SOME NUMERICAL RESULTS ON BEST UNIFORM POLYNOMIAL APPROXIMATION OF X^α ON [0,1]

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Let α be a positive number, and let $E_n(x^\alpha;[0,1])$ denote the error of best uniform approximation to x^α by polynomials of degree at most n on the interval [0, 1]. Russian mathematician S. N. Bernstein established the existence of a nonnegative constant $\beta(\alpha)$ such that $\beta(\alpha) := \lim_{n\to\infty}(2n)^{2\alpha}E_n(x^\alpha;[0,1])$ $(\alpha > 0)$.
In addition, Bernstein showed that $\pi\beta(\alpha) < \Gamma(2\alpha)|\sin(\pi\alpha)|$ $(\alpha > 0)$, and that $\Gamma(2\alpha)|\sin(\pi\alpha)|(1 - 1/(2\alpha - 1)) < \pi\beta(\alpha)$ $(\alpha > 1/2)$, so that the asymptotic behavior of $\beta(\alpha)$ is thus known when $\alpha \to \infty$.

Still, the problem of trying to determine $\beta(\alpha)$ more precisely, for all $\alpha > 0$, is intriguing. To this end, we have rigorously determined the numbers $\{E_n(x^\alpha;[0,1])\}_{n=1}^{40}$ for thirteen values of α, where these numbers were calculated with a precision of at least 200 significant digits. For each of these thirteen values of α, Richardson's extrapolation was applied to the products $\{(2n)^{2\alpha}E_n(x^\alpha;[0,1])\}_{n=1}^{40}$ to obtain estimates of $\beta(\alpha)$ to approximately 40 decimal places. Included are graphs of the points $(\alpha, \beta(\alpha))$ for the thirteen values of α that we considered.

1. Introduction

One of the first results in *constructive* approximation theory was in 1913 by the great Russian mathematician S. N. Bernstein who studied in [1] the best uniform approximation by polynomials of the specific function $|t|$ on the interval $[-1, +1]$. With π_n denoting the set of all real polynomials of degree at most n $(n = 0, 1, \cdots)$, and with

$$E_n(f(t);[a,b]) := \inf\left\{\|f(t) - g(t)\|_{L_\infty[a,b]} : g(t) \in \pi_n\right\} \tag{1.1}$$

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denoting the error of best uniform approximation of a given continuous function $f(t)$ on the finite interval $[a,b]$ by polynomials in π_n, Bernstein proved in [1] that there exists a positive constant $\beta(\frac{1}{2})$ such that

$$\beta(\tfrac{1}{2}) = \lim_{n\to\infty} 2nE_{2n}(|t|;[-1,+1]). \tag{1.2}$$

In addition, he constructively determined the following upper and lower bounds for $\beta(\frac{1}{2})$:

$$0.278 < \beta(\tfrac{1}{2}) < 0.286. \tag{1.3}$$

Bernstein [1, p. 56] remarked in a footnote that it would be very interesting to determine if the limit of $2nE_{2n}(|t|;[-1,+1])$ is a new transcendental number, or can be expressed in terms of known transcendental numbers. Without resolving this question, he remarked, "as a curious coincidence", that $1/(2\sqrt{\pi}) = 0.28209\cdots$, this number being nearly the average, namely 0.282, of the upper and lower bounds in (1.3), and this remark became known in the literature as the

$$\textbf{Bernstein Conjecture:}\quad \beta(\tfrac{1}{2}) \stackrel{?}{=} \frac{1}{2\sqrt{\pi}} = 0.28209\cdots. \tag{1.4}$$

More than 70 years later, Varga and Carpenter [18] showed in 1985, by means of high-precision computations, that the Bernstein conjecture of (1.4) is *false*, in that $\beta(\frac{1}{2})$ rigorously satisfies

$$0.28016\,85460\cdots \le \beta(\tfrac{1}{2}) \le 0.28017\,33791\cdots. \tag{1.5}$$

In addition, from high-precision extrapolations, the following estimate from [18] for $\beta(\frac{1}{2})$ to 50 significant figures is

$$\beta(\tfrac{1}{2}) = 0.28016\,94990\,23869\,13303\,64364\,91230\,67200\,00424\;\;82139\,81236. \tag{1.6}$$

But, what is $\beta(\frac{1}{2})$? Knowing $\beta(\frac{1}{2})$ to some 50 significant digits is one thing, but it would be much more mathematically satisfying to have $\beta(\frac{1}{2})$ in *closed form*, i.e., to have $\beta(\frac{1}{2})$ represented as the evaluation of some specific function. The problem of finding such a solution for $\beta(\frac{1}{2})$ has intrigued us for some time.

Indeed, this problem evidently intrigued Bernstein also for, at the end of his long paper [1], Bernstein briefly mentions without proof that for any $\alpha > 0$, it can be similarly shown that there is a nonnegative real number $\delta(\alpha)$ such that*

$$\delta(\alpha) := \lim_{n\to\infty} (2n)^{2\alpha} E_{2n}(x^\alpha;[0,1]) \quad (\alpha > 0). \tag{1.7}$$

*We remark that the expression in (1.7) above corrects a typographical error appearing on line 11 of [1, p. 56], i.e., the factor $(2n)^{2\alpha}$ in (1.7) appears incorrectly in [1, p. 56] as $(2n)^\alpha$.

It readily follows from (1.7), more generally, that

$$\delta(\alpha) = \lim_{n\to\infty} n^{2\alpha} E_n(x^\alpha; [0,1]) \quad (\alpha > 0). \tag{1.8}$$

Next, to show that the result of (1.8) can be related to the special case $\alpha = \frac{1}{2}$ of (1.2), we note that as $|t|^{2\alpha}$ is an even function on $[-1, +1]$, it is easily seen (cf. Rivlin [15, p. 43, Exercise 1.1] or [17, p. 4]) that

$$E_{2n}(|t|^{2\alpha}; [-1,+1]) = E_n(x^\alpha; [0,1]),$$

which implies from (1.8) that for $\alpha > 0$

$$\beta(\alpha) = \lim_{n\to\infty} (2n)^{2\alpha} E_{2n}(|t|^{2\alpha}; [-1,+1]) = \lim_{n\to\infty} (2n)^{2\alpha} E_n(x^\alpha; [0,1]), \tag{1.9}$$

where (cf. (1.8)) $\beta(\alpha) := 2^{2\alpha}\delta(\alpha)$.

Then, 25 years after the appearance of his paper [1] in 1913, Bernstein revised this problem in 1938 by publishing in [2] or [3] some remarkable results for the function $\beta(\alpha)$. Using a myriad of interesting but difficult techniques, Bernstein established in [2] the upper bound

$$\beta(\alpha) < \frac{\Gamma(2\alpha)|\sin(\pi\alpha)|}{\pi} \quad (\alpha > 0), \tag{1.10}$$

as well as the lower bound

$$\frac{\Gamma(2\alpha)|\sin(\pi\alpha)|}{\pi}\left(1 - \frac{1}{2\alpha - 1}\right) < \beta(\alpha) \quad (\alpha > \frac{1}{2}), \tag{1.11}$$

from which the precise asymptotic behavior as $\alpha \to \infty$, namely,

$$\lim_{\alpha\to\infty} \frac{\beta(\alpha)}{\{\Gamma(2\alpha)|\sin(\pi\alpha)|/\pi\}} = 1, \tag{1.12}$$

follows. He also established in [2] that

$$\lim_{\alpha\to 0} \beta(\alpha) = \tfrac{1}{2} =: \beta(0). \tag{1.13}$$

The results of (1.10) – (1.13) can be interpreted as an attempt by Bernstein to determine $\beta(\alpha)$ in closed form for $\alpha \geq 0$. But, aside from the specific values $\beta(0)$ and $\beta(\frac{1}{2})$ (cf. (1.13) and (1.5) – (1.6), respectively) and the elementary fact from (1.9) that $\beta(s) = 0$ for any positive integer s, we knew of no other computed or theoretical values of $\beta(\alpha)$. Because of this, it was our thought that similarly determining $\beta(\alpha)$ of (1.9), for a number of different values of α, might shed some light on our problem of determining

