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How high-precision calculations can stimulate mathematical research

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1. Introduction

It comes at no surprise that ubiquitous high-speed computers are powerful tools in the hands of modern scientists. Our goal is to show, in a few examples, how recent high-precision calculations have directly *stimulated* mathematical research in the area of rational approximation theory.

As a gradient student at Harvard University many years ago, Professor Garrett Birkhoff introduced this author to the new area of scientific computing, an area that still fascinates me. With a great debt of gratitude for this introduction, this article is dedicated to Professor Birkhoff on the occasion of his 80th birthday.

2. The "1/9" Conjecture

It is well known (cf. [30, Section 8.3]) that certain classical time-stepping procedures, such as the forward-difference, backward-difference, and the Crank–Nicolson methods, for numerically approximating the solutions of second-order linear parabolic partial differential equations, can be interpreted as specific Padé rational matrix approximations of the matrix exponential

$$\exp(-tA) := \sum_{k=0}^{\infty} (-t)^k A^k / k!, \quad t \geqslant 0, \tag{2.1}$$

where A is a real symmetric and positive-definite $N \times N$ matrix (which arises from finite difference or finite element approximations to the associated time-independent self-adjoint differential operator). These Padé rational approximations, being defined as the best *local* rational approximation of e^{-x} at $x = 0$, are generally poor approximations of e^{-x} for x large, and this leads, for reasons of stability and/or accuracy, to restrictions on the step size Δt which can be used with such time-stepping schemes.

In contrast to the local nature of Padé rational approximations of e^{-x}, are of *Chebyshev semi-discrete rational approximations*, introduced in [29], which are *global* rational approxima-

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tions of e^{-x} on $[0, +\infty)$. More precisely, if, for each pair (m, n) of nonnegative integers, $\pi_{m,n}$ denotes the collection of all real rational functions $r_{m,n}(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are, respectively, polynomials of degree m and degree n, with $q(x) > 0$ on $[0, +\infty)$, then one is interested in the error of best uniform approximation from $\pi_{m,n}$ to e^{-x} on $[0, +\infty)$, i.e.,

$$\lambda_{m,n} := \inf\{\| e^{-x} - r_{m,n}(x) \|_{L_\infty[0,+\infty)} : r_{m,n} \in \pi_{m,n}\}. \tag{2.2}$$

It is known (cf. Meinardus [16, p. 161]), after dividing out possible common factors, that there is a unique $\hat{r}_{m,n}$ in $\pi_{m,n}$ for which

$$\lambda_{m,n} = \| e^{-x} - \hat{r}_{m,n}(x) \|_{L_\infty[0,+\infty)}, \tag{2.3}$$

and $\hat{r}_{m,n}(x)$ is completely characterized by $m+n+2$ equi-oscillations of the error $e^{-x} - \hat{r}_{m,n}(x)$ on $[0, +\infty)$, i.e., there are distinct points $\{x_j\}_{j=1}^{m+n+2}$ with $0 \leqslant x_1 < x_2 < \cdots < x_{m+n+2} \leqslant \infty$ such that

$$e^{-x_j} - \hat{r}_{m,n}(x_j) = \varepsilon(-1)^j \lambda_{m,n}, \quad 1 \leqslant j \leqslant m+n+2, \tag{2.4}$$

where $\varepsilon = +1$ or $\varepsilon = -1$. A standard numerical technique for determining $\hat{r}_{m,n}$ is the second Remez algorithm (cf. Remez [20] or [16, p. 105]).

But, how rapidly does $\lambda_{n,n}$ of (2.3) approach zero, as $n \to \infty$? This is of course the very key ingredient in the use of the Chebyshev semi-discrete approximation for numerically approximating the solutions of linear parabolic partial differential equations. The first step in this direction was the following result of Cody, Meinardus and Varga [9] from 1969:

Theorem 1 [9]. *Let $\{m(n)\}_{n=1}^{\infty}$ be any sequence of nonnegative integers with $0 \leqslant m(n) \leqslant n$ for each $n \geqslant 0$. Then,*

$$\frac{1}{6} \leqslant \overline{\lim_{n\to\infty}} \left(\lambda_{m(n),n}\right)^{1/n} \leqslant \frac{1}{2.29878}. \tag{2.5}$$

While the result of (2.5) certainly gives the *geometric convergence* of $\{\lambda_{n,n}\}_{n=0}^{\infty}$ to zero as $n \to \infty$, the limited computations in Table 1, from [9], indicated a much *faster* convergence. Thus, the geometric rate of convergence to zero of $\lambda_{n,n}$ appeared to be *substantially* better than the upper bound estimate of (2.5). Subsequently, Schönhage [23] proved in 1973 that

$$\frac{1}{6\sqrt{(4n+4)\log 3 + 2 + 2\log 2}} \leqslant 3^n \lambda_{0,n} \leqslant \sqrt{2}, \quad n = 0, 1, \ldots,$$

Table 1

n	$\lambda_{n,n}$	$1/\lambda_{n,n}^{1/n}$	n	$\lambda_{n,n}$	$1/\lambda_{n,n}^{1/n}$	n	$\lambda_{n,n}$	$1/\lambda_{n,n}^{1/n}$
0	5.000(−1)	—	5	9.346(−6)	10.14	10	1.361(−10)	9.696
1	6.685(−2)	14.96	6	1.008(−6)	9.987	11	1.466(−11)	9.658
2	7.359(−3)	11.66	7	1.087(−7)	9.882	12	1.579(−12)	9.626
3	7.994(−4)	10.77	8	1.172(−8)	9.804	13	1.701(−13)	9.600
4	8.653(−5)	10.37	9	1.263(−9)	9.744	14	1.832(−14)	9.577

so that in fact

$$\lim_{n\to\infty} \lambda_{0,n}^{1/n} = \tfrac{1}{3}. \tag{2.6}$$

But then, since the number of coefficients available in the rational function $\hat{r}_{n,n}(x)$ which determines $\lambda_{n,n}$, is essentially *twice* the number of coefficients available in $\hat{r}_{0,n}(x)$ which determines $\lambda_{0,n}$, the combination of Schönhage's result (2.6) and the numbers from Table 1 (weakly) suggested the following conjecture in 1977:

Conjecture (Saff and Varga [21]).

$$\lim_{n\to\infty} \lambda_{n,n}^{1/n} \overset{?}{=} \tfrac{1}{9}. \tag{2.7}$$

Table 2
Chebyshev constants $\lambda_{n,n}$ for $n = 0,\ldots,30$ (50 significant digits)

n	$\lambda_{n,n}$
0	5.0000000000000000000000000000000000000000000000000(−01)
1	6.6831042161850463470611623827115147261452912335145(−02)
2	7.3586701695805292800125541630806037567449132444213(−03)
3	7.9938063633568782880811900971119616897657016325167(−04)
4	8.6522406952888523482243458254146735250070248312132(−05)
5	9.3457131530266464767536568207923979896088688301112(−06)
6	1.0084543748996707079345287764100020604073115263471(−06)
7	1.0874974913752479608665313072729334784854440482418(−07)
8	1.1722652116334907177954323039388804735105573142020(−08)
9	1.2632924833223141460949321009097283343341503331607(−09)
10	1.3611205233454477498707881615368423764725511956239(−10)
11	1.4663111949374871406681261995577526903481661603094(−11)
12	1.5794568370512387714867567328183815746851594910467(−12)
13	1.7011870763403529664164865499450815333370532262774(−13)
14	1.8321743782540412751555017565131565305593964959525(−14)
15	1.9731389966128034286256658020822992417697007771241(−15)
16	2.1248537104952237487996344364187178090447946797672(−16)
17	2.2881485632478919604052208612692419494718110924698(−17)
18	2.4639157377651692748310829623232282977743134908752(−18)
19	2.6531146580633127669264550346953305434632777390920(−19)
20	2.8567773835490937066908938449300680288297707203370(−20)
21	3.0760143495057905069144218639753086839478993352108(−21)
22	3.3120205005513186907513737108226141460287572456630(−22)
23	3.5660818606364245847698227997651372597237663431761(−23)
24	3.8295825821681321269364868473011895629431895000911(−24)
25	4.1340125172853630062707580554526301970561733375450(−25)
26	4.4509753557304246897932636072797330395116595664658(−26)
27	4.7921973758889041899314199978855209710518995114011(−27)
28	5.1595368582571326546650112912554530364106857672396(−28)
29	5.5549942137516226746420079038791349910276155552236(−29)
30	5.9807228828496954372714270071247982846421892890349(−30)

Table 3
Ratios $\lambda_{n-1,n-1}/\lambda_{n,n}$ for $n=1,\ldots,30$

n	$\lambda_{n-1,n-1}/\lambda_{n,n}$
1	7.4815532397221509829356536616817047817627984227696(+00)
2	9.0819455991000169708588696090116053013062015321681(+00)
3	9.2054646248528427537813883233351970910096341895810(+00)
4	9.2390013695637342229492228910895668594203903511860(+00)
5	9.2579780201008948071386386966176824253418736773867(+00)
6	9.2673633886078728002406169193047563695298406988185(+00)
7	9.2731650684028757880126091410302398681948193463184(+00)
8	9.2768895688704833336324198589706052242724880476172(+00)
9	9.2794442071765347804120940269531440575883814379160(+00)
10	9.2812683495309755120464682831533111454037182839080(+00)
11	9.2826170054814049413318434721810547537510453765262(+00)
12	9.2836420758101343365763572286526847004054823670187(+00)
13	9.2844394306651793615709775312796518943731046644594(+00)
14	9.2850718606898552364565090058954105627072638256542(+00)
15	9.2855819149043751995853520164465592457350862957931(+00)
16	9.2859992519340952301903430112062879793060351437360(+00)
17	9.2863450591648612312069660869905305380460379908323(+00)
18	9.2866347991400778934888938646138283512490395348046(+00)
19	9.2868799705918004397301050888494834728534387767879(+00)
20	9.2870892682832631479754585814309629455011369059197(+00)
21	9.2872693653333554026168814849193491692724731187991(+00)
22	9.2874254522088778823625744929007529809804239100012(+00)
23	9.2875616152014957847981034264879772236314002103896(+00)
24	9.2876811967903443960632445539185855003199200785395(+00)
25	9.2877865418013514399321174449489929772550616085232(+00)
26	9.2878800417598657237157599122881662160311796009207(+00)
27	9.2879633425048853224353278979631458037287458900494(+00)
28	9.2880378753756707950994314690770008799076337263447(+00)
29	9.2881048291364217038868850048114151489211447279412(+00)
30	9.2881651976905816378400677087169532012794506558118(+00)

Next, a numerical *update* of the estimates (from 1969) of $\{\lambda_{n,n}\}_{n=0}^{14}$ of Table 1 was carried out in 1984 by Carpenter, Ruttan and Varga [8], using Richard Brent's MP (multiple precision) package [4] with 230 significant digits. Using the (second) Remez algorithm, these calculations gave the Chebyshev constants $\{\lambda_{n,n}\}_{n=0}^{30}$ to an accuracy of about 200 significant digits. These numbers are given in Table 2, rounded to 50 significant digits.

From these numbers, the ratios $\{\lambda_{n-1,n-1}/\lambda_{n,n}\}_{n=1}^{30}$ were computed in [8], and these are given in Table 3.

To the last eleven entries of Table 3, Richardson's extrapolation (as described in (3.8) and (3.9) of Section 3) was used (with $x_n := 1/n^2$). These extrapolations are given in Tables 4–7.

