \cos(2n\theta), \tag{2.17}$$

from which it follows that Re $\{[x - i\varepsilon)/(x + i\varepsilon)]^n\}$ has $2n$ distinct zeros, namely $\{x_j(\varepsilon) := -\varepsilon \cot\left[\frac{(2j+1)\pi}{4n}\right]\}_{j=0}^{2n-1}$, which are all clustered in an ε-neighborhood of the origin in I. Next, the $2k$ distinct zeros in I of $T_{2k}(x)$ are given by $\{y_i := \cos\left[\frac{(2i+1)\pi}{4k}\right]_{i=0}^{2k-1}$. As $T_{2k}(0) = (-1)^k \neq 0$, these zeros of $T_{2k}(x)$ are *disjoint* from the zeros $\{x_j(\varepsilon)\}_{j=0}^{2n-1}$ for all $\varepsilon > 0$ sufficiently small, and hence

$$\{x_j(\varepsilon)\}_{j=0}^{2n-1} \bigcup \{y_i\}_{i=0}^{2k-1}$$

are the $2n+2k$ (distinct) zeros of Re $h_{2k,n,\varepsilon}(x)$ in I, for all $\varepsilon > 0$ sufficiently small. From this, it is not difficult to verify from (2.16) and (2.17) that there are $2n+2k+1$ distinct points $\{w_j(\varepsilon)\}_{j=0}^{2n+2k}$ of I which *interlace* these zeros and for which (cf. [3]) Re $h_{2k,n,\varepsilon}(x)$ takes on the values $1-o(1)$, with alternating sign, in these points; *i.e.*,

$$(-1)^j \text{ Re } h_{2k,n,\varepsilon}(w_j(\varepsilon)) \geq 1 - o(1) \quad (j = 0, 1, \cdots, 2n+2k), \tag{2.18}$$

as $\varepsilon \to 0$.

With the rational function $h_{2k,n,\varepsilon}$ of (2.15) and with Proposition 2, we establish the special case $m = 2k+n$ of

Theorem 4. (Levin [3]). *For any pair* (m,n) *of nonnegative integers with* $m+1 \geq n \geq 1$,

$$\gamma_{m,n} \leq \frac{1}{2} \quad (m+1 \geq n \geq 1). \tag{2.19}$$

Proof: We show that $\gamma_{2k+n,n} \leq \frac{1}{2}$. With $m := n+2k$ where $k \geq 1$ and $n \geq 1$, set $S(x) :=$ Re $h_{2k,n,\varepsilon}(x)$, $L := 2n+2k+1$, and $g(x) := h_{2k,n,\varepsilon}(x) \in \pi^c_{2k+n,n}$, and apply Proposition 2. First note that since $n \geq 1$ by hypothesis, then $L \geq m+2 = n+2k+2$. The discussion above shows that (2.2) of Proposition 2 is again valid, and from (2.18), we see (cf. (2.4)) that

$$M := \min_{0 \leq j \leq 2n+2k} |2 \text{ Re } h_{2k,n,\varepsilon}(w_j(\varepsilon))| \geq 2 - o(1), \text{ as } \varepsilon \to 0.$$

Next, a calculation shows that

$$\begin{aligned} \|S(x) - i \text{ Im } g(x)\|_{L_\infty(I)} &= \|\text{Re } h_{2k,n,\varepsilon}(x) - i \text{ Im } h_{2k,n,\varepsilon}(x)\|_{L_\infty(I)} \\ &= \|\overline{h_{2k,n,\varepsilon}(x)}\|_{L_\infty(I)} = 1, \end{aligned}$$

since, from (2.16), $\|(\frac{x+i\varepsilon}{x-i\varepsilon})^n\|_{L_\infty(I)} = 1 = \|T_{2k}\|_{L_\infty(I)}$ and $h_{2k,n,\varepsilon}(0) = 1$. Applying (2.3) of Proposition 2 then directly gives

$$\gamma_{2k+n,n} \leq \frac{1}{2-o(1)} \quad (\varepsilon \to 0). \tag{2.20}$$

Letting $\varepsilon \to 0$ in the above expression results in

$$\gamma_{2k+n,n} \leq \frac{1}{2} \quad (k = 1, 2, \cdots; n \geq 1),$$

the special case $m = 2k+n$ of (2.19). The construction for the remaining cases is similar (cf. [3]). ■