$\beta(\frac{1}{2})$ in closed form. To this end, using high-precision calculations as in [18], we present below highly accurate estimates of $\beta(\alpha)$ for the thirteen specific values of α, namely

$$\begin{cases} \alpha = \frac{j}{8} & (j = 1, 2, \cdots, 7), \\ \alpha = \frac{j}{4} & (j = 5, 6, 7), \text{ and} \\ \alpha = \frac{j}{4} & (j = 9, 10, 11). \end{cases} \tag{1.14}$$

(All the values of α in (1.14) lie in the interval (0,3), with most of these values lying in (0,1). These choices of α were made to complement the known asymptotic behavior of $\beta(\alpha)$ in (1.12) when $\alpha \to \infty$.) For each α in (1.14), each of the numbers $\{E_n(x^\alpha; [0,1])\}_{n=1}^{40}$ was determined to a precision of at least 200 significant digits. Then for each α in (1.14), the products $\{(2n)^{2\alpha} E_n(x^\alpha; [0,1])\}_{n=1}^{40}$ were extrapolated, using the Richardson extrapolation method, to obtain an estimate of $\beta(\alpha)$ to at least 40 significant digits. As can be really understood, this is a *nontrivial computation*, since for *each* choice of α and n, the number $E_n(x^\alpha; [0,1])$ is determined only after the result of several iterations of the second Remez algorithm (cf. §2). The total amount of these calculations consumed approximately 2200 cpu hours on the Alliant FX/8 and the Encore Multimax at the Argonne National Laboratory.

In Table 1.1, we give our estimates, to 40 significant digits, of $\beta(\alpha)$ which includes the values of α in (1.14), along with $\beta(0) = \frac{1}{2}$ from (1.13) and $\beta(s) = 0$ for s a positive integer.

Table 1.1 Estimates of $\beta(\alpha)$ to 40 decimal places

α	$\beta(\alpha)$
0.000	5.00000 00000 00000 00000 00000 00000 00000 00000E-01
0.125	3.92106 06865 24306 18102 87889 06500 29516 74073E-01
0.250	3.48648 23272 56100 43273 50066 60904 27053 37181E-01
0.375	3.15241 27414 61107 18764 66738 56654 82499 32994E-01
0.500	2.80169 49902 38691 33036 43649 12306 72000 04248E-01
0.625	2.36444 76483 36463 84095 46777 48284 40668 05347E-01
0.750	1.78360 33169 26983 67018 81533 55271 40747 72851E-01
0.875	1.00876 79735 91345 44168 60888 21616 98102 46483E-01
1.000	0.00000 00000 00000 00000 00000 00000 00000 00000E+00
1.250	2.74965 97507 96998 83110 43751 61314 34800 90980E-01
1.500	5.91166 95862 73252 18719 17623 47676 77064 56234E-01
1.750	7.00436 70509 81847 47876 41483 52647 24759 44023E-01
2.000	0.00000 00000 00000 00000 00000 00000 00000 00000E+00
2.250	2.48295 48477 65220 75909 82753 63988 73708 98876E+00
2.500	7.28031 92389 13027 78383 71440 91741 94820 71406E+00
2.750	1.12733 90258 80507 55856 52198 44242 73647 99596E+01
3.000	0.00000 00000 00000 00000 00000 00000 00000 00000E+00

We remark that the behavior of $\beta(\alpha)$ for small positive α particularly interested us, and by the same techniques, we similarly determined the following three values of $\beta(\alpha)$ for α near zero:

$$\begin{cases} \beta(\frac{1}{32}) = 0.45292\ 47687\ 14618\ 57962\,, \\ \beta(\frac{2}{32}) = 0.42680\ 83946\ 25923\ 80658\,, \\ \beta(\frac{3}{32}) = 0.40752\ 60096\ 52622\ 44796\,, \end{cases} \tag{1.15}$$

but to a lesser accuracy of approximately 20 significant digits (which is surely sufficient for plotting purposes).

In Figure 1.1, we have plotted as crosses, "x", all the values of $\beta(\alpha)$ from Table 1.1 and (1.15), i.e., values of α in the interval $[0,3]$. In Figure 1.2, we give an enlargement of this plot for α in the interval $[0,1]$. In addition, Figures 1.1 and 1.2 both contain a plot of the Bernstein function (cf. (1.10) – (1.13)), namely

$$B(\alpha) := \frac{\Gamma(2\alpha)|\sin(\pi\alpha)|}{\pi} \quad (\alpha \geq 0), \tag{1.16}$$

for purposes of comparison. Unfortunately, while we now know many new highly accurate estimates for $\beta(\alpha)$ of (1.9), we have been unable to find a *closed form expression* for $\beta(\alpha)$ which exactly fits our computed data. In Table 1.2, we give values of the Bernstein function $B(\alpha)$ of (1.16) to ten decimal digits for comparison with our numerical estimates of $\beta(\alpha)$ from Table 1.1 and (1.15).

Table 1.2 Estimates of $\beta(\alpha)$ and values of $(\Gamma(2\alpha)|\sin(\pi\alpha)|)/\pi$

α	$\beta(\alpha)$	$(\Gamma(2\alpha)\|\sin(\pi\alpha)\|)/\pi$
0.00000	0.50000 00000	0.50000 00000
0.03125	0.45292 47687	0.48301 32572
0.06250	0.42680 83946	0.46785 15819
0.09375	0.40752 60097	0.45415 88569
0.12500	0.39210 60687	0.44164 25034
0.25000	0.34864 82327	0.39894 22804
0.37500	0.31524 12741	0.36037 05302
0.50000	0.28016 94990	0.31830 98862
0.62500	0.23644 47648	0.26655 48303
0.75000	0.17836 03317	0.19947 11402

Table 1.2 (continued)

α	$\beta(\alpha)$	$(\Gamma(2\alpha)\lvert\sin(\pi\alpha)\rvert)/\pi$
0.87500	0.10087 67974	0.11195 27708
1.00000	0.00000 00000	0.00000 00000
1.25000	0.27496 59751	0.29920 67103
1.50000	0.59110 69586	0.63661 97724
1.75000	0.70043 67051	0.74801 67758
2.00000	0.00000 00000	0.00000 00000
2.25000	2.48295 48478	2.61805 87151
2.50000	7.28031 92389	7.63943 72684
2.75000	11.27339 02588	11.78126 42181
3.00000	0.00000 00000	0.00000 00000

It is our fervent hope that our calculations of these values of $\beta(\alpha)$ in Table 1.1 and (1.15) will give insight and will stimulate others to ponder and mathematically solve our question as to what exactly is $\beta(\alpha)$ for all $\alpha \geq 0$.

We mention that this hope, where numerical calculations could stimulate fundamental mathematical research in approximation theory, has had recent precedences in rational approximation theory. Earlier, motivated by a problem in the numerical solution of heat-conduction problems, the "1/9" Conjecture in rational approximation theory arose, and high-precision calculations were carried out on this conjecture. Subsequently, a beautiful mathematical resolution of this conjecture was given recently in 1987 by Gonchar and Rackhmanov [8]. For highlights of this, see [17, Chapter 2]. More recently, high-precision calculations of the best uniform rational approximations $E_{n,n}(|t|;[-1,+1])$ in Varga,Ruttan, and Carpenter [19] gave rise to the

$$\text{Conjecture:} \qquad \lim_{n\to\infty} e^{\pi\sqrt{2n}}\, E_{2n,2n}(|t|;[-1,+1]) \stackrel{?}{=} 8, \tag{1.17}$$

which was subsequently shown to be mathematically correct in an elegant solution by Stahl [16].

