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ASYMPTOTICS FOR THE ZEROS OF THE PARTIAL SUMS OF e^z. I.

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Dedicated to Wolfgang J. Thron on the occasion of his seventieth birthday, August 17, 1988.

ABSTRACT. We continue the work of Szegö and others on describing the convergence of the zeros, $\{z_{k,n}\}_{k=1}^n$, of the normalized partial sum $s_n(nz)$ of e^z where $s_n(z) := \sum_{j=0}^n z^j/j!$, to the Szegö curve D_∞, where

$$D_\infty := \{z \in \mathbf{C} : |ze^{1-z}| = 1 \text{ and } |z| \leq 1\}.$$

It turns out that the convergence rate of these zeros to D_∞ is exactly $O(1/\sqrt{n})$, as $n \to \infty$, whereas this convergence rate improves to $O((\log n)/n)$, as $n \to \infty$, on compact subsets of $\Delta \backslash \{1\}$, where $\Delta := \{z \in \mathbf{C} : |z| \leq 1\}$. We further show that there are new curves, D_n, now depending on n, for which the zeros of $s_n(nz)$ are $O(1/n^2)$ in distance from the curve D_n, on any compact subset of $\Delta \backslash \{1\}$.

Included also are a number of figures which illustrate these results graphically.

1. Introduction. With $s_n(z) := \sum_{j=0}^n z^j/j!$, $n \geq 1$, denoting the familiar partial sum of the exponential function e^z, we investigate here the location of the zeros of the *normalized* partial sums, $s_n(nz)$, and the rate at which these zeros tend to the Szegö curve D_∞, defined by

$$D_\infty := \{z \in \mathbf{C} : |ze^{1-z}| = 1 \text{ and } |z| \leq 1\}. \tag{1.1}$$

By way of review, the well-known Eneström-Kakeya Theorem (cf. Marden [**6**, p. 137, Exercise 2]) asserts that, for any polynomial $p_n(z) =$

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$\sum_{j=0}^{n} a_j z^j$ with $a_j > 0$, $0 \leq j \leq n$, all the zeros of $p_n(z)$ necessarily lie in the annulus

$$\min_{0\leq i<n} \left(\frac{a_i}{a_{i+1}}\right) \leq |z| \leq \max_{0\leq i<n} \left(\frac{a_i}{a_{i+1}}\right).$$

Applying the final inequality above to the partial sum $s_n(z)$ of e^z immediately shows that $s_n(z)$ has all its zeros in $|z| \leq n$, for every $n \geq 1$. A sharpened form of the Eneström-Kakeya Theorem (cf. Anderson, Saff, and Varga [**1**, Corollary 2]) actually shows that all zeros of $s_n(z)$ satisfy $|z| < n$ for any $n > 1$. Thus, if $\{z_{k,n}\}_{k=1}^{n}$ denotes the set of the zeros of the *normalized* partial sum $s_n(nz)$, then these zeros lie in the closed unit disk $\Delta := \{z \in \mathbf{C} : |z| \leq 1\}$ for every $n \geq 1$, and they lie in the *interior* of Δ for every $n > 1$. (This can be seen quite clearly in Figure 1.) Consequently, compactness considerations guarantee that the set of all zeros of all normalized partial sums $\{s_n(nz)\}_{n=1}^{\infty}$ have at least one accumulation point in Δ.

In a remarkable paper in 1924, Szegö [**11**] showed that each accumulation point z (of zeros of the normalized partial sums $\{s_n(nz)\}_{n=1}^{\infty}$) must lie on the curve D_∞ of (1.1), and, conversely, that each point of D_∞ is an accumulation point of zeros of the normalized partial sums $\{s_n(nz)\}_{n=1}^{\infty}$. Subsequently, it was shown by Buckholtz [**2**] that the zeros of $s_n(nz)$ lie *outside* the curve D_∞, for every $n \geq 1$. To indicate these results, we have graphed in Figure 2 the 16 zeros of $s_{16}(16z)$ (these zeros being represented by $\times$'s), along with the Szegö curve D_∞ (cf. (1.1)) and $\partial\Delta$, the boundary of Δ. The same is done in Figure 3 with the 27 zeros of $s_{27}(27z)$.

Figures 1, 2 and 3 tend to indicate that the zeros of $s_n(nz)$ converge in a seemingly "regular" way to the curve D_∞, and these figures also indicate that this convergence seems *slowest* in a neighborhood of the point $z = 1$ of D_∞. As a measure of the rate of convergence of these zeros to the curve D_∞, Buckholtz [**2**] established the result that the zeros $\{z_{k,n}\}_{k=1}^{n}$ of $s_n(nz)$ all lie within a distance of $(2e)/\sqrt{n}$ from D_∞, i.e., with the notation $\operatorname{dist}[\{z_{k,n}\}_{k=1}^{n}; D_\infty] := \max_{1\leq k\leq n}(\operatorname{dist}[z_{k,n}, D_\infty])$, then

$$\operatorname{dist}[\{z_{k,n}\}_{k=1}^{n}; D_\infty] \leq \frac{2e}{\sqrt{n}}, \quad \text{all } n \geq 1. \tag{1.2}$$

This implies, of course, that

$$\overline{\lim_{n\to\infty}} \{\sqrt{n} \cdot \operatorname{dist}[\{z_{k,n}\}_{k=1}^{n}; D_\infty]\} \leq 2e \doteq 5.436\,563. \tag{1.2'}$$

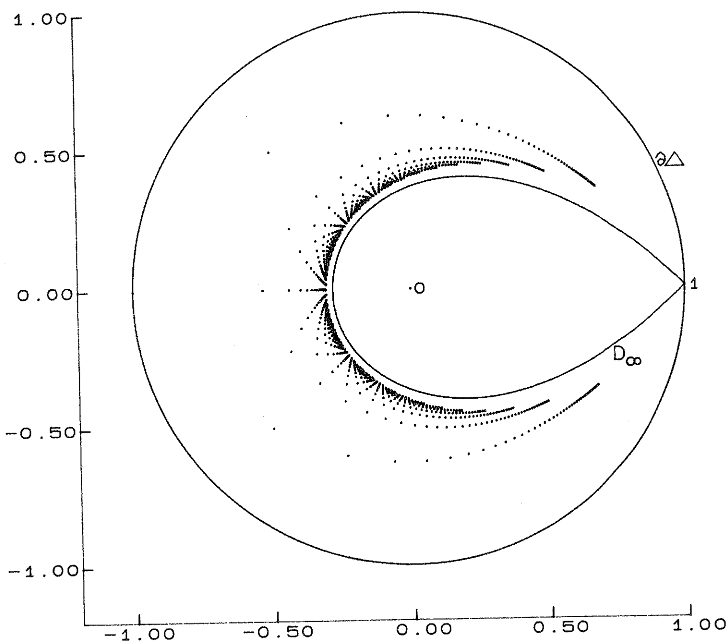

FIGURE 1. Zeros of $\{s_n(nz)\}_{n=1}^{40}$.