In the previous construction, $f(x) = 2\text{Re}\ h_{2k,n,\varepsilon}(x)$ is a function in $C_r(I)$ which is simultaneously approximated by the identically zero function in $\pi^r_{2k+n,n}$ and by $h_{2k,n,\varepsilon}(x)$ in $\pi^c_{2k+n,n}$. In Figure 3, we graph the function $2\ \text{Re}\ h_{2,5,\varepsilon}(x)$ for x in I with $\varepsilon = 0.1$, to show its 9 points of near equioscillation in I. Next, the complex error in this case is just

$$f(x) - g(x) = \overline{h_{2,5,\varepsilon}(x)} \qquad (x \in I),$$

and its path, as x increases from -1 to 1, in the complex plane is given in Figure 4. Note the interesting *near-circularity* of this path!

Finally, we give the construction of Ruttan and Varga [7] for determining the upper bound $\gamma_{m,m+2} \leq 1/3$ for any $m = 0, 1, \cdots$. For any ε satisfying $0 < \varepsilon < 1/(m+1)$, let m be any *fixed* nonnegative *even* integer and consider the following functions

$$\ell_j(z) := \ell_j(z; \varepsilon, m) := \frac{-\frac{2\varepsilon i}{3}(-1)^j}{z - 1 + \frac{2j}{m+1} - \varepsilon i} \qquad (j = 0, 1, \cdots, m+1). \tag{2.21}$$

It is evident from (2.21) that

$$\ell_j\left(1 - \frac{2j}{m+1}\right) = \frac{2}{3}(-1)^j, \text{ and } \ell_j\left(1 - \frac{2j}{m+1} \pm \varepsilon\right) = \frac{(1 \mp i)(-1)^j}{3}, \tag{2.22}$$

for all $j = 0, 1, \cdots m+1$. Since each $\ell_j(z)$ is a Möbius transformation, each ℓ_j maps the real axis $\mathbb{R}$ onto some generalized circle in the complex plane. As $\ell_j(\infty) = 0$, this generalized circle necessarily passes through the origin. Moreover, as the pole of $\ell_j(z)$, namely $1 - (2j)/(m+1) + \varepsilon i$, when reflected in $\mathbb{R}$, is the point $w_j := 1 - (2j)/(m+1) - \varepsilon i$, it follows from (2.21) that

$$\ell_j(w_j) = \frac{1}{3}(-1)^j \qquad (j = 0, 1, \cdots, m+1). \tag{2.23}$$

Thus, from the symmetry principle for Möbius transformations, the image of $\mathbb{R}$ under $w = \ell_j(z)$ is the circle with center $\frac{1}{3}(-1)^j$ and radius 1/3 (since this circle passes through the origin). It is then geometrically clear that

$$\|\text{Re}\ \ell_j\|_{L_\infty(\mathbb{R})} = \|\ell_j\|_{L_\infty(\mathbb{R})} = \frac{2}{3}, \text{ and } \|\text{Im}\ \ell_j\|_{L_\infty(RR)} = \frac{1}{3}, \quad \text{for } j = 0, 1, \cdots, m+1. \tag{2.24}$$

To extend the results of (2.24), define the real intervals $I_k(m)$ by

$$I_k(m) := \left[1 - \frac{2k+1}{m+1}, 1 - \frac{2k-1}{m+1}\right] \bigcap I \quad (k = 0, 1, \cdots, m+1), \tag{2.25}$$

so that these intervals cover $I := [-1, +1]$, i.e.,

$$\bigcup_{k=0}^{m+1} I_k(m) = I.$$

From the definitions of $\ell_j(x)$ and $I_k(m)$, it follows (as m is fixed) that

$$\|\ell_j\|_{L_\infty(I_k(m))} = O(\varepsilon) \text{ for any } k \neq j \qquad (\text{as } \varepsilon \to 0), \tag{2.26}$$

and from (2.22) that

$$\|\mathrm{Re}\ \ell_j\|_{L_\infty(I_j(m))} = \frac{2}{3}, \text{ and } \|\mathrm{Im}\ \ell_j\|_{L_\infty(I_j(m))} = \frac{1}{3} \quad (j = 0, 1, \cdots, m+1). \tag{2.27}$$