The best extrapolated numbers come from Table 5, which yields, numerically to 15 significant digits, that

$$\lim_{n\to\infty} \lambda_{n,n}^{1/n} \overset{?}{=} \frac{1}{9.28902\,54919\,2081}. \tag{2.8}$$

Table 4
Third Richardson's extrapolation

9.28902549192644262469043159980379746163462295356O5(+00)
9.28902549192463636338821127572007957255446896844 32(+00)
9.2890254919235212361782552355452310998144517528836(+00)
9.2890254919227472919000844467255805180069646466419(+00)
9.2890254919222163735816605228071443149565143016341(+00)
9.2890254919218439884743390512914155757325560186146(+00)
9.2890254919215797099009277334795760282585652815175(+00)
9.2890254919213896705910708011872612164000901127673(+00)

Table 5
Fourth Richardson's extrapolation

9.2890254919205312240649832664389025519177553659037(+00)
9.2890254919208485671587410305609475319973452742869(+00)
9.2890254919207963073654937082427113430339242329911(+00)
9.2890254919208120946294556940428805578880731390886(+00)
9.2890254919208127681771411301709359901892869271757(+00)
9.2890254919208150149547144296258736339472494913060(+00)
9.2890254919208161591023954162336683020414775499319(+00)

Table 6
Fifth Richardson's extrapolation

9.2890254919214127320587548334445830521388384447460(+00)
9.2890254919206982368598678821050851331068234703891(+00)
9.2890254910209432825305071272154597415183101617465(+00)
9.2890254919208141654584179760993333379827853346193(+00)
9.2890254919208198985165341295732097954739907627740(+00)
9.2890254919208187594380340221604743658919958649908(+00)

Table 7
Sixth Richardson's extrapolation

9.2890254919196627357020607062507403229154974205964(+00)
9.2890254919210653837136734712907208606483991578876(+00)
9.2890254919207671899154474789653160736121386135206(+00)
9.2890254919208296189900708128670580198741435560234(+00)
9.2890254919208167344095893867600558244128938244875(+00)

Using a different computational procedure, namely the Carathéodory–Fejér method, Trefethen and Gutknecht [28] numerically estimated the quantity of (2.8) as follows. Let

$$\exp[(x-1)/(x+1)] = \sum_{k=1}^{\infty}{}' c_k T_k(x), \quad x \in [-1, +1], \tag{2.9}$$

denote the Chebyshev expansion of $\exp[(x-1)/(x+1)]$ on $[-1, +1]$, where

$$c_k := \frac{2}{\pi}\int_{-1}^{+1} \frac{\exp[(x-1)/(x+1)]T_k(x)}{\sqrt{1-x^2}}\mathrm{d}x, \quad k=0, 1, \ldots, \tag{2.10}$$

and where the prime in the summation in (2.9) means that $\frac{1}{2}c_0$ is used in place of c_0. From the infinite Hankel matrix $H := [c_{i+j-1}]_{i,j=1}^{\infty}$, let

$$\sigma_n := n\text{th singular value of } H \text{ (where } \sigma_1 \geqslant \sigma_2 \geqslant \cdots).$$

It was conjectured in [28] that

$$\lambda_{n,n} \overset{?}{\sim} \sigma_n \quad \text{as } n \to \infty,$$

and, on the basis of numerical estimates of σ_n, Trefethen and Gutknecht [28] conjectured that

$$\lim_{n\to\infty} \lambda_{n,n}^{1/n} \overset{?}{=} \frac{1}{9.28903}. \tag{2.11}$$

The close numerical estimates of (2.8) and (2.11), based on entirely different numerical methods, gave *strong* evidence that the conjecture of (2.7) is *false*.

There has been a large number of research contributions to the ideas related to the "1/9" Conjecture, and, up to the year 1982, this was surveyed in the monograph of Varga [31]. These research contributions took several distinct directions, one being to find lower bound estimates of Λ_1 and another to find upper bound estimates of Λ_2, where (cf. (2.2))

$$\Lambda_1 := \varliminf_{n\to\infty} \lambda_{n,n}^{1/n}, \qquad \Lambda_2 := \varlimsup_{n\to\infty} \lambda_{n,n}^{1/n}, \tag{2.12}$$

for the geometric convergence rate, by best uniform rational approximations of the function e^{-x} on $[0, +\infty)$. The best specific lower bound for Λ_1 of (2.12) was determined by Schönhage [24] in 1982, who showed that

$$\frac{1}{13.928} < \Lambda_1 := \varliminf_{n\to\infty} \lambda_{n,n}^{1/n}, \tag{2.13}$$

and the best specific upper bound for Λ_2 of (2.9) was determined by Opitz and Scherer [19] in 1985, who showed that

$$\varlimsup_{n\to\infty} \lambda_{n,n}^{1/n} =: \Lambda_2 < \frac{1}{9.037}. \tag{2.14}$$

This result, of course, *proved* that the "1/9" Conjecture of (2.7) is *false*. Actually, (2.14) proves that the *degree* of geometric convergence to zero of the constants $\{\lambda_{n,n}\}_{n=0}^{\infty}$ is actually *better* than 1/9.

In a beautiful and deep new development, Gonchar and Rakhmanov [11] have given an *exact* solution of the "1/9" Conjecture using potential-theoretic methods in the complex plane, methods which unfortunately cannot be adequately described in a few pages. An important role in the development of this theory has been played by results of Nuttal [18] on local rational approximations, based on the theory of Abelian integrals on compact Riemann surfaces, and by the results of Stahl [25] on the asymptotic behavior of multipoint Padé approximants. For a survey of these results, see also Stahl [26].

A special case of the results of Gonchar and Rakhmanov [11] is:

Theorem 2 (Gonchar and Rakhmanov [11]). *With* $\lambda_{n,n}$ *defined in* (2.2), *there is a positive number* Λ *with* $0 < \Lambda < 1$ *such that*

$$\lim_{n \to \infty} \left(\lambda_{n,n}(\mathrm{e}^{-x})\right)^{1/n} = \Lambda. \tag{2.15}$$

This result of course establishes that the numbers Λ_1 and Λ_2 of (2.12) are *equal*. But what this number Λ numerically is and how it can be described, is very fascinating!

It turns out that Magnus [15] has earlier correctly identified in 1986 (without a complete proof) that

$$\Lambda = \exp(-\pi K'/K) = \frac{1}{9.28902\,54919\,20818\,91875\,54494\,35951\ldots}, \tag{2.16}$$

where K and K' are complete elliptic integrals of the first kind for the moduli k and $k' := \sqrt{1-k^2}$, evaluated at the point where $K = 2E$, E being the complete elliptic integral of the second kind. On the other hand, Gonchar announced, at the International Congress of Mathematicians at Berkeley in August 1986, the following result.