Based on results of Newman and Rivlin [**7**] and Saff and Varga [**9**], we will easily deduce here that the exponent, $-1/2$, of n in the upper bound of (1.2) is *best possible.* More precisely, we have below in Proposition 1 (whose proof is given in Section 2) that the limit inferior of the quantity in (1.2′) is *positive.*

Proposition 1. *If* $\{z_{k,n}\}_{k=1}^{n}$ *denotes the zeros of* $s_n(nz)$ *and if* t_1 *denotes* (cf. (2.2)) *the zero of* $\operatorname{erfc}(w) := (2/\sqrt{\pi})\int_w^\infty e^{-t^2}\,dt$ *with*

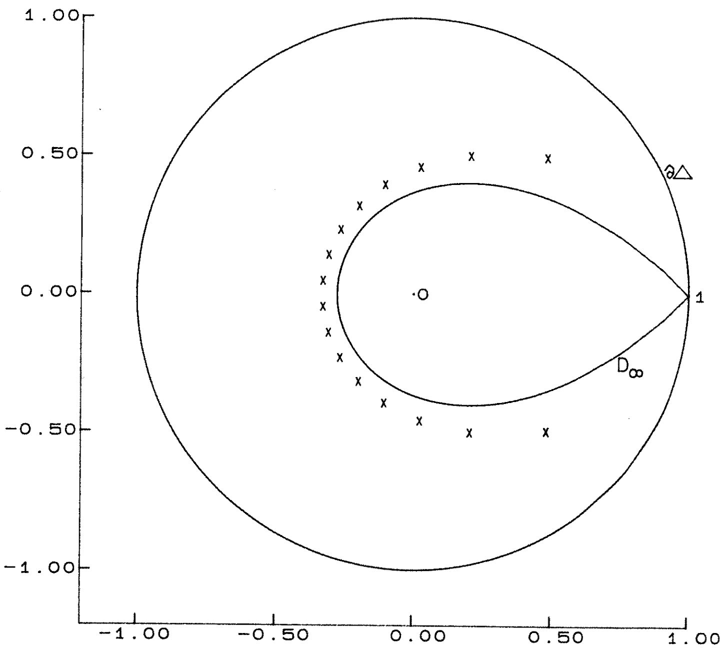

FIGURE 2. Zeros of $s_{16}(16z)$.

$\operatorname{Im} t_1 > 0$ *which is closest to the origin, then*

(1.3) $$\varliminf_{n\to\infty} \{\sqrt{n}\cdot \operatorname{dist}[\{z_{k,n}\}_{k=1}^n; D_\infty]\} \geq (\operatorname{Im} t_1 + \operatorname{Re} t_1) \doteq 0.636\ 657.$$

On examining Figures 2 and 3, we note that there is apparently *faster convergence* (to the curve D_∞) of those zeros of $\{z_{k,n}\}_{k=1}^n$ which stay uniformly away from the point $z = 1$. In fact, if we use the open circle C_δ about the point $z = 1$, i.e.,

(1.4) $$C_\delta := \{z \in \mathbf{C} : |z-1| < \delta\}, \quad 0 < \delta \leq 1,$$

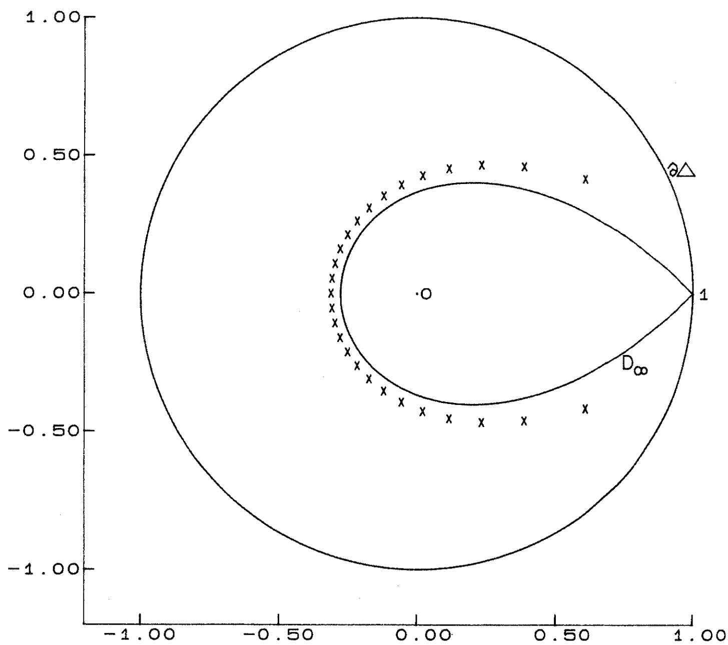

FIGURE 3. Zeros of $s_{27}(27z)$.

to exclude points of $\{z_{k,n}\}_{k=1}^{n}$ near $z = 1$, this observed faster convergence can be *quantified.* More precisely, we shall prove in Section 2 the new result of

Theorem 2. *If* $\{z_{k,n}\}_{k=1}^{n}$ *denotes the zeros of the normalized partial sums* $s_n(nz)$ *and if* δ *is any fixed number with* $0 < \delta \leq 1$, *then* (cf. (1.4))

$$\text{dist}\left[\{z_{k,n}\}_{k=1}^{n} \backslash C_\delta; D_\infty\right] = O\left(\frac{\log n}{n}\right), \qquad \textit{as } n \to \infty. \tag{1.5}$$

We remark that the result (1.5) will also be shown to be *best possible*, as a function of n.

It is natural, on seeing the seemingly "regular" way the zeros $\{z_{k,n}\}_{k=1}^{n}$ of $s_n(nz)$ are distributed in Figures 2 and 3, to conjecture that there is a smooth curve D_n (dependent on n) in the unit disk Δ which provides a much closer approximation, than does the curve D_∞, of the zeros $\{z_{k,n}\}_{k=1}^{n}$ of $s_n(nz)$. As already suggested from the work of Szegö [**11**], we set

$$D_n := \left\{ z \in \mathbf{C} : |ze^{1-z}|^n = \tau_n \sqrt{2\pi n} \left| \frac{1-z}{z} \right|, \ |z| \le 1, \text{ and } |\arg z| \ge \cos^{-1}\left(\frac{n-2}{n}\right) \right\}, \tag{1.6}$$

for all $n \ge 1$, where, from Stirling's formula,

$$\tau_n := \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} \cong 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots, \text{ as } n \to \infty. \tag{1.7}$$

We remark that $\log \tau_n$ can be expressed (cf. Henrici [**5**, p. 377]) in terms of the Binet function $J(z)$:

$$\log \tau_n = J(n) \cong \frac{n^{-1}}{12} - \frac{n^{-3}}{360} + \frac{n^{-5}}{1260} - \frac{n^{-7}}{1680} + \frac{n^{-9}}{1188} - \cdots.$$

Our next result, which will be proved in Section 3, shows that D_n of (1.6) is a well-defined curve in the closed unit disk Δ, for any $n \ge 1$.