Next, consider the complex rational function $g(x)$ defined by

$$h(x) = h(x; \varepsilon, m) := \sum_{j=0}^{m+1} \ell_j(x). \tag{2.28}$$

On rationalizing $h(x)$, we find that

$$h(x) = \frac{\frac{-2\varepsilon i}{3} \sum_{j=0}^{m+1} (-1)^j \prod_{\substack{k=0 \\ k \neq j}}^{m+1} \left\{x - 1 + \frac{2k}{m+1} - \varepsilon i\right\}}{\prod_{k=0}^{m+1} \left\{x - 1 + \frac{2k}{m+1} - \varepsilon i\right\}}, \tag{2.29}$$

so that h is at least an element of $\pi^c_{m+1,m+2}$. However,the numerator of $h(x)$ of (2.29) is

$$\frac{-2\varepsilon i}{3} \left\{ x^{m+1} \sum_{j=0}^{m+1} (-1)^j + \text{ lower terms in } x^s (0 \leq s \leq m) \right\}$$

But since m is by hypothesis even, then $\sum_{j=0}^{m+1} (-1)^j = 0$, which implies that h is an element of $\pi^c_{m,m+2}$. More precisely, it can be verified that the coefficient of x^m in the numerator of $h(x)$ of (2.29) is

$$-\frac{2(m+2)\varepsilon i}{3(m+1)} \neq 0,$$

so that h is an element of $\pi^c_{m,m+2}$, but not an element of $\pi^c_{s,m+2}$ for any $s < m$. It is interesting to mention here that (2.28) is just the *partial fraction decomposition* of $h(x)$.

With the rational function h of (2.29) and with Proposition 2, we have

Theorem 5. **([7]).** *For each nonnegative integer m,*

$$\gamma_{m,m+2} \leq \frac{1}{3}. \tag{2.30}$$

Proof: For a fixed nonnegative *even* integer m, consider the real continuous function $s(u)$ on RR, defined by

$$s(u) := \begin{cases} \frac{1-u^2}{1+u^2} & , \quad -1 \leq u \leq +1, \\ 0 & , \quad \text{otherwise}, \end{cases} \tag{2.31}$$

so that $s(0) = 1, s(\pm 1) = 0$, and $0 < s(u) < 1$ for $0 < |u| < 1$. Recalling that $0 < \varepsilon < 1/(m+1)$, set

$$S_{\varepsilon,m}(x) := \frac{1}{3}\sum_{j=0}^{m+1}(-1)^j s\left(\frac{x-1+\frac{2j}{m+1}}{\varepsilon^2}\right) \qquad (x \in RR). \tag{2.32}$$

It follows from (2.32) that $S_{\varepsilon,m}(x)$ is a real continuous function on RR with

$$S_{\varepsilon,m}(1-\frac{2j}{m+1}) = \frac{(-1)^j}{43}, \text{ and } S_{\varepsilon,m}(1-\frac{2j}{m+1} \pm \varepsilon^2) = 0 (j = 0, 1, \cdots, m+1). \tag{2.33}$$

Geometrically, we note that $S_{\varepsilon,m}(x)$ has $m+2$ alternating *spikes* on I, with one spike for each of the disjoint intervals $[1 - \frac{2j}{m+1} - \varepsilon^2, 1 - \frac{2j}{m+1} + \varepsilon^2]$ $(j = 0, 1, \cdots, m+1)$.

With the above definitions, set $n := m+2$, $L := m+2$, $S(x) := S_{\varepsilon,m}(x)$ of (2.32), and $g(x) := h(x; \varepsilon, m)$ of (2.28) and apply Proposition 2. With these definitions, we obtain from (2.22), (2.26) - (2.28) and (2.33) that

$$(-1)^j[S_{\varepsilon,m}(1-\frac{2j}{m+1}) + \text{ Re } g(1-\frac{2j}{m+1})] = 1 + O(\varepsilon) \qquad (j = 0, 1, \cdots, m+1), \tag{2.34}$$

as $\varepsilon \to 0$, so that (cf. (2.4))

$$M := \min_{0 \leq j \leq m+1} |S_{\varepsilon,m}(1-\frac{2j}{m+1}) + \text{ Re } g(1-\frac{2j}{m+1})| = 1 + O(\varepsilon). \tag{2.35}$$

