Analysis 11, 165 - 190 (1991)

Analysis

SOME 2-PERIODIC TRIGONOMETRIC INTERPOLATION PROBLEMS ON EQUIDISTANT NODES

A. SHARMA, J. SZABADOS[1] and R.S. VARGA[2]

Received: February 11, 1991

ABSTRACT: The problem to be considered here is the determination of necessary and sufficient conditions for the uniqueness of 2-periodic lacunary trigonometric interpolation on equidistant nodes. Our main results are new necessary and sufficient conditions in particular cases which depend only on the total number of even and odd integers in the derivatives which define the trigonometric interpolation process. As such, these new conditions can be readily checked, as they avoid the evaluation of determinants.

AMS Classification: 42A15.

1. Introduction.

Given any positive integer n, define

$$x_k = x_k(n) := \frac{k\pi}{n} \qquad (k = 0, 1, \cdots, 2n-1), \tag{1.1}$$

so that $\{x_k\}_{k=0}^{2n-1}$ is a set of $2n$ equidistant *nodes* in $[0, 2\pi)$. Next, with p and q arbitrary

[1] Research supported by the Hungarian National Foundation of Research, Grant #1801.

[2] Research supported by the National Science Foundation.

positive integers, assume that the associated vectors

$$\mathbf{m} := (m_1, m_2, \cdots, m_p), \text{ and } \mathbf{m}' := (m'_1, m'_2, \cdots, m'_q), \tag{1.2}$$

have nonnegative integer components which satisfy

$$0 =: m_1 < m_2 < \cdots < m_p, \text{ and } 0 \le m'_1 < m'_2 < \cdots < m'_q. \tag{1.3}$$

We consider here the following *2-periodic trigonometric interpolation problem* for the vectors $(\mathbf{m}; \mathbf{m}')$. For arbitrary data consisting of complex numbers $\{\alpha_{j,\nu}\}_{j=0,\nu=1}^{n-1,p}$ and $\{\alpha'_{j,\nu}\}_{j=0,\nu=1}^{n-1,q}$, we ask if there is a *unique* complex trigonometric polynomial of the form

$$t_M(x) = a_0 + \sum_{k=1}^{M} (a_k \cos kx + b_k \sin kx), \tag{1.4}$$

or of the form

$$t_M(x) = a_0 + \sum_{k=1}^{M-1} (a_k \cos kx + b_k \sin kx) + a_M \cos(Mx + \frac{\varepsilon\pi}{2}) \tag{1.4'}$$

(where $\varepsilon = 0$ or $\varepsilon = 1$), such that

$$\begin{cases} t_M^{(m_\nu)}(x_{2j}) = \alpha_{j,\nu} & (j = 0, 1, \cdots, n-1;\ \nu = 1, 2, \cdots, p), \text{ and} \\ t_M^{(m'_\nu)}(x_{2j+1}) = \alpha'_{j,\nu} & (j = 0, 1, \cdots, n-1;\ \nu = 1, 2, \cdots, q). \end{cases} \tag{1.5}$$

Note that as the number of nodes in (1.1) is *even* (namely, $2n$), we see that the interpolation conditions of (1.5) are broken into interpolation conditions on *two* disjoint sets of n nodes, from which the term, *2-periodic* trigonometric interpolation, is derived.

The total number of interpolation conditions in (1.5) is evidently

$$N := n(p+q). \tag{1.6}$$

If N is odd (which implies that n is odd and that $p+q$ is odd), then the sought trigonometric interpolant $t_M(x)$ is necessarily of the form (1.4) (which has an odd number of parameters), and in this case, $M = (N-1)/2$. If N is even (which implies that at least one of the numbers, n and $p+q$, is even), the sought trigonometric interpolant $t_M(x)$ is necessarily of the form

(1.4′) with $M = N/2$, where $\varepsilon(= 0 \text{ or } 1)$ is to be appropriately determined. To summarize,

$$(1.7)\qquad \begin{cases} M = (N-1)/2 & \text{if } N \text{ is odd, and} \\ M = N/2 & \text{if } N \text{ is even.} \end{cases}$$

We say that this $(\mathbf{m}; \mathbf{m}')$ 2-periodic trigonometric interpolation problem of (1.5) is *regular*, if, for arbitrary data, the interpolation problem (1.5) admits a *unique* solution $t_M(x)$, where $t_M(x)$ is of the appropriate form (1.4) or (1.4′).

This $(\mathbf{m}; \mathbf{m}')$ 2-periodic trigonometric interpolation problem on $2n$ equidistant nodes is a *special case* of the more general s-periodic trigonometric interpolation problem on sn equidistant nodes in $[0, 2\pi)$, considered in Sharma, Smith, and Tzimbalario [5]. In [5], necessary and sufficient conditions for the regularity of this s-periodic trigonometric interpolation problem were derived, in terms of the nonvanishing of several determinants (of possibly large order). In this form, these necessary and sufficient conditions are not in general easy to apply.