Theorem 3 (Gonchar and Rakhmanov [11]). *The number* Λ *of* (2.15) *can be characterized in a number-theoretic way as follows. Define*

$$f(z) := \sum_{j=1}^{\infty} a_j z^j, \tag{2.17}$$

where

$$a_j := \left| \sum_{d \mid j} (-1)^d d \right|, \quad j = 1, 2, \ldots, \tag{2.18}$$

so that $f(z)$ *is analytic in* $|z| < 1$. *Then,* Λ *is the unique positive root of the equation*

$$f(\Lambda) = \tfrac{1}{8}. \tag{2.19}$$

Using Newton's method, Carpenter [7] has computed Λ from (2.19) to high precision, and, to 101 significant digits, $1/\Lambda$ is given by

$$\begin{aligned} \frac{1}{\Lambda} = 9.&28902\,54919\,20818\,91875\,54494\,35951\,74506\,10316\,94867\,75012 \\ &44082\,39700\,61421\,72937\,52472\,86507\,07052\,41587\,06142\,47144\ldots, \end{aligned} \tag{2.20}$$

which confirms the numerical approximations of (2.8) and (2.11).

In a truly interesting development, Magnus wrote to Gonchar in late 1986 that Λ of (2.15) is also the unique positive solution (less than unity) of

$$\sum_{n=1}^{\infty} (2n+1)^2 (-\Lambda)^{n(n+1)/2} = 0, \tag{2.21}$$

which is equivalent to the formulation of (2.19), and, moreover, that *exactly one hundred years earlier*, Halphen [12] in 1886 had computed the value of Λ from (2.21) to six significant figures!

(Halphen had arrived at the equation in (2.21) in this studies of variations of theta functions.) It is thus fitting and proper that the "1/9" constant be called the *Halphen constant*!

3. The Bernstein Conjecture

A central issue in approximation theory is the relationship between the smoothness of a given function and the behavior of its error of best uniform approximation, on the interval $[-1, +1]$, by either polynomials or rational functions. For example, it is known (cf. Jackson [14]) that if $f(x)$ and all its derivatives less than order k are continuous on $[-1, +1]$ with $f^{(k)}(x) \in \mathrm{Lip}\,\alpha$, then there exists a positive constant M such that

$$E_n(f) \leqslant M 6^{k+1} n^{-(k+\alpha)}, \quad n > k \geqslant 1, \tag{3.1}$$

where

$$E_n(f) := \min\{\| f - g \|_{L_\infty[-1,+1]}: g \in \pi_n\}. \tag{3.2}$$

(Here, π_n denotes the set of all polynomials of degree at most n.) Hence, roughly speaking, the smoother f is, the faster its error, of best uniform approximation by polynomials of degree n, tends to zero as $n \to \infty$.

In the opposite direction, suppose we consider continuous functions on $[-1, +1]$ which are *not* continuously differentiable on $[-1, +1]$. Perhaps the first example which comes to mind of such a function might be $|x|$, and it is of interest to know just how the lack of differentiability of $|x|$ at $x = 0$ affects the asymptotic behavior of its best uniform error, $E_n(|x|)$, as $n \to \infty$. This particular problem was treated in considerable depth by S.N. Bernstein [2] who, by means of a long and difficult proof, established the following result.

Theorem 4 (Bernstein [2]). *There exists a positive constant* β *(*β *for Bernstein) such that*

$$\lim_{n\to\infty} 2nE_{2n}(|x|) = \beta, \tag{3.3}$$

where β *satisfies*

$$0.278 < \beta < 0.286. \tag{3.4}$$

In addition to this above result, Bernstein noted in [2, p. 56], as a "curious coincidence", that the constant

$$\frac{1}{2\sqrt{\pi}} = 0.28209\,47917\ldots \tag{3.5}$$

also satisfied the bounds of (3.4) and is very nearly the *average*, namely, 0.282, of the upper and lower bounds for β of (3.4). This observation has, over the years, become known as the

Bernstein Conjecture (1913).

$$\beta \overset{?}{=} \frac{1}{2\sqrt{\pi}} = 0.28209\,47917\ldots. \tag{3.6}$$

Table 8
The numbers $2nE_{2n}(|x|)$ for $n=1,\ldots,52$

n	$2nE_{2n}(\mid x\mid)$	n	$2nE_{2n}(\mid x\mid)$
1	0.25000 00000 00000 00000	27	0.28010 92365 22206 18525
2	0.27048 35791 11137 10107	28	0.28011 34608 89950 28384
3	0.27557 43724 01175 38604	29	0.28011 72562 49499 61792
4	0.27751 78246 75052 69646	30	0.28012 06787 72662 82833
5	0.27845 11855 35508 60152	31	0.28012 37757 31660 88450
6	0.27896 79174 64958 70636	32	0.28012 65871 38731 91844
7	0.27928 29449 58518 02460	33	0.28012 91470 43904 51720
8	0.27948 88375 94507 44771	34	0.28013 14845 70012 61069
9	0.27963 06574 10128 20125	35	0.28013 36247 44030 04676
10	0.27973 24337 71973 82968	36	0.28013 55891 69271 11713
11	0.27980 79172 88743 87383	37	0.28013 73965 72336 69662
12	0.27896 54321 23793 27279	38	0.28013 90632 50782 89591
13	0.27991 02543 15557 69036	39	0.28014 06034 41582 48218
14	0.27994 58584 85782 13247	40	0.28014 20296 25997 94087
15	0.27997 46066 86407 49231	41	0.28014 33527 83104 08169
16	0.27999 81519 56316 72827	42	0.28014 45826 01611 08707
17	0.28001 76771 33297 25379	43	0.28014 57276 57645 50097
18	0.28003 40474 14993 50964	44	0.28014 67955 64600 41624
19	0.28004 79072 85905 85156	45	0.28014 77930 99959 13546
20	0.28005 97447 60423 15265	46	0.28014 87263 13048 74446
21	0.28006 99348 31809 43067	47	0.28014 96006 16931 43684
22	0.28007 88694 75287 53423	48	0.28015 04208 67046 95023
23	0.28008 64787 57075 57049	49	0.28015 11914 28744 92326
24	0.28009 32459 38808 50547	50	0.28015 19162 35465 27355
25	0.28009 92184 52382 83558	51	0.28015 25988 39017 81632
26	0.28010 45159 86556 70489	52	0.28015 32424 53163 84249