On the other hand, consider $S_{\varepsilon,m}(x) - i$ Im $g(x)$ on I. For the particular interval $I_k(m)$ of (2.25), it follows from (2.26) that

$$S_{\varepsilon,m}(x) - i \text{ Im } g(x) = S_{\varepsilon,m}(x) - i \text{ Im } \ell_k(x) + O(\varepsilon) \qquad (x \in I_k(m)).$$

Moreover, a short calculation shows from (2.27) and (2.32) that

$$\|S_{\varepsilon,m}(x) - i \text{ Im } \ell_k(x)\|_{L_\infty(I_k(m))} = \frac{1}{3} + O(\varepsilon) \qquad (k = 0, 1, \cdots, m+1),$$

so that with (2.26),

$$\|S_{\varepsilon,m}(x) - i \text{ Im } g(x)\|_{L_\infty(I)} = \frac{1}{3} + O(\varepsilon) \qquad (\text{as } \varepsilon \to 0). \tag{2.36}$$

Thus, it follows from (2.3) of Proposition 2 that

$$\gamma_{m,m+2} \le \frac{1}{3} + O(\varepsilon) \qquad (\text{as } \varepsilon \to 0),$$

and on letting $\varepsilon \to 0$, we have

$$\gamma_{m,m+2} \le \frac{1}{3},$$

the desired result of (2.30) when m is even.

For the case when m is odd, one simply modifies the definition of (2.21) to

$$\ell_j(z) = \ell_j(z;\varepsilon,m) := \frac{\frac{-2\varepsilon i}{3}\mu_j(-1)^j}{z - 1 + \frac{2j}{m+1} - \varepsilon\mu_j i} \quad (j = 0, 1, \cdots, m+1), \tag{2.37}$$

where the numbers $\{\mu_j\}_{j=0}^{m+1}$ are any $m+2$ fixed positive numbers satisfying $0 < \mu_j < 1$, and

$$\sum_{j=0}^{m+1}(-1)^j\mu_j = 0 \text{ and } \sum_{j=0}^{m+1} j(-1)^j\mu_j \ne 0. \tag{2.38}$$

An application of Proposition 2 again yields the desired result. ∎

In the above construction, $f_{\varepsilon,2m}(x) := S_{\varepsilon,2m}(x) + \text{ Re } h(x;\varepsilon,2m)$ is the function in $C_r(I)$ which is simultaneously approximated by the identically zero function in $\pi^r_{2m,2m+2}$ and by $h(x;\varepsilon,2m)$ in $\pi^c_{2m,2m+2}$. In Figure 5, we graph the function $S_{\varepsilon,2}(x) + \text{ Re } h(x;\varepsilon,2)$ for x in I with $\varepsilon = 0.1$, to show its 4 spikes and its 4 points of near equioscillation in I. Next, the complex error in this case is just $S_{\varepsilon,2}(x) - i\ Im\ h(x;\varepsilon,2)$, and its path, as x increases from -1 to 1, in the complex plane is given in Figure 6. This path is confined, from (2.36), to the disk $\{z \in \mathbb{C} : |z| \le \frac{1}{3} + O(\varepsilon)\}$, but this path does not exhibit near-circularity.

The constructions and the figures of this section collectively show how *differently* the functions in $C_r(I)$ must be chosen in order to obtain sharp upper bounds for $\gamma_{m,n}$ in the three cases of (2.9), (2.19), and (2.30).

§3. Lower Bounds for $\gamma_{m,n}$

To determine lower bounds for the $\gamma_{m,n}$, we describe the following two results of Ruttan and Varga [8, Theorems 4 and 5].

For a given real or complex polynomial p, let ∂p denote the *exact* degree of p. If $R = p/q$ is continuous on I where p and q are real polynomials, it is evident that $Re\ R = R$ can have at most ∂p sign changes in I, since each sign change of R corresponds to a zero of p. But, if $R = p/q$ is a continuous *complex-valued* function on I, the number of possible sign changes of $Re\ R$ on I depends not only on ∂p and ∂q, but also on the *magnitude* of the oscillations of $Re\ R$ in I. This is discussed in the following result of [8]. For additional notation, $\lfloor x \rfloor$ denotes the greatest integers N satisfying $N \leq x$.