Proposition 3. *For each positive integer n, and for each fixed real number θ with $|\theta| \ge \cos^{-1}((n-2)/n)$, there is a unique positive number $r = r_n(\theta)$ such that $z = re^{i\theta}$ lies on the curve D_n of* (1.6).

With Proposition 3, we shall prove in Section 3 the new result of

Theorem 4. *If $\{z_{k,n}\}_{k=1}^{n}$ denotes the zeros of the normalized partial sums $s_n(nz)$ and if δ is any fixed number with $0 < \delta \le 1$, then* (cf. (1.4))

$$\text{dist}\,[\{z_{k,n}\}_{k=1}^{n} \backslash C_\delta; D_n] = O\left(\frac{1}{n^2}\right), \qquad \text{as } n \to \infty. \tag{1.8}$$

As in our previous new results, the result (1.8) will also be shown to be *best possible*, as a function of n.

To illustrate the result of Theorem 4, we have graphed the 16 zeros of $s_{16}(16z)$, along with the curve D_{16}, in Figure 4. The same is done in Figure 5 for the 27 zeros of $s_{27}(27z)$ and the curve D_{27}. Up to plotting accuracy, it appears that the zeros of $s_{16}(16z)$ and $s_{27}(27z)$ lie respectively *on* the curves D_{16} and D_{27}!

In a subsequent paper, we will establish formal series for the zeros $z_{k,n}$ of the normalized partial sum $s_n(nz)$. Specifically, expansions about a point $z \neq 1$ on D_∞ or D_n (cf. (1.1) and (1.6)), as well as an expansion about the point $z = 1$, will be carried out to high orders of precision.

2. Proofs of Proposition 1 and Theorem 2. We begin with the

Proof of Proposition 1. As shown in Newman and Rivlin [**7**],

$$\left\{ \frac{s_n(n+\sqrt{2n}w)}{e^{n+\sqrt{2n}w}} \right\}_{n=1}^{\infty} \tag{2.1}$$

$$\text{converges uniformly to} \qquad \frac{1}{\sqrt{\pi}} \int_w^\infty e^{-t^2}\, dt =: \frac{1}{2}\operatorname{erfc}(w),$$

as $n \to \infty$, on any compact set in the upper-half complex plane (i.e., $\operatorname{Im} w \geq 0$). If t_1 denotes the zero of $\operatorname{erfc}(w)$ in the upper half-plane (i.e., $\operatorname{Im} t_1 > 0$) which is closest to the origin, then it is known numerically (cf. Fettis, Caslin, and Cramer [**4**]) that

$$t_1 \doteq -1.354\ 810 + 1.991\ 467i. \tag{2.2}$$

Now, as the only zeros of $s_n(n+\sqrt{2n}w)e^{-n-\sqrt{2n}w}$ are values of w for which $s_n(n+\sqrt{2n}w)$ vanishes, then the uniform convergence in (2.1) implies, with Hurwitz's Theorem, that $s_n(n+\sqrt{2n}w)$ has a zero in any small closed disk with center t_1, for all n sufficiently large. Consequently, as shown in Saff and Varga [**9**], $s_n(n+\sqrt{2n}w)$ has a zero of the form

$$n+\sqrt{2n}(t_1+o(1)) = n\left\{1+\sqrt{\frac{2}{n}}(t_1+o(1))\right\}, \qquad \text{as } n \to \infty,$$

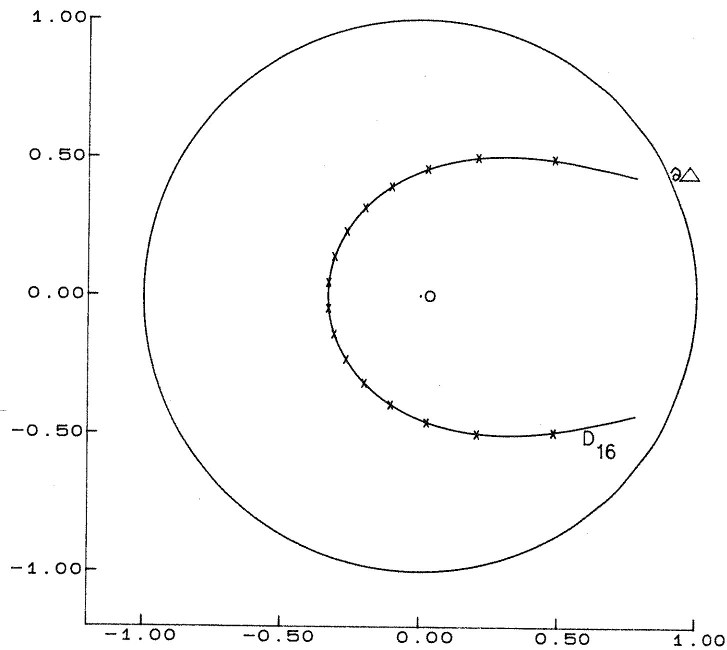

FIGURE 4. D_{16} and the zeros of $s_{16}(16z)$.

or, equivalently, $s_n(nz)$ has a zero $z_{1,n}$ of the form

$$z_{1,n} := 1 + \sqrt{\frac{2}{n}}(t_1 + o(1)), \qquad \text{as } n \to \infty. \tag{2.3}$$

Next, if z lies on D_∞ with $\operatorname{Re} z = 1-\delta$, where $\delta > 0$ is small, it easily follows from the definition of D_∞ in (1.1) that $(1-\delta)^2+(\operatorname{Im} z)^2 = e^{-2\delta}$, so that

$$|\operatorname{Im} z| = \delta\left\{1 - \frac{2}{3}\delta + O(\delta^2)\right\}, \qquad \text{as } \delta \to 0. \tag{2.4}$$