In the more than 70 years since Bernstein's work appeared, the truth of this conjecture remained unresolved, despite numerical attacks by several authors (cf. Bell and Shah [1], Bojanic and Elkins [3], and Salvati [22]). The reasons that this conjecture remained open so long were probably due to the fact that

(i) the accurate determination of the numbers $E_{2n}(|x|)$, for n large, is numerically *nontrivial*, and
(ii) that the convergence of $2nE_{2n}(|x|)$ to β, guaranteed by (3.3), is quite *slow*.

Recently, it was shown by Varga and Carpenter [33] in 1985 that the Bernstein Conjecture is *false*; this is a consequence of the following improved bounds of [33] for β:

$$0.28016\,85460\ldots = l_{20} \leqslant \beta \leqslant 2\mu_{100} = 0.28017\,33791\ldots. \tag{3.7}$$

Since the upper bound for β in (3.7) is *less* than $1/(2\sqrt{\pi}) = 0.28209\,47917\ldots$, the Bernstein Conjecture (3.6) is therefore *false*! Based on calculations involving the second Remez algorithm, the numbers $\{2nE_{2n}(|x|)\}_{n=1}^{52}$ were determined by Varga and Carpenter [33] to 95 significant digits, where the calculations of $E_{2n}(|x|)$ were carried out to a precision of 100 significant digits. These numbers are given in Table 8.

The numbers $\{2nE_{2n}(|x|)\}_{n=1}^{52}$ appearing in Table 8 indicate that the convergence of these numbers to the Bernstein constant β is quite slow. A typical scheme for improving the convergence rate of slowly convergent sequences is the *Richardson extrapolation method* (cf. Brezinski [5, p. 7]), which can be described as follows. If $\{S_n\}_{n=1}^{N}$, where $N > 2$, is a given (finite) sequence of real numbers, set $T_0^{(n)} := S_n$, $1 \leqslant n \leqslant N$, and regard $\{T_0^{(n)}\}_{n=1}^{N}$ as the zeroth column, consisting of N numbers, of the Richardson extrapolation table. The first column of the Richardson extrapolation table, consisting of $N-1$ numbers, is defined by

$$T_1^{(n)} := \frac{x_n T_0^{(n+1)} - x_{n+1} T_0^{(n)}}{x_n - x_{n+1}}, \quad 1 \leqslant n \leqslant N-1, \tag{3.8}$$

and inductively, the $(k+1)$st column of the Richardson extrapolation table, consisting of $N-k-1$ numbers, is defined by

$$T_{k+1}^{(n)} := \frac{x_n T_k^{(n+1)} - x_{n+k+1} T_k^{(n)}}{x_n - x_{n+k+1}}, \quad 1 \leqslant n \leqslant N-k-1, \tag{3.9}$$

for each $k = 0, 1, \ldots, N-2$, where $\{x_n\}_{n=1}^{N}$ are given constants. In this way, a triangular table, consisting of $\frac{1}{2}N(N+1)$ entries, is created. In our case of $\{2nE_{2n}(|n|)\}_{n=1}^{52}$, a triangular table of 1,378 entries was created. As for the choice of the numbers $\{x_n\}_{n=1}^{52}$ in (3.8)–(3.9), preliminary calculations indicated that

$$2nE_{2n}(|x|) \doteq \beta + K/n^2 + \text{lower-order terms},$$

so we chose $x_n := 1/n^2$. We remark that the potential loss of accuracy in the subtractions in the numerators and denominators of the fractions defined in (3.8) and (3.9) *suggested* that the calculations of $2nE_{2n}(|x|)$ be done to very high precision (95 significant digits).

The Richardson extrapolation of $\{2nE_{2n}(|x|)\}_{n=1}^{52}$ produced *unexpectedly beautiful results*. Rather than presenting here the complete extrapolation table of 1,378 entries (giving each entry to, say, 95 significant digits), it seems sufficient to mention that of the last 20 columns of this table, all but 3 of the 210 entries in these columns agreed with the first 45 digits of the following approximation of β:

$$\beta = 0.28016\,94990\,23869\,13303\,64364\,91230\,67200\,00424\,82139\,81236. \tag{3.10}$$

The success of this Richardson extrapolation (with $x_n := 1/n^2$) applied to $\{2nE_{2n}(|x|)\}_{n=1}^{52}$ strongly suggests that $2nE_{2n}(|x|)$ admits an asymptotic series expansion (cf. Henrici [13, p. 355]) of the form

$$2nE_{2n}(|x|) \overset{?}{\approx} \beta - \frac{K_1}{n^2} + \frac{K_2}{n^4} - \frac{K_3}{n^6} + \cdots, \quad n \to \infty, \tag{3.11}$$

where the constants K_j are independent of n. Assuming that (3.11) is valid, it follows that

$$n^2(2nE_{2n}(|x|) - \beta) \approx -K_1 + \frac{K_2}{n^2} - \frac{K_3}{n^4} + \cdots, \quad n \to \infty. \tag{3.12}$$

Thus, with the known high-precision approximations of $2nE_{2n}(|x|)$ of Table 8, and with an estimate for β determined from the last entry of the Richardson extrapolation table for $\{2nE_{2n}(|x|)\}_{n=1}^{52}$, we can again apply Richardson extrapolation to $\{n^2(2nE_{2n}(|x|) - \beta)\}_{n=1}^{52}$

Table 9
$\{K_j\}_{j=1}^{10}$ for equation (3.11) (10 significant digits)

j	K_j	j	K_j
1	0.04396 75288 8	6	0.59543 53151
2	0.02640 71687 7	7	2.92591 5470
3	0.03125 34264 6	8	18.49414 033
4	0.05889 00165 7	9	146.94301 23
5	0.16010 69971	10	1438.03271 7

(with $x_n = 1/n^2$) to obtain an extrapolated estimate for K_1 of (3.11). This bootstrapping procedure can be continued to produce, via Richardson extrapolation, estimates for the successive constants K_j of (3.11). As might be suspected, there is a progressive loss of numerical accuracy in the successive determination of the constants K_j.