Recently, however, two papers (cf. Sharma, Szabados, and Varga [6] and Sharma and Varga [7]) have treated special cases of 2-periodic trigonometric interpolation on equidistant nodes, and, in each of these latter two papers, new necessary and sufficient conditions for regularity were derived which depend *only* on the total number of even and odd integers in the components of the vectors $\mathbf{m}$ and $\mathbf{m}'$ of (1.2). As such, these necessary and sufficient conditions can be *easily checked*, in contrast with the necessary and sufficient determinantal conditions of [5]. Thus, our goal here is to extend the results of [6] and [7] by finding necessary and sufficient conditions, for the regularity of the 2-periodic lacunary trigonometric interpolation problem, which similarly depend only on the total number of even and odd integers in the components of each of the vectors $\mathbf{m}$ and $\mathbf{m}'$ of (1.2). To this end, we use throughout the notation of (cf. (1.3))

$$(1.8)\qquad \begin{cases} E & := \text{number of even integers in the components of } \mathbf{m} = (0, m_2, \cdots, m_p), \\ O & := \text{number of odd integers in the components of } \mathbf{m} = (0, m_2, \cdots, m_p), \\ E' & := \text{number of even integers in the components of } \mathbf{m}' = (m_1', m_2', \cdots, m_q'), \\ O' & := \text{number of odd integers in the components of } \mathbf{m}' = (m_1', m_2', \cdots, m_q'), \end{cases}$$

so that

$$E + O = p \text{ and } E' + O' = q. \tag{1.9}$$

2. Statements of Theorems 1 and 2.

With the notation of section 1, we now state our basic results, Theorem 1 (when N is odd) and Theorem 2 (when N is even). We begin with the case when N is odd.

THEOREM 1. *Let N be odd, so that* (cf. (1.6)) $n =: 2r+1$ *and* $p+q =: 2s+1$ *are both odd. Then, a necessary condition for the* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem* (1.5) *to be regular is that* (cf. (1.8))

$$E + E' - 1 = s = O + O'. \tag{2.1}$$

Moreover, if the components of m and m′ *are, respectively, alternately even and odd integers, i.e.* (cf. (1.2) and (1.3)),

$$m_i + m_{i+1} \text{ is odd } (i = 0, 1, \cdots, p-1), \text{ and } m'_i + m'_{i+1} \text{ is odd } (i = 0, 1, \cdots, q-1), \tag{2.2}$$

then (2.1) *is both necessary and sufficient for this* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem to be regular.*

It is not yet known if condition (2.1) is, by itself, *both* necessary and sufficient for the regularity of this trigonometric interpolation problem when N is odd, though we tend to believe that it *is*. Indeed, with N odd and with (2.1) holding, there are examples showing that the trigonometric interpolation problem is regular, *without* (2.2) holding. Explicit examples of this will be given in section 4.

We continue with the case when N of (1.6) is even. Here, there are three cases to be dealt with:

$$\begin{cases} i) & n \text{ is even } (n =: 2r), \text{ and } p+q \text{ is odd } (p+q =: 2s+1), \\ ii) & n \text{ is even } (n =: 2r), \text{ and } p+q \text{ is even } (p+q =: 2s+2), \\ iii) & n \text{ is odd } (n =: 2r+1), \text{ and } p+q \text{ is even } (p+q =: 2s+2). \end{cases} \tag{2.3}$$

Because N is now even, the sought interpolant $t_M(x)$ is necessarily of the form (1.4′), and $\varepsilon(=0$ or $1)$ must also be appropriately chosen in each case of (2.3). We also use the notation $[[x]]$ to denote the integer part of the real number x.

THEOREM 2. *i) Let N be even, with* (cf. (2.3*i*)) $n =: 2r$ *even and* $p + q =: 2s + 1$ *odd. Then, a necessary condition for the* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem* (1.5) *to be regular is that*

$$E + E' - 1 = s = O + O'. \tag{2.4i}$$

Moreover, if the alternation condition (2.2) *holds, then* (2.4*i*) *is both necessary and sufficient for this* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem to be regular, with* ε *of* (1.4′) *being given by*

$$\varepsilon = \begin{cases} 0, & if\ p\ is\ odd; \\ 1, & if\ p\ is\ even. \end{cases} \tag{2.5i}$$

ii) Let N be even, with (cf. (2.3*ii*)) $n =: 2r$ *even and* $p + q =: 2s + 2$ *even. Then, a necessary condition for the* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem* (1.5) *to be regular is that*

$$E = [[(p+1)/2]], \text{ and } E' = [[(q+1)/2]]. \tag{2.4ii}$$

Moreover, if the alternation condition (2.2) *holds, then* (2.4*ii*) *is both necessary and sufficient for this* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem to be regular, with* ε *of* (1.4′) *being given by*

$$\varepsilon = \begin{cases} 0, & if\ p\ and\ q\ are\ both\ odd; \\ 1, & if\ p\ and\ q\ are\ both\ even. \end{cases} \tag{2.5ii}$$