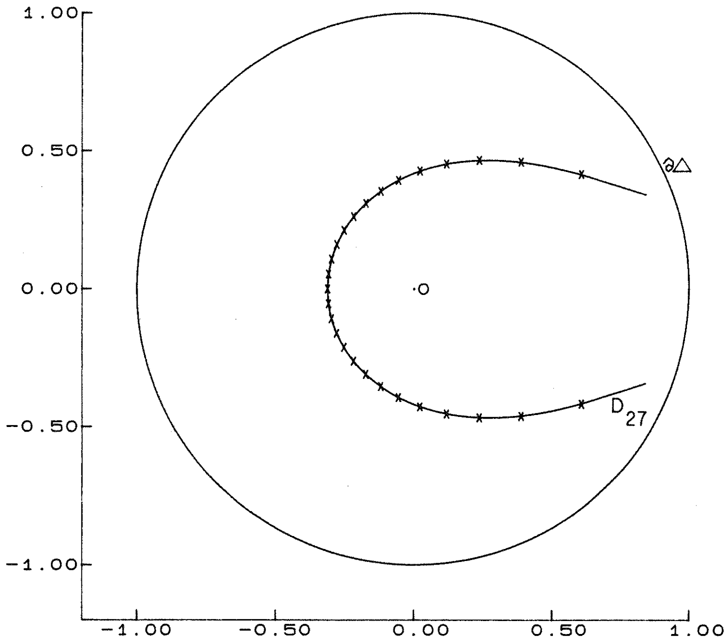

FIGURE 5. D_{27} and the zeros of $s_{27}(27z)$.

Note that (2.4) establishes that the curve D_∞, in the upper-half plane, makes an angle of $\pi/4$ with the real axis as it passes through $z = 1$. (This can also be seen from Figures 2 and 3.) From (2.3) and (2.4), it readily follows (cf. Carpenter [**3**, p. 137]) that, for n large, the distance of $z_{1,n}$ of (2.3) to the curve D_∞ satisfies

$$\text{dist}\,[z_{1,n}; D_\infty] = \frac{1}{\sqrt{n}}\{\text{Im}\, t_1 + \text{Re}\, t_1 + o(1)\}, \qquad \text{as } n \to \infty,$$

whence

$$\lim_{n\to\infty}\{\sqrt{n}\cdot \text{dist}\,[z_{1,n}; D_\infty]\} = \text{Im}\, t_1 + \text{Re}\, t_1 \doteq 0.636\ 657, \tag{2.5}$$

the last result following from the numerical estimates of (2.2). But, by definition, as $\text{dist}\,[z_{1,n}; D_\infty] \leq \text{dist}\,[\{z_{k,n}\}_{k=1}^n; D_\infty]$, we have the desired result of (1.3). □

For the proof of Theorem 2 of Section 1, we need the following construction. From the definition $s_n(z) := \sum_{j=0}^n z^j/j!$, it is well-known (cf. Szegö [**11**]) and is easy to verify (by differentiation) that

$$e^{-z}s_n(z) = 1 - \frac{1}{n!}\int_0^z \zeta^n e^{-\zeta}\,d\zeta. \tag{2.6}$$

Replacing ζ by $n\zeta$ and z by nz in the above expression results in

$$e^{-nz}s_n(nz) = 1 - \frac{n^{n+1}}{n!}\int_0^z \zeta^n e^{-n\zeta}\,d\zeta. \tag{2.7}$$

Using the definition of τ_n of (1.7) in (2.7) then gives

$$e^{-nz}s_n(nz) = 1 - \frac{\sqrt{n}}{\tau_n\sqrt{2\pi}}\int_0^z (\zeta e^{1-\zeta})^n\,d\zeta. \tag{2.8}$$

Next, from Szegö [**11**], it is known that $w = \zeta e^{1-\zeta}$ is *univalent* in $|\zeta| < 1$. (For a proof of this, see the special case $\sigma = 0$ in Saff and Varga [**10**, Lemma 4.1]). Since we are ultimately interested only in the zeros of $s_n(nz)$ (which, from Section 1, must lie in $|z| < 1$ for all $n > 1$), we make the change of variables $w = \zeta e^{1-\zeta}$ in (2.8), which gives

$$e^{-nz}s_n(nz) = 1 - \frac{\sqrt{n}}{\tau_n\sqrt{2\pi}}\int_0^{ze^{1-z}} w^{n-1}\left(\frac{\zeta(w)}{1-\zeta(w)}\right)dw. \tag{2.9}$$

The form of the above integral brings us to the following result, which is again motivated by the original work of Szegö [**11**]. Consider the integral

$$\int_0^A w^{n-1}G(w)\,dw, \tag{2.10}$$

where the path of integration in (2.10) is taken to be the complex line segment joining 0 and A. Assuming that $G(w)$ is analytic in an open region containing this line segment $[0, A]$, then expanding $G(w)$ in a Taylor's series about the point $w = A$ gives

$$\int_0^A w^{n-1} G(w)\, dw = \sum_{m=0}^{\infty} \frac{G^{(m)}(A)}{m!} \int_0^A w^{n-1} (w - A)^m \, dw. \tag{2.11}$$

Since the integral associated with the m^{th} term of the above sum is (after a change of variables) just the Beta function, this term then satisfies

$$\frac{G^{(m)}(A)}{m!} \int_0^A w^{n-1}(w - A)^m \, dw = \frac{(-1)^m A^{m+n} G^{(m)}(A)}{\prod_{j=0}^{m}(n+j)},$$

for all $m \geq 0$ and $n \geq 1$. Thus, the integral of (2.10) can be expressed as

$$\int_0^A w^{n-1} G(w)\, dw = A^n \sum_{m=0}^{\infty} \frac{(-1)^m A^m G^{(m)}(A)}{\prod_{j=0}^{m}(n+j)}. \tag{2.12}$$

We connect the two integrals of (2.9) and (2.10) by setting $A := ze^{1-z}$, $F(\zeta) := \zeta/(1-\zeta)$, and $G(w) := F(\zeta(w))$, where $w := \zeta e^{1-\zeta}$. If z is any *interior point* of the closed unit disk Δ, then $G(w)$, so defined, is indeed analytic in an open region containing the line segment $[0, ze^{1-z}]$, and the representation of (2.12) is valid. A short calculation of the explicit values of $G^{(m)}(ze^{1-z})$, for $0 \leq m \leq 4$, allows us in this case to give the first few terms of (2.12):

(2.13)

$$\int_0^{ze^{1-z}} w^{n-1} \left(\frac{\zeta(w)}{1 - \zeta(w)} \right) dw =$$

$$\frac{z(ze^{1-z})^n}{n(1-z)} \left\{ 1 - \frac{1}{(n+1)(1-z)^2} + \frac{z(4-z)}{(n+1)(n+2)(1-z)^4} \right.$$
$$- \frac{z^2(27 - 14z + 2z^2)}{(n+1)(n+2)(n+3)(1-z)^6}$$
$$\left. + \frac{z^3(256 - 203z + 58z^2 - 6z^3)}{\prod_{j=1}^{4}(n+j) \cdot (1-z)^8} - \cdots \right\}, \text{ all } n \geq 1.$$