In Table 9, we tabulate estimates of $\{K_j\}_{j=1}^{10}$, rounded to ten significant digits. As Table 9 indicates, the latter constants K_j begin to grow quite rapidly. Because these constants all turned out to be *positive*, we have the following new conjecture:

Conjecture (Varga and Carpenter [33]). $2nE_{2n}(|x|)$ *admits an asymptotic expansion of the form*

$$2nE_{2n}(|x|) \overset{?}{\approx} \beta - \frac{K_1}{n^2} + \frac{K_2}{n^4} - \frac{K_3}{n^6} + \cdots, \quad n \to \infty, \tag{3.13}$$

where the constants K_j *(independent of* n*) are all positive.*

As of this writing, the above conjecture is still *unsolved*!

4. The "8" conjecture

Since the previous section of this paper was devoted to the Bernstein Conjecture, i.e., to the problem of best uniform *polynomial* approximation to $|x|$ on $[-1, +1]$, it is natural to finally consider in this section the corresponding problem of best uniform *rational* approximation to $|x|$ on $[-1, +1]$. As in Section 1, let $\pi_{n,n}$ denote, for any nonnegative integer n, the set of all real rational functions $r_{n,n}(x) = p(x)/q(x)$ with $p \in \pi_n$ and $q \in \pi_n$. (Here, it is assumed that p and q have no common factors, that q does not vanish on $[-1, +1]$, and that q is normalized by $q(0) = 1$.) Then for any real-valued function $f(x)$ defined on $[-1, +1]$, we define, in analogy with (3.2),

$$E_{n,n}(f) := \inf\{\| f - r_{n,n} \|_{L_\infty[-1,+1]}: r_{n,n} \in \pi_{n,n}\}. \tag{4.1}$$

Interestingly, while Bernstein [2] considered in depth in 1913 the asymptotic behavior of best uniform *polynomial* approximation to $|x|$ on $[-1, +1]$, it was only pointed out fifty years later in 1964 by D.J. Newman [17] how decisively *different* best uniform *rational* approximation to $|x|$ on $[-1, +1]$ is, in that Newman constructively showed that

$$\frac{1}{2\,e^{9\sqrt{n}}} \leqslant E_{n,n}(|x|) \leqslant \frac{3}{e^{\sqrt{n}}}, \quad n = 4, 5, \ldots. \tag{4.2}$$

Newman's inequalities in (4.2) generated much research interest, and, in the spirit of Bernstein's earlier work on the asymptotic behavior of $E_n(|x|)$ as $n \to \infty$, a good part of this research interest focused on the analogous problem of sharpened asymptotic results for $E_{n,n}(|x|)$ as $n \to \infty$.

For the general theory for the asymptotic behavior of $E_{n,n}(f)$, important contributions have been made by Gonchar [10] and others. For specifically $E_{n,n}(|x|)$, the best results to date have been found by Bulanov [6], who proved that

$$E_{n,n}(|x|) \geqslant e^{-\pi\sqrt{n+1}}, \quad n = 0, 1, \ldots, \tag{4.3}$$

and by Vjacheslavov [35], who proved that there exist positive constants M_1 and M_2 such that

$$M_1 \leqslant e^{\pi\sqrt{n}} E_{n,n}(|x|) \leqslant M_2, \quad n = 1, 2, \ldots . \tag{4.4}$$

Obviously, (4.3) and (4.4) imply both that

$$e^{\pi(1-\sqrt{2})} = 0.27218\ldots \leqslant e^{\pi\sqrt{n}} E_{n,n}(|x|) \leqslant M_2, \quad n = 1, 2, \ldots, \tag{4.5}$$

and if

$$\underline{M} := \varliminf_{n\to\infty} e^{\pi\sqrt{n}} E_{n,n}(|x|), \qquad \overline{M} := \varlimsup_{n\to\infty} e^{\pi\sqrt{n}} E_{n,n}(|x|), \tag{4.6}$$

that

$$1 \leqslant \underline{M} \leqslant \overline{M}. \tag{4.7}$$

The result of (4.4) clearly gives the asymptotically *sharp* multiplier, namely π, for $\sqrt{n}$ in the asymptotic behavior of $E_{n,n}(|x|)$ as $n \to \infty$. What only remains then is the determination of the best *asymptotic* constants $\underline{M}$ and $\overline{M}$ in (4.7).

To give insight into this problem, we now describe very recent high-precision calculations of Varga, Ruttan and Carpenter [34] for the numbers $\{E_{n,n}(|x|)\}_{n=1}^{40}$. As in the polynomial case of Section 3, for each nonnegative integer n, the best uniform approximation to $|x|$ on $[-1, +1]$ from $\pi_{n,n}$, say $\hat{r}_{n,n}(x)$, is unique (cf. [16, p. 158]), so that

$$E_{n,n}(|x|) = \| |x| - \hat{r}_{n,n}(x) \|_{L_\infty[-1,+1]}, \quad n = 1, 2, \ldots . \tag{4.8}$$

Furthermore, since $|x|$ is even in $[-1, +1]$, so is $\hat{r}_{n,n}(x)$, and this can be shown to imply that

$$E_{2n,2n}(|x|) = E_{2n+1,2n+1}(|x|), \quad n = 1, 2, \ldots . \tag{4.9}$$

Thus, it suffices, for our purposes, to consider only the manner in which the sequence $\{E_{2n,2n}(|x|)\}_{n=0}^{\infty}$ decreases to zero.