Theorem 6. ([8]). *Let $\phi = p/q$ be a complex rational function, with no poles in I, which satisfies $\|\text{Im}\ \phi\|_{L_\infty(I)} \leq 1$. Assume that there exist real numbers $d > 0$ and $\{x_j\}_{j=1}^{L}$, with $-1 \leq x_1 < x_2 < \cdots < x_L \leq 1$, for which (with fixed $\lambda = 1$ or $\lambda = -1$)*

$$\lambda(-1)^j\ \text{Re}\ \phi(x_j) \geq d \qquad (j = 1, 2, \cdots, L). \tag{3.1}$$

If $\partial q \leq \partial p$ and if $d \geq 1$, then

$$L \leq \partial p + 1. \tag{3.2}$$

Similarly, if $\partial q > \partial p$, then

$$L \leq \partial q, \quad whenever \quad d \geq 1, \tag{3.3}$$

and

$$L \leq \lfloor \frac{\partial p + \partial q + 1}{2} \rfloor, \quad whenever \quad d \geq 2. \tag{3.4}$$

Proof: We shall establish (3.2) using a geometrical argument, suggested by the work of Levin [3]. Assuming $\partial q \leq \partial p$ and $d \geq 1$, let B denote the closed rectangle in the complex plane with vertices $\pm\ d \pm i$, so that the circle $C := \{z : |z| = 1\}$ is a subset of B, as indicated in Figure 7. Condition 3.1 and the hypothesis $\|Im\ \phi\|_L \leq 1$ imply that the curve (in the extended complex plane) $\Gamma_1 := \{z = \phi(x) : x \in \mathbb{R}\}$ intersects the vertical sides of B, and hence the circle C, in $2(L-1)$ points as x increases from x_1 to x_L. (Here, points where Γ_1 is tangent to C are counted twice.) If $\phi(t)$, for t in I, is such an intersection of the Γ_1 and C, then

$$|\phi(t)|^2 = \left|\frac{p(t)}{q(t)}\right|^2 = 1,$$

so that t is a zero of the polynomial real polynomial

$$P(x) := |p(x)|^2 - |q(x)|^2.$$

The above discussion shows that there are at least $2(L-1)$ zeros of $P(x)$ in I. Since $\partial p \geq \partial q$, then $\partial P \leq 2\partial p$. Thus, $2(L-1) \leq \partial P \leq 2\partial p$, which gives $L \leq \partial p + 1$, the desired result of (3.2). The proof of the remainder of Theorem 5 is similarly geometrical in nature, but involves many separate cases (cf. [8]). ■

We remark, as shown in [8], that the results of (3.2) - (3.4) of Theorem 6 are *sharp*, in the following sense: *i)* there exist complex rational functions, satisfying the appropriate hypotheses, for which the upper bounds for L given in (3.2) - (3.4) are attained; *i.e.*, equality can hold in (3.2) - (3.4); and *ii)* for any positive d with $d < 1 (< 2)$, there exist rational functions satisfying all hypotheses of Theorem 6 except the hypotheses on d in (3.3) ((3.4)), for which the bound on L in (3.3) ((3.4)) is *exceeded.*

With Theorem 6, the following lower bounds for $\gamma_{m,n}$ can be determined.

Theorem 7. ([8]). *Let (m,n) be a pair of nonnegative integers with $n \geq 1$, let $f \in C^r(I)\backslash\pi^r_{m,n}$, and let $r_{m,n}$ and $R_{m,n}$ be respectively rational functions of best uniform approximation of f from $\pi^r_{m,n}$ and $\pi^c_{m,n}$ on I. Then,*

$$\frac{\|f - R_{m,n}\|_{L_\infty(I)}}{\|f - r_{m,n}\|_{L_\infty(I)}} > \frac{1}{2} \quad \text{if} \quad m+1 \geq n \geq 1, \tag{3.5}$$

and

$$\frac{\|f - R_{m,n}\|_{L_\infty(I)}}{\|f - r_{m,n}\|_{L_\infty(I)}} > \frac{1}{3} \quad \text{if} \quad m+2 \geq n \geq 1. \tag{3.5}$$