In the next section of this paper, we give in detail a description of how our high-precision calculations of the numbers $E_n(x^\alpha;[0,1])$ were obtained. Finally, in §3 we discuss our use of the Richardson extrapolation method to accelerate the convergence of the products $\left\{(2n)^{2\alpha}E_n(x^\alpha;[0,1])\right\}_{n=1}^{40}$. In this final section, we further give numerical evidence for a new conjecture concerning the asymptotic behavior of the products $(2n)^{2\alpha}E_n(x^\alpha;[0,1])$, as $n \to \infty$. (This conjecture extends the associated conjecture of [18] for the case $\alpha = \frac{1}{2}$ to the general case $\alpha > 0$.)

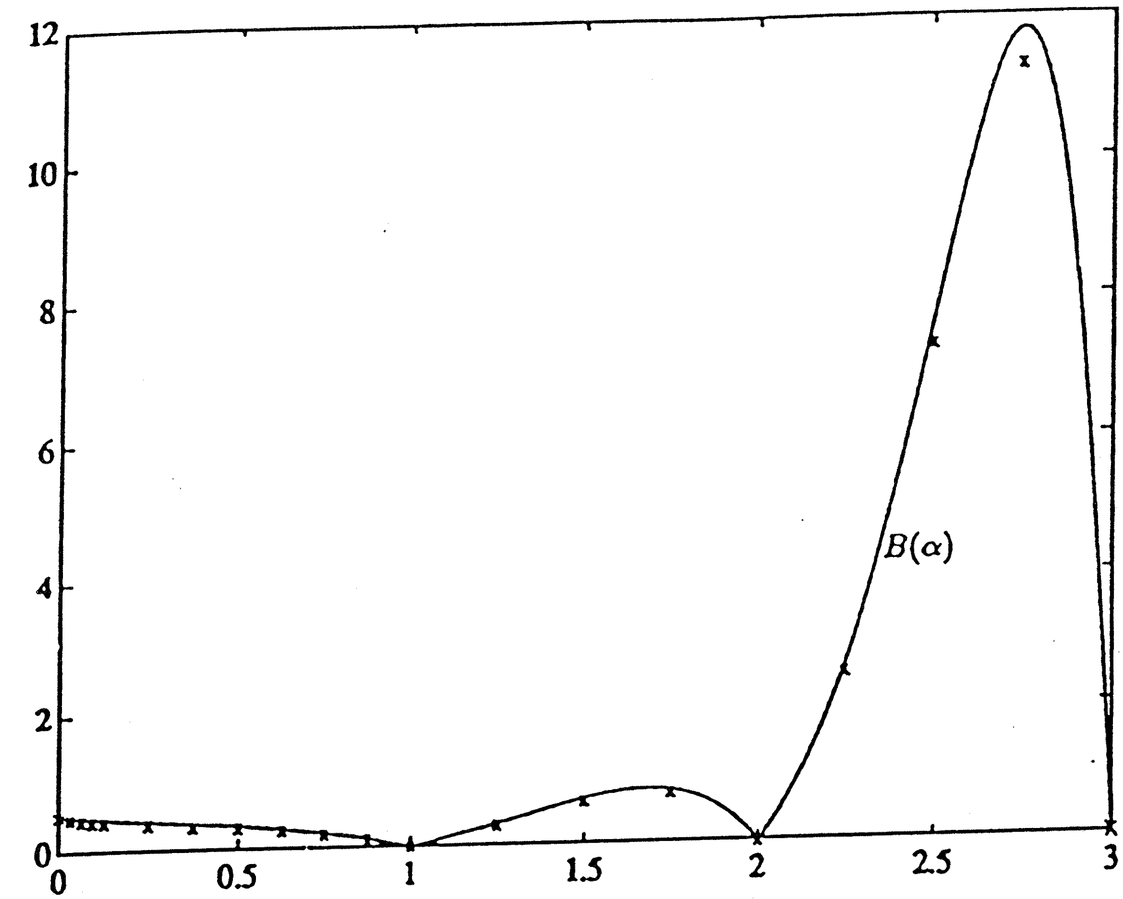

Figure 1.1 The points $(\alpha, \beta(\alpha))$ and the function $B(\alpha)$ for $0 \leq \alpha \leq 3$

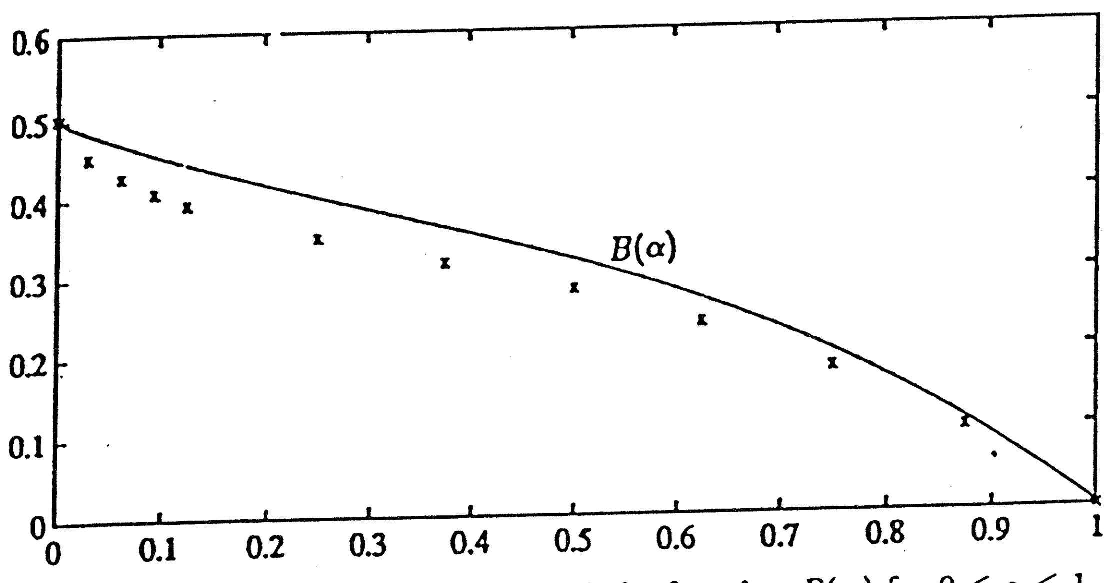

Figure 1.2 The points $(\alpha, \beta(\alpha))$ and the function $B(\alpha)$ for $0 \leq \alpha \leq 1$

2. Computing the Products $\{(2n)^{2\alpha}E_n(x^\alpha;[0,1])\}_{n=1}^{40}$

For any $\alpha > 0$ not an integer, we consider the best uniform approximation from π_m to x^α on $[0,1]$ for each nonnegative integer m. It is well known (cf. Rivlin [15, p. 28]) that there is a unique p_m^* in π_m such that

$$E_m(x^\alpha;[0,1]) := \|x^\alpha - p_m^*(x)\|_{L_\infty[0,1]}. \tag{2.1}$$

Moreover, for each nonnegative integer m, it is easily verified that

$$W_m(\alpha) := \text{span}\,\{1, x, \cdots, x^m; x^\alpha\} \qquad (\alpha > 0 \text{ not an integer}) \tag{2.2}$$

is a *Haar space* of dimension $m+2$ on the interval $[0,1]$, i.e., any function not identically zero in $W_m(\alpha)$ has at most $m+1$ distinct zeros in $[0,1]$. Thus, in the terminology of Loeb [11], x^α is *hypernormal* on $[0,1]$. Consequently (cf. [11] or Meinardus [13, p. 165]), for any nonnegative integer m, the unique best uniform approximation p_m^* in π_m for which (2.1) is valid, has the property that $\partial p_m^* = m$ (where ∂g denotes the exact degree of a polynomial g), and, moreover, the largest alternation set for $x^\alpha - p_m^*(x)$ on $[0,1]$ consists of $m+2$ points, i.e., there exist $m+2$ points $\{x_j\}_{j=0}^{m+1}$ in $[0,1]$ with $0 \le x_0 < x_1 < \cdots < x_{m+1} = 1$, such that

$$x_i^\alpha - p_m^*(x_i) = \sigma(-1)^i E_m(x^\alpha;[0,1]) \qquad (i = 0, 1, \cdots, m+1), \tag{2.3}$$

with σ satisfying $\sigma = +1$ or $\sigma = -1$.