Next, if $\hat{h}_{n,n}(t) \in \pi_{n,n}$ is the best uniform approximation to $\sqrt{t}$ on [0, 1] from $\pi_{n,n}$ for each $n = 1, 2, \ldots$, i.e., if

$$\begin{aligned} E_{n,n}(\sqrt{t}; [0, 1]) &:= \inf_{r_{n,n} \in \pi_{n,n}} \| \sqrt{t} - r_{n,n}(t) \|_{L_\infty[0,1]} \\ &= \| \sqrt{t} - \hat{h}_{n,n}(t) \|_{L_\infty[0,1]}, \end{aligned} \tag{4.10}$$

then it can be easily shown that

$$E_{2n,2n}(|x|) = E_{n,n}(\sqrt{t}; [0, 1]), \quad n = 1, 2, \ldots, \tag{4.11}$$

Table 10
The numbers $E_{2n,2n}(|x|)$ and $e^{\pi\sqrt{2n}}E_{2n,2n}(|x|)$ for $n = 21, \ldots, 40$ (25 significant digits)

n	$E_{2n,2n}(\|x\|; [-1, +1])$	$e^{\pi\sqrt{2n}}E_{2n,2n}(\|x\|; [-1, +1])$
21	9.601122612842236480898718 4E−9	6.675616512649122885656417 9
22	5.970823398705558055298613 7E−9	6.703214288224997725642425 7
23	3.752381381641316369086450 2E−9	6.729109963476020911052099 8
24	2.381499690721783089227969 4E−9	6.753473365851186986196498 3
25	1.525473289510979374814720 7E−9	6.776451379185256903334534 8
26	9.856763349496452995813741 3E−10	6.798171795031113669577074 1
27	6.421358050726624692365324 8E−10	6.818746400291279675079678 8
28	4.215884842992714575828506 1E−10	6.838273474222969818037143 6
29	2.788324165133927541106021 4E−10	6.856839824093862326770264 3
30	1.857072001162821795312570 7E−10	6.874522457133671117247554 0
31	1.245078325074423591090236 0E−10	6.891389963299101763905561 5
32	8.400599755776278634321604 9E−11	6.907503666267325308041961 3
33	5.702211575728862026377444 7E−11	6.922918587292003040007665 6
34	3.892950581599345944390982 3E−11	6.937684256909916668184585 7
35	2.672443556645653736397589 4E−11	6.951845402139240175290985 3
36	1.844299509252544160250377 7E−11	6.965442531166209461463720 4
37	1.279244840924708988199301 0E−11	6.978512433145669705344080 0
38	8.916358294918686087120193 9E−12	6.991088607329832331986247 5
39	6.243828154996281262473042 4E−12	7.003201633058588770167246 1
40	4.392048409181786189839103 7E−12	7.014879490023366905666533 7

where

$$\hat{r}_{2n,2n}(x) = \hat{h}_{n,n}(x^2), \quad n = 1, 2, \ldots . \tag{4.12}$$

From (4.12), our estimates of $\{E_{2n,2n}(|x|)\}_{n=1}^{40}$ were obtained directly from high-precision calculations of $\{E_{n,n}(\sqrt{t}; [0, 1])\}_{n=1}^{40}$. These calculations, as in Section 3, involved the (second) Remez algorithm, where Brent's MP package [4] was used with up to 250 significant digits, and, allowing for guard digits and the possibility of small rounding errors, we believe that the numbers $\{E_{n,n}(\sqrt{t}; [0, 1])\}_{n=1}^{40}$ are accurate to 200 significant digits. The numbers $\{E_{2n,2n}(|x|)\}_{n=21}^{40}$ and $\{e^{\pi\sqrt{2n}}E_{2n,2n}(|x|)\}_{n=21}^{40}$ are given in Table 10, truncated to 25 digits.

As in Section 3, we performed several different extrapolation techniques, such as Richardson's extrapolation, Aitken's Δ^2 extrapolation, etc. (cf. Brezinski [5]), on the numbers $\{e^{\pi\sqrt{2n}}E_{2n,2n}(|x|)\}_{n=21}^{40}$. Our best results were obtained from Richardson extrapolation with $x_n := 1/\sqrt{n}$, and the results of the ninth and tenth Richardson extrapolations $\{\tau_n := e^{\pi\sqrt{2n}}E_{2n,2n}(|x|)\}_{n=21}^{40}$ are given in Table 11. The ninth and tenth Richardson extrapolations of Table 11 are, respectively, strictly decreasing and strictly increasing. Based on these extrapolations in Table 11, Varga, Ruttan and Carpenter [34] then made the numerically very plausible new conjecture:

Conjecture [34].

$$\lim_{n\to\infty} e^{\pi\sqrt{2n}}E_{2n,2n}(|x|) \overset{?}{=} 8. \tag{4.13}$$

Table 11
Extrapolation of $\{\tau_n\}_{n=21}^{40}$

9th Richardson extrapolation	10th Richardson extrapolation
8.0000000004818513852150904	7.9999999993370575957653022
8.0000000002792857242205205	7.9999999996174919169009855
8.0000000001662223537658992	7.9999999997766671415448461
8.0000000001018861846283786	7.9999999998673924596859198
8.0000000000644065954058002	7.9999999999194597844657179
8.0000000000419621984410583	7.9999999999496419688750299
8.0000000000280990775511207	7.9999999999673808389086599
8.0000000000192489204346099	7.9999999999779992400786189
8.0000000000134077625530325	7.9999999999845068292101649
8.0000000000094285808538428	7.9999999999886129550248035
8.0000000000066398157884231	

It is interesting to mention that Professor Herbert Stahl asked the authors of [34] for samples of their numerical results concerning the distribution of the extreme points of the error of best uniform rational approximation, namely $E_{n,n}(\sqrt{x};\,[0,\,1])$ of $\sqrt{x}$ on [0, 1], and these numerical results were apparently of great utility to him; Stahl was able to *theoretically* establish in [27] that conjecture (4.13) is *correct*!