Moreover, on estimating the Cauchy remainder for the sections of the Taylor series in (2.11), it readily follows from (2.12) that, for each nonnegative integer N,

$$\begin{aligned}(2.14)\quad &\int_0^{ze^{1-z}} w^{n-1}\left(\frac{\zeta(w)}{1-\zeta(w)}\right)dw \\ &= (ze^{1-z})^n \sum_{m=0}^{N} \frac{(-1)^m (ze^{1-z})^m G^{(m)}(ze^{1-z})}{\prod_{j=0}^{m}(n+j)} + O\left(\frac{1}{n^{N+2}}\right),\end{aligned}$$

as $n \to \infty$, *uniformly* on any compact subset of $\Delta\backslash\{1\}$. (The motivation for this result, of course, comes from Szegö [**11**], who derived (2.14) for the case $N = 0$.)

As a consequence of the case $N = 0$ in (2.14) and (2.13), we have

$$(2.15)\quad \int_0^{ze^{1-z}} w^{n-1}\left(\frac{\zeta(w)}{1-\zeta(w)}\right)dw = \frac{z(ze^{1-z})^n}{n(1-z)}\left\{1+O\left(\frac{1}{n}\right)\right\}, \quad \text{as } n \to \infty,$$

uniformly on any compact subset of $\Delta\backslash\{1\}$. Thus, if z is a zero of the normalized partial sum $s_n(nz)$ of e^z, then, from (2.9) and (2.15), we have

$$1 - \frac{\sqrt{n}}{\tau_n\sqrt{2\pi}} \cdot \frac{z(ze^{1-z})^n}{n(1-z)}\left\{1+O\left(\frac{1}{n}\right)\right\} = 0,$$

or, equivalently,

$$(2.16)\quad (ze^{1-z})^n = \tau_n\sqrt{2\pi n}\left(\frac{1-z}{z}\right)\left\{1+O\left(\frac{1}{n}\right)\right\}, \qquad \text{as } n \to \infty,$$

uniformly on any compact subset of $\Delta\backslash\{1\}$.

We are now in a position to give the

Proof of Theorem 2. From Szegö [**11**], it is known that $w = ze^{1-z}$ maps the interior of the Szegö curve, D_∞, conformally onto the interior of $|w| < 1$, and it also maps, in a 1-1 fashion, the points of D_∞ onto $|w| = 1$, such that the argument of $w = ze^{1-z}$, as z traverses D_∞ in the positive sense starting at $z = 1$, increases monotonically from 0 to 2π. Szegö [**11**] also showed that the zeros of $s_n(nz)$ are asymptotically (as $n \to \infty$) *uniformly distributed* in angle under the mapping $w = ze^{1-z}$. More precisely, let ϕ_1 and ϕ_2 be any real numbers with $0 < \phi_1 < \phi_2 < 2\pi$, and let z_1 and z_2 be respectively the inverse images of $w_1 = e^{i\phi_1}$ and $w_2 = e^{i\phi_2}$, under the mapping $w = ze^{1-z}$, so that z_1 and z_2 are points of D_∞. Let S be the sector in the z-plane, defined by

$$S = \{z \in \mathbf{C} : \arg z_1 \leq \arg z \leq \arg z_2, \text{ where } 0 \leq \arg z \leq 2\pi\}.$$

Then, if μ_n is the number of zeros of $s_n(nz)$ in S, Szegö showed that

$$\lim_{n\to\infty} \frac{\mu_n}{n} = \frac{\phi_2 - \phi_1}{2\pi}.$$

This result implies that, for n large, the zeros of $s_n(nz)$ are roughly *uniformly distributed* in angle in the w-plane, under the mapping $w = ze^{1-z}$.

This can be used as follows. If we take the n uniformly distributed points $\{\exp[i(2k-1)\pi/n]\}_{k=1}^n$ on $|w| = 1$, let $\{\tilde{z}_{k,n}\}_{k=1}^n$ be the unique inverse images of these points in the z-plane under the mapping $w = ze^{1-z}$, i.e.,

$$\tilde{z}_{k,n}e^{1-\tilde{z}_{k,n}} = \exp[i(2k-1)\pi/n], \qquad k = 1, 2, \ldots, n. \tag{2.17}$$

By definition, the points $\{\tilde{z}_{k,n}\}_{k=1}^n$ lie on D_∞, and we have graphed, in Figure 6, the points $\{\tilde{z}_{k,16}\}_{k=1}^{16}$ as $*$'s on D_∞, along with the zeros $\{z_{k,16}\}_{k=1}^{16}$ of $s_{16}(16z)$. The same is done in Figure 7 for $\{\tilde{z}_{k,27}\}_{k=1}^{27}$ and the zeros $\{z_{k,27}\}_{k=1}^{27}$ of $s_{27}(27z)$.

From (2.17), we have

$$(\tilde{z}_{k,n}e^{1-\tilde{z}_{k,n}})^n = -1, \qquad k = 1, 2, \ldots, n. \tag{2.18}$$

Regarding $\tilde{z}_{k,n}$ as an approximation of $z_{k,n}$, write $z_{k,n}$, a zero of $s_n(nz)$, as $z_{k,n} = \tilde{z}_{k,n} + \delta_{k,n}$ and insert this in (2.16). On using (2.18), a straightforward calculation shows, on taking logarithms and dividing by n, that

$$\begin{aligned} &-(1 - \frac{1}{\tilde{z}_{k,n}})\delta_{k,n} + O(\delta_{k,n}^2) \\ &= \frac{1}{n}\log\{\tau_n\sqrt{2\pi n}(1 - \frac{1}{\tilde{z}_{k,n}})\} + O(\frac{\delta_{k,n}}{n}) + O(\frac{1}{n^2}), \end{aligned} \tag{2.19}$$