The minimization problem (2.1) was solved using the following essentially standard implementation of the *second Remez algorithm* (cf. Remez [14] or Meinardus [13, p. 105]):

Step 1: Let $S := \{t_j\}_{j=0}^{m+1}$ be a set of $m+2$ distinct points in $[0,1]$ satisfying

$$0 \le t_0 < t_1 < \cdots < t_{m+1} \le 1. \tag{2.4}$$

Step 2: Find the unique polynomial $h_m(t) = \sum_{j=0}^{m} a_j t^j$ and a constant λ such that

$$h_m(t_j) + (-1)^j \lambda = t_j^\alpha \qquad (j = 0, 1, \cdots, m+1), \tag{2.5}$$

which is a linear system of $m+2$ equations in the $m+2$ unknowns λ and $\{a_j\}_{j=0}^{m}$. Because the associated coefficient matrix for (2.5) is a nonsingular Vandermonde matrix of order $m+2$, there are unique values λ and $\{a_j\}_{j=0}^{m}$ which solve (2.5). Note that since $\alpha > 0$ is not an integer, then $\lambda \neq 0$. From (2.5), $h_m(t)$ is then the best uniform approximation from π_m to x^α on this discrete set S, with an alternating error $|\lambda|$ in successive points t_j of S. Thus, in analogy with (2.1), we can write

$$\|x^\alpha - h_m(x)\|_{L_\infty(S)} =: E_m(x^\alpha; S) = |\lambda| > 0, \tag{2.6}$$

and as S is a subset of $[0,1]$, then clearly

$$\|x^\alpha - h_m(x)\|_{L_\infty[0,1]} - |\lambda| \geq 0. \tag{2.7}$$

Step 3: With a preassigned small $\varepsilon > 0$, if $\|x^\alpha - h_m(x)\|_{L_\infty[0,1]} - |\lambda| \leq \varepsilon$, the iteration is terminated. Otherwise, as $x^\alpha - h_m(x)$ is an element of $W_m(\alpha)$ which takes on, by construction, the value $\lambda \neq 0$ with alternating sign in $m+2$ points of $[0,1]$, then $x^\alpha - h_m(x)$ has precisely $m+1$ distinct zeros in $[0,1]$, and a new set S', from the set of local extrema in $[0,1]$ (with alternating signs) of $x^\alpha - h_m(x)$, can be determined. With this new set S', Steps 2 and 3 are repeated until convergence is reached.

Starting with the particular set $S^{(0)} := \left\{t_j^{(0)}\right\}_{j=0}^{m+1}$, where

$$t_j^{(0)} := \frac{1}{2}\left\{1 + \cos\left(\frac{(m+1-j)\pi}{m+1}\right)\right\} \qquad (j = 0, 1, \cdots, m+1) \tag{2.8}$$

are the $m+2$ extreme points of the Chebyshev polynomial $T_{m+1}(2t-1)$ on $[0,1]$, the above algorithm, with $\varepsilon = 10^{-213}$, was used in conjunction with Brent's multiple-precision (MP) package [5], where the products $\left\{(2n)^{2\alpha}E_n(x^\alpha;[0,1])\right\}_{n=1}^{40}$ were computed with a precision of at least 200 significant digits. Because of the known quadratic convergence of this second Remez algorithm (cf. [13, p. 113]), at most *ten* iterations of the algorithm were needed for convergence in all the cases considered. In Tables 2.2 – 2.14, we list the numbers $\{E_n(x^\alpha;[0,1])\}_{n=1}^{40}$ and the products $\left\{(2n)^{2\alpha}E_n(x^\alpha;[0,1])\right\}_{n=1}^{40}$, each rounded to 25 decimal digits, for $\alpha = \frac{j}{8}$ $(j = 1, 2, \cdots, 7)$, $\alpha = \frac{j}{4}$ $(j = 5, 6, 7)$, and $\alpha = \frac{j}{4}$ $(j = 9, 10, 11)$.

These computations were performed at the Argonne National Laboratory using the p4 package and up to eight processors on the Alliant FX/8 and the Encore Multimax. Table 2.1 gives the speedup and efficiency on the Encore Multimax for $\alpha = \frac{1}{2}$ and $n = 10$. For a FORTRAN tutorial on the p4 package, see Lusk and Overbeek [12].

Table 2.1 Best uniform polynomial approximation of $x^{\frac{1}{2}}$ on $[0,1]$

The degree of the polynomial is 10		
Number of Processors	Speedup	Efficiency
1	1.00	1.00
2	1.37	0.69
3	2.15	0.72
4	2.96	0.74
5	3.85	0.77
6	4.75	0.79
7	5.69	0.81
8	6.70	0.84

Figure 2.1 gives graphs of the error functions $E^{(j)}(x) = x^{\frac{1}{2}} - p_n^{(j)}(x)$, for $j = 1$ and 9, and $n = 5$, where $E^{(j)}(x)$ is the error function at the j^{th} iteration of the second Remez algorithm (cf. Step 3 above). For $n = 5$, *nine* iterations of the Remez algorithm were needed for convergence. The error function $E^{(1)}(x) = x^{\frac{1}{2}} - p_n^{(1)}(x)$ is denoted by a dashed line, while the error function $E^{(9)}(x) = x^{\frac{1}{2}} - p_n^{(9)}(x)$ is denoted by a solid line. In the graph of $E^{(9)}(x)$, there are seven alternation points (denoted by small dark disks) in the interval [0,1].

Figure 2.2 gives graphs of the zeros of the polynomials of best uniform approximation to x^α on [0,1], for $n = 30$. The innermost zeros correspond to the case $\alpha = \frac{5}{4}$, the middle zeros correspond to the case $\alpha = \frac{6}{4}$, and the outermost zeros correspond to the case $\alpha = \frac{7}{4}$. In each of these three cases, there are two zeros on the real axis (one on either side of the origin). These zeros were computed with a precision of at least 200 significant digits by using Jenkins's algorithm [10] and Brent's MP package [5].

It follows from a general result of Blatt and Saff [4, Cor. 2.2] that every point of $[0, 1]$ is a limit point of the zeros of the polynomials of best uniform approximation to x^α on $[0, 1]$. It also follows from their result [4, Thm. 3.4] that, for n large, these zeros are roughly uniformly distributed in angle, which is quite evident from Figure 2.2. Moreover, Figure 2.2 suggests a *further* uniformity in α as well!

The high-precision coefficients of the polynomials of best uniform approximation to x^α on [0,1], together with the extreme points on [0,1] of these polynomials, are much too lengthy to reproduce here. These numbers are, however, available upon request.