With the apparent success of the Richardson extrapolations (with $x_n := 1/\sqrt{n}$) of the numbers $\{e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;[-1,\,+1])\}_{n=10}^{40}$, it is consistent with conjecture (3.13) to make the following new conjecture:

Conjecture [34]. $e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;\,[-1,\,+1])$ *admits an asymptotic series expansion of the form*

$$e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;\,[-1,\,+1]) \approx 8 + \frac{K_1}{\sqrt{n}} + \frac{K_2}{n} + \frac{K_3}{n^{3/2}} + \cdots, \quad n \to \infty. \tag{4.14}$$

Assuming that (4.14) is valid, it would follow that

$$\sqrt{n}\left\{e^{\pi\sqrt{n}}E_{2n,2n}(|x|;\,[-1,\,+1]) - 8\right\} \approx K_1 + \frac{K_2}{\sqrt{n}} + \frac{K_3}{n} + \cdots, \quad n \to \infty. \tag{4.15}$$

With the known high-precision approximations of the numbers $\tau_n := e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;\,[-1,\,+1])$ of the second column of Table 10, we can similarly perform Richardson extrapolation (with $x_n := 1/\sqrt{n}$) on the numbers $\sqrt{n}\,(\tau_n - 8)$, to estimate the constant K_1 of (4.15). In Table 12, we similarly give the eight and ninth columns of the Richardson extrapolation method, applied to the numbers (cf. (3.3)) of $\{\sqrt{n}\,(\tau_n - 8)\}_{n=21}^{40}$, for the particular choice $x_n := 1/\sqrt{n}$ $(n = 21, 22, \ldots, 40)$, these numbers again having been truncated to 25 decimal digits. Here, we similarly see strict monotonicity of the numbers in each of these two columns, and it appears that

$$-6.66432\,44072\,27\ldots \leqslant K_1 \leqslant -6.66432\,44071\,90\ldots. \tag{4.16}$$

This bootstrapping procedure can be continued to produce, via Richardson extrapolation, estimates for the successive constants K_j in (4.14). As might be expected, there is a progressive

Table 12
Extrapolation of $\{\sqrt{n}\,(\tau_n - 8)\}_{n=21}^{40}$

8th Richardson extrapolation	9th Richardson extrapolation
−6.6643244082814322566680373	−6.6643244056422235361938739
−6.6643244078503439726613513	−6.6643244062953468702991632
−6.6643244076053130326767931	−6.6643244066650174044155402
−6.6643244074621918084852786	−6.6643244068769002380708173
−6.6643244073760385664278707	−6.6643244070004868069058853
−6.6643244073225197839348864	−6.6643244070742680700833267
−6.6643244072882341387675361	−6.6643244071196396734608119
−6.6643244072656467633538892	−6.6643244071485601513855522
−6.6643244072504158643384467	−6.6643244071677648651080498
−6.6643244072399678761510565	−6.6643244071810864075910701
−6.6643244072327288717801809	−6.6643244071907349895094533
−6.6643244072277039192319918	

loss of accuracy in the successive determination of the constants K_j. In Table 13, we tabulate estimates of $\{K_j\}_{j=1}^{5}$, where each number is truncated to 10 decimal digits.

Note that as K_1 is negative in Table 13, it would follow from conjecture (4.14) that the product $\tau_n := e^{\pi\sqrt{n}} E_{2n,2n}(|x|; [-1, +1])$ would be eventually *increasing* to the value 8, as $n \to \infty$, which turns out to be consistent with the behavior of the numerical values in the second column of Table 10. Then, one might ask how large n_0 would have to be so that the inequality,

$$\tau_n \geqslant 8 - 0.1 = 7.9, \quad \text{all } n \geqslant n_0, \tag{4.17}$$

is valid. Surprisingly, using the constants of Table 13 in the series of (4.14), the answer to (4.17) appears to be

$$n_0 \doteq 4{,}386. \tag{4.18}$$

This would indicate that to *numerically* extend the second column of Table 10 to values of τ_n which satisfy (4.17) would be computationally nearly impossible!

It is of interest to observe how the error curve $-\sqrt{t} + r_{32,32}^{*}(t)$, associated with $E_{32,32}(\sqrt{t}; [0, 1])$, behaves on the interval [0, 1]. In Fig. 1, we graph $-\sqrt{t} + r_{32,32}^{*}(t)$ on [0, 1]. Though it is very difficult to count from Fig. 1, there are exactly 66 *extreme points* in [0, 1], i.e., there are distinct points $\{t_j\}_{j=1}^{66}$ with

$$0 = t_1 < t_2 < \cdots < t_{66} = 1$$

Table 13
$\{K_j\}_{j=1}^{5}$ for equation (4.14) (10 significant digits)

j	K_j
1	−6.66432 44072
2	+2.77582 62379
3	−0.14601 15270
4	−0.35994 22092
5	+0.07289 48673

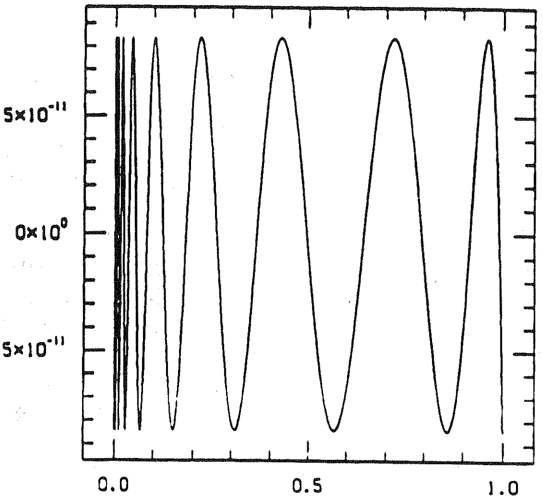

Fig. 1. $-\sqrt{t} + r^*_{32,32}(t)$.

for which

$$-\sqrt{t_j} + r^*_{32,32}(t_j) = (-1)^{j+1} E_{32,32}(\sqrt{t};\ [0,\ 1]), \quad j = 1, 2, \ldots, 66.$$

It is clear from Fig. 1 that there is a severe *bunching up* of these extreme points in the neighborhood of $t = 0$, for the case $n = 32$, and, in fact, this bunching up of the extreme points near $t = 0$ becomes progressively *worse* as n increases. As can be imagined, this is another reason for working numerically with very high precision (at least 200 significant digits) in such computation!

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