Consequently,

$$\gamma_{m,n} = \frac{1}{2} \quad \text{if} \quad m+1 \geq n \geq 1, \tag{3.7}$$

and

$$\gamma_{m,m+2} = 1/3. \tag{3.8}$$

Proof: Let $s := \|f - R_{m,n}\|_{L_\infty(I)} / \|f - r_{m,n}\|_{L_\infty(I)}$, so that $0 \leq s \leq 1$, and set $e := f - r_{m,n}$, $R_{m,n} := p_1/q_1$, and $r_{m,n} := p_2/q_2$, where the pairs (p_1, q_1) and (p_2, q_2) are assumed to contain no common factors. Since $f \notin \pi^r_{m,n}$, we may assume, upon multiplication by a suitable nonzero constant, that $\|e\|_{L_\infty(I)} = 1$, so that

$$s = \|f - R_{m,n}\|_{L_\infty(I)}. \tag{3.9}$$

Since $r_{m,n}$ is the best uniform approximant of f from $\pi^r_{m,n}$ on I, there exist (cf. (1.5)) at least

$$L \geq 2 + \max[m + \partial q_2; n + \partial p_2] = m + m + 2 - \min[m - \partial p_2; n - \partial q_2] \quad (3.10)$$

distinct points $\{x_j\}_{j=1}^L$, with $-1 \leq x_1 < x_2 < \cdots < x_L \leq 1$, such that $e(x_j) = (-1)^j\lambda$ for all $1 \leq j \leq L$ (with fixed $\lambda = 1$ or $\lambda = -1$). Again, upon multiplication by -1 if necessary, we may assume that $\lambda = 1$, so that $e(x_1) = -1$.

With these normalizations, then

$$s = \|f - R_{m,n}\|_{L_\infty(I)} \geq |f(x_j) - R_{m,n}(x_j)| = |(-1)^j + r_{m,n}(x_j) - R_{m,n}(x_j)|$$

for all $1 \leq j \leq L$, which implies, on taking real parts, that

$$(-1)^j \{Re\ R_{m,n}(x_j) - r_{m,n}(x_j)\} \geq 1 - s \qquad (j = 1, 2, \cdots, L). \qquad (3.11)$$

Next, set $\phi(x) := (R_{m,n}(x) - r_{m,n}(x))/s = p(x)/q(x)$, where p and q are complex polynomials with no common factors. From (3.11), it follows that

$$(-1)^j\ Re\ \phi(x_j) \geq (1 - s)/s =: d \qquad (j = 1, 2, \cdots L), \qquad (3.12)$$

and since (cf. (3.9)) $s = \|f - R_{m,n}\|_{L_\infty(I)} \geq \|Im\ R_{m,n}\|_{L_\infty(I)}$, we similarly have

$$\|Im\ \phi\|_{L_\infty(I)} \leq 1. \qquad (3.13)$$

Consider the sought result of (3.5) of Theorem 7, which, with our reductions and definitions, is the statement that

$$m + 1 \geq n \geq 1 \text{ implies } s > 1/2,$$

or equivalently, using the contra-positive, that $s \leq 1/2$ implies $m + 1 < n$; *i.e.*,

$$s \leq 1/2 \text{ implies } m + 2 \leq n. \qquad (3.14)$$

To establish (3.14), assume that s satisfies $0 < s \leq 1/2$, which gives from (3.12) that $d \geq 1$. Because all the hypothesis of Theorem 6 are fulfilled, we see from (3.2) and (3.3) of Theorem 6 that

$$L \leq \partial p + 1 \text{ if } \partial q \leq \partial p, \text{ and } L \leq \partial q \text{ if } \partial q > \partial p. \qquad (3.15)$$

Supposing that $\partial q > \partial p$, the second part of (3.15) gives $L \leq \partial q$, which can be expressed from (3.10) and the definitions $\phi = p/q = (R_{m,n} - r_{m,n})/s = (p_1/q_1 - p_2/q_2)/s$, as

$$m + m + 2 - \min[m - \partial p_2; n - \partial q_2] \le \partial q = \partial q_1 + \partial q_2,$$

or equivalently, as

$$\{n - \partial q_1\} + \{(n - \partial q_2) - \min[m - \partial p_2; n - \partial q_2]\} \le n - (m + 2). \qquad (3.16)$$