Table 2.2 The numbers $\left\{E_n(x^{\frac{1}{8}};[0,1])\right\}_{n=1}^{40}$ and the products $\left\{(2n)^{\frac{1}{4}}E_n(x^{\frac{1}{8}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{1}{8}};[0,1])$	$(2n)^{\frac{1}{4}}E_n(x^{\frac{1}{8}};[0,1])$
1	3.25061 25074 87074 67924 96791E-01	3.86565 15220 20465 11260 56734E-01
2	2.76150 31636 19477 56038 05105E-01	3.90535 52265 26873 33084 14186E-01
3	2.50074 11823 92040 64282 53365E-01	3.91387 14633 16022 93554 90478E-01
4	2.32904 52719 18662 58415 35389E-01	3.91697 16402 40032 39094 59871E-01
5	2.20349 50093 39596 87398 41689E-01	3.91842 98052 32129 01695 61450E-01
6	2.10574 24648 79833 28223 10718E-01	3.91922 83396 69610 32938 37563E-01
7	2.02638 59148 93118 78035 43392E-01	3.91971 20766 95181 38166 57091E-01
8	1.96001 34789 36635 49477 29651E-01	3.92002 69578 73270 98954 59302E-01
9	1.90324 58460 34424 28397 31860E-01	3.92024 32604 39412 08769 73143E-01
10	1.85384 18469 46552 96105 35779E-01	3.92039 81918 49731 19796 56528E-01
11	1.81024 44353 79056 85532 88508E-01	3.92051 29375 97544 03801 84001E-01
12	1.77133 11325 06794 29013 58591E-01	3.92060 02762 95127 64989 24387E-01
13	1.73626 80061 27020 41260 05362E-01	3.92066 82853 84276 49900 86341E-01
14	1.70441 98051 20307 84723 57395E-01	3.92072 22728 79461 09515 37814E-01
15	1.67529 22998 24192 29229 64142E-01	3.92076 58429 72185 54474 78658E-01
16	1.64849 39683 37033 18169 42066E-01	3.92080 15123 70940 47946 61591E-01
17	1.62370 97668 10908 85046 05285E-01	3.92083 10814 45673 43683 99437E-01
18	1.60068 27042 91669 04868 12329E-01	3.92085 58656 12882 34696 80135E-01
19	1.57920 06061 87634 51593 37378E-01	3.92087 68440 13578 88385 41443E-01
20	1.55908 64156 54444 16094 73875E-01	3.92089 47576 16884 63653 35012E-01
21	1.54019 09617 98407 29980 10553E-01	3.92091 01754 92631 59342 02779E-01
22	1.52238 74823 43275 89005 41781E-01	3.92092 35405 29909 55585 17763E-01
23	1.50556 74164 70474 85826 99317E-01	3.92093 52015 75815 53553 16537E-01
24	1.48963 71320 40592 04078 26812E-01	3.92094 54364 15407 70318 63163E-01
25	1.47451 53501 93545 65626 44243E-01	3.92095 44684 58368 44490 94276E-01
26	1.46013 10973 33818 04292 04726E-01	3.92096 24790 27471 16049 00523E-01
27	1.44642 20607 67641 61236 88881E-01	3.92096 96165 25365 09135 38932E-01
28	1.43333 32567 13289 83584 80543E-01	3.92097 60033 54063 41282 04326E-01
29	1.42081 59425 11973 84201 14453E-01	3.92098 17411 95359 93257 29494E-01
30	1.40882 67215 42159 54512 14575E-01	3.92098 69150 81299 77891 52150E-01
31	1.39732 68015 25676 72413 95525E-01	3.92099 15965 61461 87384 36639E-01
32	1.38628 13759 22269 79117 85810E-01	3.92099 58461 89017 82822 04973E-01
33	1.37565 91048 40609 74168 18212E-01	3.92099 97154 88001 60397 77349E-01
34	1.36543 16769 74304 85117 28928E-01	3.92100 32485 21918 12990 05600E-01
35	1.35557 34379 41934 60828 61825E-01	3.92100 64831 53409 99243 44816E-01
36	1.34606 10733 84635 91321 14816E-01	3.92100 94520 62610 25619 46486E-01
37	1.33687 33374 79490 36797 19733E-01	3.92101 21835 75600 51505 97090E-01
38	1.32739 08193 26498 45605 62452E-01	3.92101 47023 42388 47129 31085E-01
39	1.31939 57410 82219 26018 54961E-01	3.92101 70298 94849 65112 86498E-01
40	1.31107 17828 33869 87246 95589E-01	3.92101 91851 08320 09918 77773E-01

Table 2.3 The numbers $\left\{E_n(x^{\frac{1}{4}};[0,1])\right\}_{n=1}^{40}$
and the products $\left\{(2n)^{\frac{1}{2}}E_n(x^{\frac{1}{4}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{1}{4}};[0,1])$	$(2n)^{\frac{1}{2}}E_n(x^{\frac{1}{4}};[0,1])$
1	2.36235 19685 52887 18393 85199E-01	3.34087 01930 26272 39277 58483E-01
2	1.72155 20693 53434 23175 24514E-01	3.44310 41387 06868 46350 49028E-01
3	1.41513 44484 23286 49573 64690E-01	3.46635 73160 71970 65124 32197E-01
4	1.22858 91806 58371 69612 20483E-01	3.47497 49637 43835 76361 54110E-01
5	1.10017 51759 47278 04268 06995E-01	3.47905 93811 69893 45338 54704E-01
6	1.00496 61792 14542 40903 35165E-01	3.48130 49645 75914 49164 08276E-01
7	9.30782 28589 35067 22815 22752E-02	3.48266 84154 91773 97088 34803E-01
8	8.70889 30422 98988 17493 53323E-02	3.48355 72169 19595 26997 41329E-01
9	8.21226 35742 02892 54425 11162E-02	3.48416 83573 26083 65640 94517E-01
10	7.79181 67820 41235 08955 10075E-02	3.48460 63985 73572 84424 92057E-01
11	7.42989 78211 46256 18165 23723E-02	3.48493 09834 18216 08916 08061E-01
12	7.11409 00748 72941 12346 19491E-02	3.48517 81335 27376 15410 66328E-01
13	6.83537 41951 97706 11255 30736E-02	3.48537 06404 02168 04912 27078E-01
14	6.58702 02497 27380 98911 01097E-02	3.48552 34923 45045 03442 21083E-01
15	6.36389 13984 71150 98536 00647E-02	3.48564 68724 55746 99266 23958E-01
16	6.16198 99344 58244 29689 13250E-02	3.48574 78946 06939 25707 40318E-01
17	5.97815 19597 52110 80664 19274E-02	3.48583 16497 38971 69450 22314E-01
18	5.80983 64310 48405 21159 11771E-02	3.48590 18586 29043 12695 47063E-01
19	5.65497 59472 69631 60724 89992E-02	3.48596 12915 80166 92687 43042E-01
20	5.51186 90072 56646 44583 46393E-02	3.48601 20454 84431 15273 77117E-01
21	5.37910 07745 81170 01589 62851E-02	3.48605 57310 66543 37107 86252E-01
22	5.25548 38464 65544 14357 71552E-02	3.48609 36020 99995 15069 74266E-01
23	5.14001 33212 96438 48956 92524E-02	3.48612 66462 69212 71024 46608E-01
24	5.03183 22576 34555 49267 50208E-02	3.48615 56501 54823 54735 25193E-01
25	4.93020 47995 95714 55809 92872E-02	3.48618 12464 32593 32272 61660E-01
26	4.83449 50359 77580 84248 23412E-02	3.48620 39486 38658 28247 18803E-01
27	4.74415 02015 93314 12358 51536E-02	3.48622 41771 07670 86018 55003E-01
28	4.65868 72048 71055 97265 86897E-02	3.48624 22785 55006 51761 13974E-01
29	4.57768 17304 83914 70678 70822E-02	3.48625 85410 22395 34351 48942E-01
30	4.50075 93549 48194 16810 85234E-02	3.48627 32054 00318 43844 06859E-01
31	4.42758 82500 71716 97980 37970E-02	3.48628 64743 94687 48956 53421E-01
32	4.35787 31494 57048 15726 64285E-02	3.48629 85195 65638 52581 31428E-01
33	4.29135 03275 76624 28860 84541E-02	3.48630 94868 97960 46745 19530E-01
34	4.22778 33965 51692 37086 85511E-02	3.48631 95012 43044 68696 75310E-01
35	4.16695 97677 90752 24531 10308E-02	3.48632 86698 86236 05750 73279E-01
36	4.10868 76576 83979 00547 32782E-02	3.48633 70854 30976 58509 08495E-01
37	4.05279 35411 86147 48926 16554E-02	3.48634 48281 45275 92024 10976E-01
38	3.99911 99762 13718 23753 60697E-02	3.48635 19678 82075 99421 14286E-01
39	3.94752 37366 89766 12357 92298E-02	3.48635 85656 59694 80740 99703E-01
40	3.89787 42037 92779 51632 01329E-02	3.48636 46749 69409 43408 78964E-01