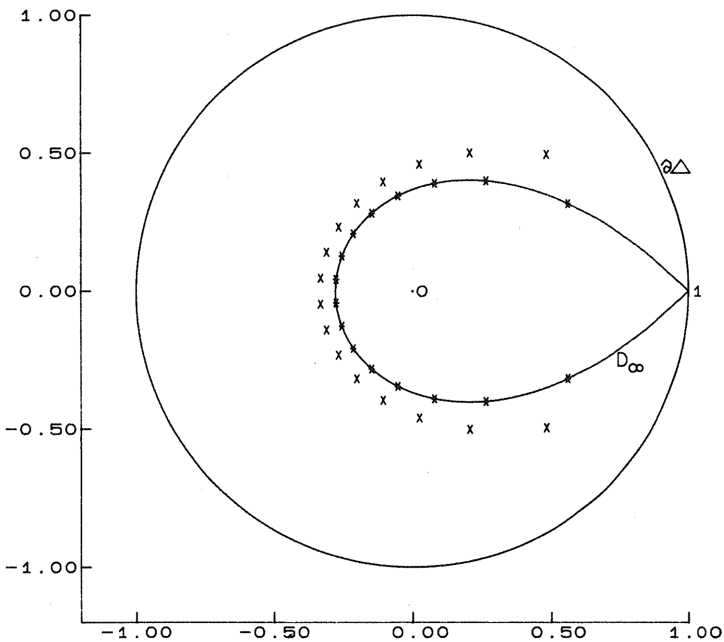

FIGURE 6. $\{z_{k,16}\}_{k=1}^{16}$ and $\{\tilde{z}_{k,16}\}_{k=1}^{16}$.

provided that we consider only those points $z_{k,n}$ which lie outside of C_δ (where δ is a fixed number with $0 < \delta \leq 1$ and where C_δ is defined in (1.4)). (For the logarithm in (2.19), we choose its single-valued extension on $\mathbf{C}\backslash[0, +\infty)$ for which $\log(-1) = i\pi$.)

For the set $\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta$, there evidently exists a positive constant c_1, dependent only on δ, such that

$$0 < c_1 \leq \left|1 - \frac{1}{\tilde{z}_{k,n}}\right|,$$

for all points $\tilde{z}_{k,n}$ associated with points of $\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta$. Thus, we

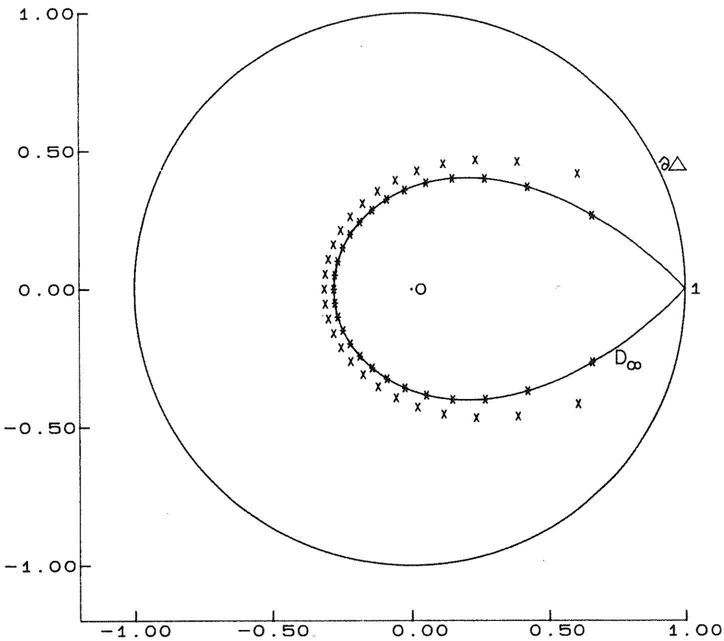

FIGURE 7. $\{z_{k,27}\}_{k=1}^{27}$ and $\{\tilde{z}_{k,27}\}_{k=1}^{27}$.

deduce from (2.19) that

$$\delta_{k,n} = O\left(\frac{\log \tau_n \sqrt{2\pi n}}{n}\right) = O\left(\frac{\log n}{n}\right), \text{ as } n \to \infty, \tag{2.20}$$

for all points of $\{z_{k,n}\}_{k=1}^{n} \backslash C_\delta$. But because $\tilde{z}_{k,n}$ is not necessarily the closest point of D_∞ to $z_{k,n}$, then $\text{dist}\,[z_{k,n}; D_\infty] \leq |\delta_{k,n}|$ for all points of $\{z_{k,n}\}_{k=1}^{n} \backslash C_\delta$, which from (2.20) gives the desired result of Theorem 2. □

As remarked in Section 1, the result of (1.5) of Theorem 2 is *best possible* as a function of n. To establish this, let $n = 2m+1$ be any odd positive integer, and let $-\mu$ denote the *negative* real point of the Szegö curve D_∞, i.e., μ is the unique positive root of $\mu e^{1+u} = 1$ and

$$\mu \doteq 0.278\,464.$$

From (2.18), we see that

$$\tilde{z}_{m+1,2m+1} = -\mu, \qquad m = 0, 1, \dots, \tag{2.21}$$

and it similarly follows that $z_{m+1,2m+1}$ is the unique real (negative) zero of the odd polynomial $s_{2m+1}((2m+1)z)$. Using (2.21), it follows from (2.19) that

(2.22)
$$\lim_{m\to\infty}\left\{\left(\frac{2m+1}{\log(2m+1)}\right)\cdot \delta_{m+1,2m+1}\right\} = -\frac{1}{2(1+\frac{1}{\mu})} \doteq -0.108\,905.$$

Since $z_{m+1,2m+1}$ lies *outside* the curve D_∞, note that $\delta_{m+1,2m+1} < 0$, that $-\delta_{m+1,2m+1} = \operatorname{dist}[z_{m+1,2m+1}; D_\infty]$, and that $z_{m+1,2m+1}$ lies outside C_δ for *any* $0 < \delta \le 1$. Thus, (2.22) becomes

(2.23)
$$\lim_{m\to\infty}\left\{\left(\frac{2m+1}{\log(2m+1)}\right)\cdot \operatorname{dist}[z_{m+1,2m+1}; D_\infty]\right\} = \frac{1}{2(1+\frac{1}{\mu})} \doteq 0.108\,905.$$

But as $\operatorname{dist}[z_{m+1,2m+1}; D_\infty] \le \operatorname{dist}[\{z_{k,2m+1}\}_{k=1}^{2m+1}\backslash C_\delta; D_\infty]$, then, from (2.23),

(2.24)
$$0 < \frac{1}{2(1+\frac{1}{\mu})} \le \varliminf_{m\to\infty}\left\{\left(\frac{2m+1}{\log(2m+1)}\right)\cdot \operatorname{dist}[\{z_{k,2m+1}\}_{k=1}^{2m+1}\backslash C_\delta; D_\infty]\right\}.$$

For the case when $n = 2m$ is an even positive integer, it can be similarly shown that the analogue of (2.23) is

$$\lim_{m\to\infty}\left\{\frac{2m}{\log(2m)}\cdot \operatorname{dist}[z_{m,2m}; D_\infty]\right\} = \frac{1}{2(1+\frac{1}{\mu})} \doteq 0.108\,905, \tag{2.25}$$

so that (cf. (2.24))

$$0 < \frac{1}{2(1+\frac{1}{\mu})} \le \varliminf_{m\to\infty}\left\{\frac{2m}{\log(2m)}\cdot \operatorname{dist}[\{z_{k,2m}\}_{k=1}^{2m}\backslash C_\delta; D_\infty]\right\}. \tag{2.26}$$

Combining (2.24) and (2.26) gives

$$0.108\,905 \leq \underline{\lim}_{n\to\infty} \left\{ \frac{n}{\log n} \cdot \operatorname{dist}\left[\{z_k, n\}_{k=1}^{n} \backslash C_\delta; D_\infty\right] \right\} \tag{2.27}$$

for any δ with $0 < \delta \leq 1$, which is the desired sharpness of Theorem 2.