But as each term in braces of (3.16) is clearly nonnegative, then $n-(m+2) \ge 0$, giving $m + 2 \le n$, the desired result of (3.14), under the assumption that $\partial q > \partial p$. However, as similar arguments show that the assumption $\partial q \le \partial p$ leads to a contradiction, then (3.5) of Theorem 7 is valid. But then, it is evident from (3.5) that

$$\gamma_{m,n} \ge \frac{1}{2} \text{ if } m + 1 \ge n \ge 1,$$

and as the reverse inequality also holds from (2.19), then, as deduced in Levin [3],

$$\gamma_{m,n} = \frac{1}{2} \text{ if } m + 1 \ge n \ge 1,$$

the desired result of (3.7). Similar arguments (cf. [8]) establish (3.6), from which, with (3.20), the desired result of (3.8) follows. ■

We note that the strict inequality in (3.5) gives us the *stronger* result that there is *no* f in $C_r(I)\backslash\pi^r_{m,n}$ for which

$$\frac{\|f - R_{m,n}\|_{L_\infty(I)}}{\|f - r_{m,n}\|_{L_\infty(I)}} = \gamma_{m,n} = \frac{1}{2} \text{ if } m + 1 \ge n \ge 1, \qquad (3.17)$$

where $r_{m,n} \in \pi^r_{m,n}$ and $R_{m,n} \in \pi^c_{m,n}$ are respectively the best uniform approximation from $\pi^r_{m,n}$ and $\pi^c_{m,n}$ of f on I. Thus, $\gamma_{m,n}$ is a *true infimum* (and not a minimum) as defined in (1.11), when $m + 1 \ge n \ge 1$. But the same is also true, for the same reasons, for $\gamma_{m,m+2} = 1/3$ for all $m \ge 0$, and for $\gamma_{m,n} = 0$ for all $n \ge m + 3$. This is *why*, in retrospect, that all the constructions of §5.2 depended on a small parameter $\varepsilon > 0$.

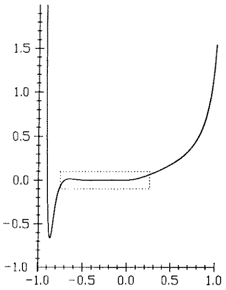

Figure 1. Re $h_{5,\epsilon}(x)$

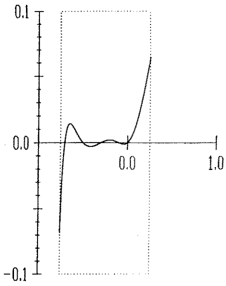

Figure 2. Magnification

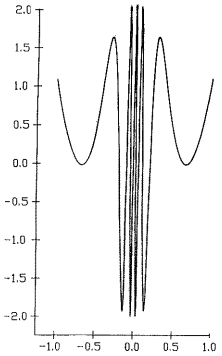

Figure 3. 2Re $h_{2,5,\epsilon}(x)$

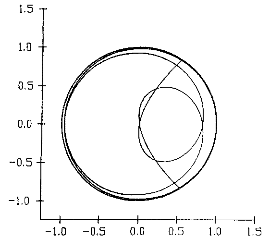

Figure 4. $\overline{h_{2,5,\epsilon}(x)}$

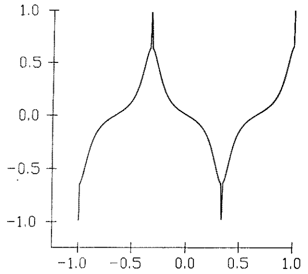

Figure 5. $S_{\epsilon,2}(x) + \mathrm{Re}\ h(x;\epsilon,2)$

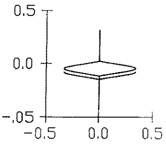

Figure 6. $S_{\epsilon,2} - i\ \mathrm{Im}\ h(x;\epsilon,2)$

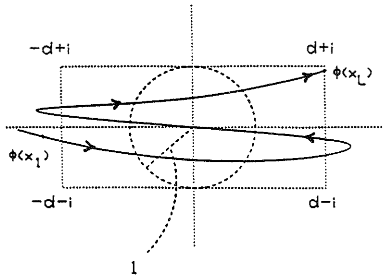

Figure 7. Rectangle B

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