Table 2.4 The numbers $\left\{E_n(x^{\frac{3}{8}};[0,1])\right\}_{n=1}^{40}$ and the products $\left\{(2n)^{\frac{3}{4}}E_n(x^{\frac{3}{8}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{3}{8}};[0,1])$	$(2n)^{\frac{3}{4}}E_n(x^{\frac{3}{8}};[0,1])$
1	1.73487 73710 33531 31898 95805E-01	2.91770 43244 13769 90067 07448E-01
2	1.08881 44520 03970 20386 43065E-01	3.07963 23298 63688 04251 04245E-01
3	8.13387 98719 42837 00491 31229E-02	3.11825 18729 67257 71251 51648E-01
4	6.58587 41685 17873 45080 94852E-02	3.13278 73678 85633 88615 11356E-01
5	5.58330 58608 08861 98230 36238E-02	3.13972 36167 10298 15503 94538E-01
6	4.87567 25909 44760 10659 95575E-02	3.14355 06981 87253 71433 22908E-01
7	4.34656 34054 60756 11306 08698E-02	3.14587 92296 34213 90729 43537E-01
8	3.93424 89374 66344 59773 81070E-02	3.14739 91499 73075 67819 04856E-01
9	3.60281 32950 00570 46976 22223E-02	3.14844 51774 96947 92650 95306E-01
10	3.32986 87171 91745 48399 35790E-02	3.14919 53983 66262 37691 55305E-01
11	3.10069 67146 81401 31791 30410E-02	3.14975 15601 56162 92033 11986E-01
12	2.90520 16026 52247 84958 55395E-02	3.15017 51868 56524 14302 85155E-01
13	2.73621 48484 90751 32749 41307E-02	3.15050 52402 13713 57138 16117E-01
14	2.58849 78839 22525 34150 36432E-02	3.15076 73599 54383 00886 97099E-01
15	2.45812 83991 95828 19616 44130E-02	3.15097 89750 89857 33985 93379E-01
16	2.34210 79349 74708 68142 60784E-02	3.15115 22666 52019 98453 15962E-01
17	2.23810 26641 29726 46673 65386E-02	3.15129 59548 39322 48172 05332E-01
18	2.14426 72730 91245 86035 06546E-02	3.15141 64147 33597 51793 71027E-01
19	2.05912 23323 27295 34346 58932E-02	3.15151 83940 25613 58524 20920E-01
20	1.98146 70482 91600 66388 38797E-02	3.15160 54870 45516 06732 87775E-01
21	1.91031 59891 58151 44085 04647E-02	3.15168 04552 41475 34662 67740E-01
22	1.84485 24059 86596 15819 66677E-02	3.15174 54484 12208 63521 47539E-01
23	1.78439 32638 73782 49827 10965E-02	3.15180 21603 16007 96340 40995E-01
24	1.72836 26795 39408 12262 51961E-02	3.15185 19399 89879 75141 45284E-01
25	1.67627 14886 93975 03715 05173E-02	3.15189 58726 17548 06511 34242E-01
26	1.62770 13470 06656 89088 35612E-02	3.15193 48391 04983 89107 58458E-01
27	1.58229 22278 98859 95565 57115E-02	3.15196 95605 36416 35540 38997E-01
28	1.53973 24958 81005 37470 04776E-02	3.15200 06317 41446 50422 34577E-01
29	1.49975 09542 17667 65501 35812E-02	3.15202 85469 17496 08886 56083E-01
30	1.46211 04214 31145 49887 30691E-02	3.15205 37193 85731 35947 21098E-01
31	1.42660 25028 17463 18435 19291E-02	3.15207 64969 66566 24131 73293E-01
32	1.39304 33042 06255 80971 85582E-02	3.15209 71740 50425 87345 75602E-01
33	1.36126 98947 09746 23545 92522E-02	3.15211 60011 51194 72009 22680E-01
34	1.33113 73693 67608 15700 00541E-02	3.15213 31925 24846 89698 94543E-01
35	1.30251 63956 89931 00033 17824E-02	3.15214 89322 88411 11807 90339E-01
36	1.27529 11531 33332 51083 98288E-02	3.15216 33793 67350 86333 85028E-01
37	1.24935 75936 44666 35272 14323E-02	3.15217 66715 20857 58198 17314E-01
38	1.22462 19660 98375 77508 95931E-02	3.15218 89286 36340 20556 36175E-01
39	1.20099 95588 41620 84669 59248E-02	3.15220 02554 40888 27403 64255E-01
40	1.17841 36234 52769 31538 97987E-02	3.15221 07437 44701 98776 15305E-01