3. Proofs of Proposition 3 and Theorem 4. In (2.16), we have a relationship which holds, uniformly in n as $n \to \infty$, for any zero z of $s_n(nz)$ which lies in a compact subset of $\Delta/\{1\}$. On taking moduli and n-th roots in (2.16), it is eminently clear *why*, as $n \to \infty$, the Szegö curve, D_∞ of (1.1), emerges as the *only* possible place where the zeros of $\{s_n(nz)\}_{n=1}^{\infty}$ can asymptotically accumulate. As we know from Proposition 1, the distance of the zeros of $s_n(nz)$ to the curve D_∞ is $O(1/\sqrt{n})$ as $n \to \infty$, and this distance can be improved in Theorem 2 to $O((\log n)/n)$ on compact subsets of $\Delta\backslash\{1\}$.

But, it is natural to ask if there is a way of defining a curve, say D_n, now depending on n, for which the zeros of $s_n(nz)$ lie *substantially* closer to D_n than to the curve D_∞. Of course, any smooth curve through the zeros $\{z_{k,n}\}_{k=1}^{n}$ of $s_n(nz)$ would trivially answer this question. It turns out that it is possible to define such a curve D_n, without explicit knowledge of the zeros $\{z_{k,n}\}_{k=1}^{n}$ of $s_n(nz)$. In fact, one obtains the definition of the curve in D_n in (1.6) by dropping the term $O(1/n)$ in (2.16) and taking moduli throughout!

We begin by showing first that D_n of (1.6) is a properly defined curve for all $n \geq 1$.

Proof of Proposition 3. We recall from Szegö [**11**] that $w = ze^{1-z}$ is univalent in $|z| < 1$. For any $n \geq 1$, fix θ to be any real number in $(0, 2\pi)$ with $|\theta| \geq \cos^{-1}((n-2)/n)$, and consider the function, defined on the ray $\{z = re^{i\theta} : r \geq 0\}$, by

$$h_1(r;\theta) := |ze^{1-z}| = re^{1-r\cos\theta}. \tag{3.1}$$

As differentiation shows, $h_1(r;\theta)$ is *strictly increasing* in r on the interval $[0,1]$. Similarly, on setting

$$h_2(r;\theta) := \tau_n\sqrt{2\pi n}\left|\frac{1-z}{z}\right| = \tau_n\sqrt{2\pi n}\left\{\frac{1-2r\cos\theta + r^2}{r^2}\right\}^{1/2}, \tag{3.2}$$

where τ_n is defined in (1.7), it similarly follows that $h_2(r;\theta)$ is *strictly decreasing* in r on $(0,1]$. Moreover, $h_1(0;\theta) = 0$ while $h_2(0;\theta) = +\infty$, and $h_1(1;\theta) = e^{1-\cos\theta}$ while $h_2(1;\theta) = \tau_n\sqrt{4\pi n}(1-\cos\theta)^{1/2}$. Thus, if

$$h_1(1;\theta) > (h_2(1;\theta))^{1/n}, \qquad \text{for all } n \geq 1, \tag{3.3}$$

we can conclude that there is a unique $r_n = r_n(\theta)$ in (0,1) for each $n \geq 1$, such that $h_1(r_n;\theta) = (h_2(r_n;\theta))^{1/n}$. (In other words, with (3.3) we will have that the ray $\{z = re^{i\theta} : r \geq 0\}$ intersects the curve D_n of (1.6) in a unique point in the open unit disk, for each $n \geq 1$ and each $|\theta| > \cos^{-1}((n-2)/n)$.)

Now (3.3) is equivalent, from (3.1) and (3.2), to

$$e^{1-\cos\theta} > \tau_n^{1/n}(4\pi n)^{1/2n}(1-\cos\theta)^{1/2n}. \tag{3.4}$$

Calling $t = 1-\cos\theta$, so that $0 \leq t \leq 2$, (3.4) becomes

$$e^{2nt} > \tau_n^2 4\pi nt, \qquad 0 \leq t \leq 2. \tag{3.5}$$

On setting $2nt =: u$, (3.5) then becomes

$$e^u - 2\pi\tau_n^2 u > 0, \qquad 0 \leq u < +\infty. \tag{3.6}$$

Consider the related equation

$$f(u) := e^u - 8u, \qquad 0 \leq u < +\infty. \tag{3.7}$$

Now it follows from (3.7) that $f(u)$ is strictly increasing on the interval $(\log 8 \doteq 2.079, +\infty)$, and that $f(\log 8) \doteq -8.636 < 0$. Thus, $f(u)$ has a unique zero, say u_1, on the interval $[\log 8, +\infty)$, given by $u_1 \doteq 3.261\,686$, and, moreover,

$$f(u) \geq 0 \quad \text{on } [u_1, +\infty). \tag{3.8}$$

From (3.6), we can write $e^u - 2\pi\tau_n^2 u = f(u) + 2\pi[4/\pi - \tau_n^2]u$. Since τ_n can be verified to be strictly decreasing for all $n \geq 1$, with $1 < \tau_n \leq \tau_1 \doteq 1.084$ for all $n \geq 1$, it can also be verified that $2\pi[4/\pi - \tau_n^2] > 0$ for all $n \geq 1$. Since (3.8) holds, then

$$e^u - 2\pi\tau_n^2 u > 0 \quad \text{for all } u \geq u_1, \text{ and all } n \geq 1. \tag{3.9}$$

As $u = 2nt = 2n(1-\cos\theta)$, (3.9) implies that (3.4) is valid for all θ with $|\theta| \geq \cos^{-1}(1-u_1/2n)$, for all $n \geq 1$. But as $u_1 \doteq 3.261\,686$, then $1-u_1/2n > 1-\frac{2}{n} = (n-2)/n$, whence

$$\cos^{-1}\left(\frac{n-2}{n}\right) > \cos^{-1}\left(1-\frac{u_1}{2n}\right). \tag{3.10}$$

In other words, for each $n \geq 1$ and for each fixed value of θ with $|\theta| \geq \cos^{-1}((n-2)/n)$, there is a *unique* point where the ray $\{z = re^{i\theta} : r \geq 0\}$ intersects the curve D_n, for every $n \geq 1$. □

We remark that the restriction that $|\arg z| \geq \cos^{-1}((n-2)/n)$, in the definition of D_n in (1.6), comes from the fact that $s_n(nz)$ has all its zeros in the sector $|\arg z| > \cos^{-1}((n-2)/n)$ (cf. Saff and Varga [**8**]).