iii) Let N be even, with (cf. (2.3*iii*)) $n =: 2r + 1$ *odd and* $p + q =: 2s + 2$ *even. Then, a necessary condition for the* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem* (1.5) *to be regular is that*

$$E + E' - (O + O') = 0 \text{ or } 2. \tag{2.4iii}$$

Moreover, if the alternation condition (2.2) *holds, then* (2.4*iii*) *is both necessary and sufficient for this* $(\mathbf{m};\mathbf{m}')$ *interpolation problem to be regular, with* ε *of* (1.4′) *being given by*

$$\varepsilon = \begin{cases} 0, & if \quad E+E' = s+2 \quad and \quad O+O' = s; \\ 1, & if \quad E+E' = s+1 \quad = \quad O+O'. \end{cases} \tag{2.5iii}$$

Having stated above our main results, we next outline the remaining portions of this paper. In section 3, we give some useful results on determinants having special forms. Then in section 4, we derive in detail necessary and sufficient *determinantal* conditions for the regularity of this $(\mathbf{m};\mathbf{m}')$ 2-periodic trigonometric interpolation problem when N is odd, from which the necessary condition of (2.1) of Theorem 1 is obtained. The remainder of section 4 gives a detailed treatment of the sufficiency part of Theorem 1 when N is odd. Because the methods used in establishing the results of Theorem 2 (when N is even), are similar to those used in establishing Theorem 1 (when N is odd), only the first part *i*) of Theorem 2 is given in detail in section 5.

To conclude this section, we now comment on the relationship of Theorem 1 and 2 here to known results in the literature. In [1], Cavaretta, Sharma, and Varga considered what can be analogously called the *1-periodic trigonometric interpolation problem* on the equidistant nodes $\{x_k(n)\}_{k=0}^{n-1}$, with $x_k(n) := 2k\pi/n$, where the single vector

$$\mathbf{m} := (m_1, m_2, \cdots, m_q) \tag{2.6}$$

has nonnegative integer components with $0 =: m_1 < m_2 < \cdots < m_q$. Here, n is either an odd or even positive integer, and for arbitrary complex numbers $\{\alpha_{k\nu}\}_{k=0,\nu=0}^{n-1,q}$, the associated 1-periodic lacunary interpolation problem was defined by

$$t_M^{m_\nu}(x_k) = \alpha_{k,\nu} \qquad (k = 0,1,\cdots,n-1;\ \nu = 0,1,\cdots,q), \tag{2.7}$$

where $t_M(x)$ is a trigonometric polynomial of the appropriate form (1.4) or (1.4′). In [1], necessary and sufficient conditions, for the interpolation problem (2.7) to be regular, were derived, and remarkably, these conditions depend only on the number of odd and even integers in $\mathbf{m}$. (For a recent survey of lacunary trigonometric interpolation, see Chapter 11 of Lorentz, Jetter, and Riemenschneider [4].)

Curiously, if $\mathbf{m} = \mathbf{m}'$ in (1.2), one would expect that this would cover completely the case treated in [1]. This, however, is not the case because the number of nodes in (1.1) is always even, namely $2n$, whereas the case treated in [1] allows the total number of nodes to be either odd or even. In this sense, the results in this paper can be viewed as being *complementary* to those of [1]. In addition, the necessary and sufficient conditions of Theorems 1 and 2 of this paper require the alternation condition of (2.2), a condition which doesn't appear in [1].

3. Some Lemmas on Determinants.

In subsequent sections, we shall need results for deciding whether certain determinants are nonzero. For this purpose, we collect in this section some elementary but useful results on determinants. We begin with the following well known result for generalized Vandermonde determinants.

LEMMA 1. *Let* $\{t_j\}_{j=1}^s$ *be positive real numbers with* $0 < t_1 < t_2 < \cdots < t_s$, *and let* $\{\mu_j\}_{j=1}^s$ *be nonnegative integers with* $0 \leq \mu_1 < \mu_2 < \cdots < \mu_s$. *Then, the determinant* D *of order* s, *defined by*

$$D := \begin{vmatrix} t_1^{\mu_1} & t_2^{\mu_1} & \cdots & t_s^{\mu_1} \\ t_1^{\mu_2} & t_2^{\mu_2} & & t_s^{\mu_2} \\ \vdots & & & \vdots \\ t_1^{\mu_s} & t_2^{\mu_s} & \cdots & t_s^{\mu_s} \end{vmatrix}, \tag{3.1}$$

satisfies $D > 0$. *Similarly, if the positive real numbers* $\{t_j\}_{j=1}^s$ *in* (3.1) *are strictly decreasing, then* $D \neq 0$ *with*

$$sgn D = (-1)^{s(s-1)/2}. \tag{3.2}$$

Proof. The first part of Lemma 1 is of course the well known result for generalized Vandermonde determinants (cf. Gantmacher [2, p. 99]). For the second part of Lemma 1, assume $\{t_j\}_{j=1}^s$ is a strictly decreasing sequence of positive real numbers. On interchanging the