Table 2.5 The numbers $\left\{E_n(x^{\frac{1}{2}};[0,1])\right\}_{n=1}^{40}$ and the products $\left\{2nE_n(x^{\frac{1}{2}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{1}{2}};[0,1])$	$2nE_n(x^{\frac{1}{2}};[0,1])$
1	1.25000 00000 00000 00000 00000E-01	2.50000 00000 00000 00000 00000E-01
2	6.76208 99277 78427 52693 01688E-02	2.70483 59711 11371 01077 20675E-01
3	4.59290 62066 86256 43414 24906E-02	2.75574 37240 11753 86048 54943E-01
4	3.46897 28084 38158 70584 45601E-02	2.77517 82467 50526 96467 56481E-01
5	2.78451 18553 55086 01522 28751E-02	2.78451 18553 55086 01522 28751E-01
6	2.32473 26455 41322 55307 47788E-02	2.78967 91746 49587 06368 97346E-01
7	1.99487 81782 75128 74717 46620E-02	2.79282 94495 85180 24604 45268E-01
8	1.74680 52349 65671 54823 39079E-02	2.79488 83759 45074 47717 42527E-01
9	1.55350 36522 78490 00696 06363E-02	2.79630 65741 01282 01252 91453E-01
10	1.39866 21688 59869 14844 94695E-02	2.79732 43377 19738 29689 89390E-01
11	1.27185 41694 94290 33563 40915E-02	2.79807 91728 87438 73839 50014E-01
12	1.16610 59671 82471 96998 22401E-02	2.79865 43212 37932 72795 73762E-01
13	1.07657 79012 13683 42447 86005E-02	2.79910 25431 55576 90364 43613E-01
14	9.99806 63637 77933 30272 38241E-03	2.79945 85818 57821 32476 26708E-01
15	9.33248 68895 46916 41050 78704E-03	2.79974 60668 64074 92315 23611E-01
16	8.74994 22486 34897 75855 32187E-03	2.79998 15195 63167 28273 70300E-01
17	8.23581 40333 32272 17056 86402E-03	2.80017 67713 32972 53799 33377E-01
18	7.77872 35393 05375 26799 72603E-03	2.80034 04741 49935 09647 90137E-01
19	7.36968 17706 81732 93581 51537E-03	2.80047 90728 59058 51560 97584E-01
20	7.00149 36190 10578 81636 37648E-03	2.80059 74476 04231 52654 55059E-01
21	6.66833 17817 09748 34944 82384E-03	2.80069 93483 18094 30676 82601E-01
22	6.36542 65789 83807 59630 73182E-03	2.80078 76947 52875 34237 52200E-01
23	6.08883 64947 19034 14122 99798E-03	2.80086 47875 70755 70496 57907E-01
24	5.83527 59570 58510 53066 43695E-03	2.80093 24593 88085 05471 88973E-01
25	5.60198 43690 47656 71171 20585E-03	2.80099 21845 23828 35585 60292E-01
26	5.38662 53074 33782 78643 85926E-03	2.80104 51598 65567 04894 80682E-01
27	5.18720 80837 44558 98625 33989E-03	2.80109 23652 22061 85257 68354E-01
28	5.00202 60873 20540 78300 00200E-03	2.80113 46088 99502 83848 00112E-01
29	4.82960 78663 70683 06763 25939E-03	2.80117 25624 94996 17922 69045E-01
30	4.66867 79795 44380 47231 57306E-03	2.80120 67877 26628 28338 94384E-01
31	4.51812 54150 26788 45969 04817E-03	2.80123 77573 16608 84500 80987E-01
32	4.37697 79240 42686 22563 12526E-03	2.80126 58713 87319 18440 40017E-01
33	4.24438 10158 16735 10914 80065E-03	2.80129 14704 39045 17203 76843E-01
34	4.11958 06554 41361 92204 59561E-03	2.80131 48457 00126 10699 12501E-01
35	4.00190 89249 14714 95376 79571E-03	2.80133 62474 40300 46763 75700E-01
36	3.89077 20717 95432 18239 95251E-03	2.80135 58916 92711 17132 76581E-01
37	3.78564 04942 20766 17066 80897E-03	2.80137 39657 23366 96629 43864E-01
38	3.68604 03059 31353 89360 76349E-03	2.80139 06325 07828 95914 18025E-01
39	3.59154 61979 69006 18185 45295E-03	2.80140 60344 15824 82184 65330E-01
40	3.50177 53703 24974 26089 62568E-03	2.80142 02962 59979 40871 70054E-01

Table 2.6 The numbers $\left\{E_n(x^{\frac{5}{8}};[0,1])\right\}_{n=1}^{40}$ and the products $\left\{(2n)^{\frac{5}{4}}E_n(x^{\frac{5}{8}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{5}{8}};[0,1])$	$(2n)^{\frac{5}{4}}E_n(x^{\frac{5}{8}};[0,1])$
1	8.56645 82174 20506 05128 94161E-02	2.03745 86125 05998 52948 72411E-01
2	3.98832 53055 92170 97502 68410E-02	2.205613 74953 29706 82337 30572E-01
3	2.46263 41002 85478 13783 76902E-02	2.31253 83940 31673 16583 77529E-01
4	1.73502 52197 58018 34401 90662E-02	2.33436 23802 70889 47802 92050E-01
5	1.31863 92765 34615 67886 74134E-02	2.34490 90747 30128 34844 16834E-01
6	1.05252 67139 74341 96148 38477E-02	2.35076 75384 62292 12653 47809E-01
7	8.69381 29915 50372 98089 22437E-03	2.35434 62740 77294 78915 66548E-01
8	7.36465 06443 14552 33237 22986E-03	2.35668 82061 80656 74635 91355E-01
9	6.36075 87259 04069 21183 82768E-03	2.35830 27301 08786 34440 52216E-01
10	5.57860 36894 61179 51118 48636E-03	2.35946 20925 43903 91675 74542E-01
11	4.95385 12759 79454 92711 65022E-03	2.36032 23302 58038 14733 18128E-01
12	4.44455 09614 20568 78709 61656E-03	2.36097 80112 91596 01834 51965E-01
13	4.02225 18978 42024 53212 86014E-03	2.36148 91266 62504 25682 99244E-01
14	3.66701 83712 29253 74117 10891E-03	2.36189 52072 63284 07997 01335E-01
15	3.36449 08647 19817 63888 39483E-03	2.36222 31528 37818 82273 40345E-01
16	3.10407 94814 54856 29725 78162E-03	2.36249 17791 23246 12311 48873E-01
17	2.87781 32791 91449 45571 71183E-03	2.36271 45651 13584 60443 23213E-01
18	2.67958 45694 17517 06371 10509E-03	2.36290 13704 01437 77671 59948E-01
19	2.50463 91730 44923 50997 25521E-03	2.36305 95410 96103 89683 90530E-01
20	2.34922 44053 45154 27910 15228E-03	2.36319 46408 86409 44175 92829E-01
21	2.21034 08668 77184 70415 92540E-03	2.36331 09456 59041 25822 65423E-01
22	2.08556 41419 42231 87308 85484E-03	2.36341 17851 58317 21562 98934E-01
23	1.97291 45570 53180 06692 96416E-03	2.36349 97834 30203 84612 63969E-01
24	1.87076 05977 83603 10939 84636E-03	2.36357 70309 10531 40869 27181E-01
25	1.77774 62971 52898 60588 85711E-03	2.36364 52094 89351 78839 79550E-01
26	1.69273 59592 11511 88182 04100E-03	2.36370 56846 78522 83563 54441E-01
27	1.61477 15947 64033 70304 14781E-03	2.36375 95744 09693 35083 63154E-01
28	1.54303 97990 68065 04878 97997E-03	2.36380 78009 95422 39725 50132E-01
29	1.47684 57256 33846 19162 91417E-03	2.36385 11308 01817 41360 94515E-01
30	1.41559 24513 83066 47608 71837E-03	2.36389 02048 44370 74856 64633E-01
31	1.35876 44794 28844 86316 22399E-03	2.36392 55626 04520 29963 70763E-01
32	1.30591 44471 40939 97463 61801E-03	2.36395 76607 30469 95448 99856E-01
33	1.25665 23389 80074 61157 73860E-03	2.36398 68878 40364 57821 50013E-01
34	1.21063 66726 77617 20900 61703E-03	2.36401 35763 29137 58054 38572E-01
35	1.16756 72519 72893 22062 43157E-03	2.36403 80118 52513 54292 41062E-01
36	1.12717 91718 92179 95286 11647E-03	2.36406 04409 96041 36392 26436E-01
37	1.08923 78322 44156 70669 48295E-03	2.36408 10775 15459 97806 01173E-01
38	1.05353 47678 12082 77646 89617E-03	2.36410 01074 44616 15116 52465E-01
39	1.01988 41440 60420 46095 20144E-03	2.36411 76932 99817 78260 65131E-01
40	9.88119 79822 36023 15155 74049E-04	2.36413 39775 58755 33541 41055E-01