We now come to the

Proof of Theorem 4. For any fixed δ with $0 < \delta \leq 1$, we consider the set $\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta$ (where C_δ is defined in (1.4) and where $\{z_{k,n}\}_{k=1}^{n}$ again denotes the zeros of $s_n(nz)$). Then, for any zeros $z_{k,n}$ of $\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta$, (2.16) is valid, i.e.,

$$(ze^{1-z})^n = \tau_n\sqrt{2\pi n}\left(\frac{1-z}{z}\right)\left\{1+O\left(\frac{1}{n}\right)\right\}, \quad \text{as } n\to\infty, \tag{3.11}$$

where the constant implicit in $O(1/n)$ depends only on δ. On the other hand, from Proposition 3, we have that, for any θ with $\theta_n \leq \theta \leq 2\pi-\theta_n$, where $\theta_n := \cos^{-1}((n-2)/n)$ for any $n \geq 1$, there is a unique $r_n(\theta)$ in (0,1) such that $z = r_n(\theta)e^{i\theta}$ lies on the curve D_n. This implies from (1.6) that there is a real $\Psi(n,\theta)$ such that

$$\frac{z(ze^{1-z})^n}{\tau_n\sqrt{2\pi n}(1-z)} = e^{i\Psi(n,\theta)}, \tag{3.12}$$

where (cf. (3.1) and (3.2))

$$\Psi(n,\theta) := n[\theta - r_n(\theta)\sin\theta] + \theta + \tan^{-1}\left[\frac{r_n(\theta)\sin\theta}{1-r_n(\theta)\cos\theta}\right], \tag{3.13}$$

for all $\theta_n \le \theta \le 2\pi - \theta_n$. It turns out that $\Psi(n,\theta)$ is a strictly increasing function of θ on $[\theta_n, 2\pi-\theta_n]$, and that there are exactly n distinct values of θ in $(\theta_n, 2\pi-\theta_n)$ for which $\Psi(n,\theta) = 0 (\mathrm{mod}\ 2\pi)$. If we denote these n particular points on D_n by $\{\hat{z}_{k,n}\}_{k=1}^{n}$, it follows from (3.12) that

$$(3.14)\qquad (\hat{z}_{k,n}e^{1-\hat{z}_{k,n}})^n = \tau_n\sqrt{2\pi n}\left(\frac{1-\hat{z}_{k,n}}{\hat{z}_{k,n}}\right), \qquad k = 1,2,\ldots,n.$$

With $\{z_{k,n}\}_{k=1}^{n}$ denoting the zeros (with increasing arguments) of $s_n(nz)$, express $z_{k,n}$ as $z_{k,n} = \hat{z}_{k,n} + \delta_{k,n}$. Thus, for any zero in $\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta$, we have from (3.11) that

$$(3.15)\qquad (z_{k,n}e^{1-z_{k,n}})^n = \tau_n\sqrt{2\pi n}\left(\frac{1-z_{k,n}}{z_{k,n}}\right)\left\{1+O\left(\frac{1}{n}\right)\right\}.$$

Replacing $z_{k,n}$ by $\hat{z}_{k,n} + \delta_{k,n}$ in (3.15) and using (3.14), this becomes, on taking logarithms and dividing by n,

$$(3.16)\qquad \log\left(1+\frac{\delta_{k,n}}{\hat{z}_{k,n}}\right) - \delta_{k,n}$$
$$= \frac{1}{n}\log\left(1-\frac{\delta_{k,n}}{1-\hat{z}_{k,n}}\right) - \frac{1}{n}\log\left(1+\frac{\delta_{k,n}}{\hat{z}_{k,n}}\right) + O\left(\frac{1}{n^2}\right),$$

as $n \to \infty$. On expanding these various terms (with the assumption that $\delta_{k,n}$ is sufficiently small), one easily determines that

$$(3.17)\qquad \delta_{k,n} = O\left(\frac{1}{n^2}\right), \qquad \text{as } n \to \infty,$$

for all points of $\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta$. Again, because $\hat{z}_{k,n}$ is not necessarily the closest point of D_n to $z_{k,n}$, then $\operatorname{dist}[z_{k,n}; D_n] \le \delta_{k,n}$ for all points of $\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta$, and the desired result of Theorem 4 follows from (3.17). □

To show that the result of (3.17) is *best possible*, as a function of n, consider (as in Section 2) the special sequences $\{z_{m+1,2m+1}\}_{m=1}^{\infty}$ and $\{z_{m,2m}\}_{m=1}^{\infty}$ of zeros of $s_n(nz)$. To be precise, it can be similarly shown (cf. (2.27)) that

$$(3.18)\qquad \frac{\mu}{(1+\mu)^3} = 0.133\,261\cdots \le \varliminf_{n\to\infty}\left\{n^2 \cdot \operatorname{dist}\left[\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta; D_n\right]\right\}$$

for any δ with $0 < \delta \leq 1$, which establishes the sharpness of Theorem 4.

Having completed the proof of Theorem 4, we now remark that the result of Theorem 4 can be easily *generalized* in the following way. In essence, only the first term (corresponding to the case $N = 0$) of the sum in (2.14) was used in approximating the integral in (2.14), to derive (2.16), and from this, Theorem 4 resulted with the error bound $O(1/n^2)$. Now, it is clear from the error bound in (2.14) that increasing N (the upper bound of the sum in (2.14)) not only improves the approximation to the integral of (2.14), but it also serves to *define* a new curve, say $D_n^{(N)}$, $N \geq 0$, in Δ (cf. (1.6)), which gives a better approximation of the zeros of $s_n(nz)$. In particular, with the same basic proof as that of Theorem 4, it can be shown that, for any nonnegative integer N,

$$\text{dist}\,[\{z_{k,n}\}_{k=1}^{n} \backslash C_\delta; D_n^{(N)}] = O\left(\frac{1}{n^{N+2}}\right), \qquad \text{as } n \to \infty,$$

for each fixed δ with $0 < \delta \leq 1$.

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