Table 2.7 The numbers $\left\{E_n(x^{\frac{3}{4}};[0,1])\right\}_{n=1}^{40}$
and the products $\left\{(2n)^{\frac{3}{2}}E_n(x^{\frac{3}{4}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{3}{4}};[0,1])$	$(2n)^{\frac{3}{2}}E_n(x^{\frac{3}{4}};[0,1])$
1	5.27343 75000 00000 00000 00000E-02	1.49155 33665 65373 68428 30311E-01
2	2.10499 41369 32078 98811 84740E-02	1.68399 53095 45663 19049 47792E-01
3	1.18077 79420 13000 16429 15090E-02	1.73538 20744 79284 49365 72463E-01
4	7.75843 03959 08188 44049 57467E-03	1.75553 23981 79363 41865 38537E-01
5	5.58246 91986 80911 02507 98277E-03	1.76533 17635 56671 88376 35929E-01
6	4.25986 76577 44793 48453 22760E-03	1.77079 37320 16018 72132 53034E-01
7	3.38684 34457 92223 03980 17226E-03	1.77413 70915 51353 23743 95222E-01
8	2.77551 22996 51750 31034 51760E-03	1.77632 78717 77120 19862 09126E-01
9	2.32800 44363 25493 47884 23229E-03	1.77783 95414 42772 43656 75017E-01
10	1.98889 94337 68184 66050 65861E-03	1.77892 57337 26600 64198 91089E-01
11	1.72472 79498 24355 97384 23183E-03	1.77973 20545 98104 75202 54440E-01
12	1.51421 57504 63441 42391 29027E-03	1.78034 68555 78047 42028 84127E-01
13	1.34326 44555 21703 26451 44801E-03	1.78082 62343 61395 60650 31758E-01
14	1.20220 18461 12590 06908 85230E-03	1.78120 71818 89327 55938 18876E-01
15	1.08419 54311 90818 81842 09057E-03	1.78151 48832 21754 89105 50599E-01
16	9.84296 48562 36347 49176 78151E-04	1.78176 69623 87351 59725 99724E-01
17	8.98842 26819 74662 20507 48822E-04	1.78197 60491 50441 56178 61438E-01
18	8.25070 08546 65244 31477 87944E-04	1.78215 13846 07692 77199 22196E-01
19	7.60861 09335 82573 42912 29810E-04	1.78229 98557 20532 35437 05871E-01
20	7.04566 00868 58775 27745 71266E-04	1.78242 66795 05080 65390 23616E-01
21	6.54883 94183 21856 18638 54082E-04	1.78253 58661 17556 67912 45245E-01
22	6.10776 51210 28255 33349 73375E-04	1.78263 05388 38191 74441 33492E-01
23	5.71405 90320 40865 20633 30799E-04	1.78271 31593 24491 33633 46630E-01
24	5.36089 47779 92470 80129 49516E-04	1.78278 56888 66618 15107 13538E-01
25	5.04266 04666 75800 20723 29464E-04	1.78284 97056 03889 34079 39993E-01
26	4.75470 47133 03977 83535 03120E-04	1.78290 64909 24811 07787 65024E-01
27	4.49314 31208 87924 89774 44627E-04	1.78295 70939 70426 61899 19414E-01
28	4.25470 92009 59948 64448 81724E-04	1.78300 23803 58846 71504 85932E-01
29	4.03663 83633 97386 98970 37043E-04	1.78304 30693 91498 65232 55660E-01
30	3.83657 67922 72660 60362 75536E-04	1.78307 97627 50146 41803 56890E-01
31	3.65250 92458 72075 08152 29990E-04	1.78311 29668 37654 98492 85112E-01
32	3.48270 13873 46932 78007 09207E-04	1.78314 31103 21629 58339 63114E-01
33	3.32565 33735 98010 00009 51936E-04	1.78317 05580 22746 85672 11694E-01
34	3.18006 22407 50505 46984 17017E-04	1.78319 56219 92768 87502 84756E-01
35	3.04479 12172 21207 53491 08453E-04	1.78321 85704 13710 97720 85119E-01
36	2.91884 45329 71644 25701 85133E-04	1.78323 96347 94410 74649 74723E-01
37	2.80134 66197 09326 97390 22336E-04	1.78325 90158 26779 70296 82413E-01
38	2.69152 48420 79778 57294 92670E-04	1.78327 68881 79563 80617 18026E-01
39	2.58869 50858 89504 45744 18379E-04	1.78329 34044 44302 12511 91127E-01
40	2.49224 96715 56483 10256 60075E-04	1.78330 86984 00582 50211 60107E-01

Table 2.8 The numbers $\left\{E_n(x^{\frac{7}{8}};[0,1])\right\}_{n=1}^{40}$ and the products $\left\{(2n)^{\frac{7}{4}}E_n(x^{\frac{7}{8}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{7}{8}};[0,1])$	$(2n)^{\frac{7}{4}}E_n(x^{\frac{7}{8}};[0,1])$
1	2.45434 93986 12976 07421 87500E-02	8.25541 44442 95046 75759 65186E-02
2	8.34746 06218 59568 01190 45596E-03	9.44407 36182 47320 77598 37409E-02
3	4.24868 26581 56727 15605 72621E-03	9.77279 93516 15869 55371 29480E-02
4	2.60244 74754 05414 79124 92647E-03	9.90351 69733 53562 27996 55449E-02
5	1.77250 26925 20828 17196 23793E-03	9.96751 51301 56243 61837 56651E-02
6	1.29293 55897 52071 59011 38915E-03	1.00033 17901 43122 46386 00750E-01
7	9.89401 42657 89072 20978 67617E-04	1.00252 81930 15314 32909 85150E-01
8	7.84351 15022 72376 70692 63043E-04	1.00396 94722 90864 21848 65669E-01
9	6.38886 96639 12825 51076 47339E-04	1.00496 49433 58227 90692 05234E-01
10	5.31688 94761 54876 79930 10117E-04	1.00568 07216 14153 15313 79459E-01
11	4.50246 08769 24362 39007 98476E-04	1.00621 23402 74043 54475 01812E-01
12	3.86807 55787 32884 19431 69683E-04	1.00661 78428 00515 14724 90342E-01
13	3.36354 85280 67543 89731 66182E-04	1.00693 41192 27184 66584 30052E-01
14	2.95517 30374 72720 14164 68463E-04	1.00718 55132 85200 20865 41768E-01
15	2.61959 91453 85459 87585 57004E-04	1.00738 86091 10897 65694 13542E-01
16	2.34021 79600 90179 93213 40468E-04	1.00755 50174 98722 66853 06756E-01
17	2.10494 21454 72886 48282 56809E-04	1.00769 30622 23668 20360 23636E-01
18	1.90479 73809 13474 12252 45572E-04	1.00780 88356 71608 48127 53286E-01
19	1.73300 22053 95899 31361 27938E-04	1.00790 68794 26300 44985 39720E-01
20	1.58435 14009 49230 21957 47720E-04	1.00799 06346 04793 60298 24680E-01
21	1.45479 33506 59923 61165 20638E-04	1.00806 27467 91170 75723 51553E-01
22	1.34113 44551 43426 29040 88890E-04	1.00812 52768 16658 51393 12795E-01
23	1.24082 86352 52370 43998 17505E-04	1.00817 98491 84650 95408 02947E-01
24	1.15182 49582 53035 64301 67861E-04	1.00822 77583 54343 76933 36687E-01
25	1.07245 56897 48139 28100 22263E-04	1.00827 00460 16238 56120 61623E-01
26	1.00135 29260 81233 48769 06316E-04	1.00830 75580 63334 14974 50412E-01
27	9.37385 73692 79339 02816 25974E-05	1.00834 09871 31499 54546 92214E-01
28	8.79612 22961 73059 68251 62569E-05	1.00837 09047 38696 32309 69970E-01
29	8.27242 60715 59833 96602 10531E-05	1.00839 77858 30067 45607 29498E-01
30	7.79610 42079 23276 25482 67851E-05	1.00842 20277 11833 28116 88292E-01
31	7.36149 99688 63152 54181 05640E-05	1.00844 39647 93074 94535 53285E-01
32	6.96378 56273 93910 50760 41378E-05	1.00846 38801 63256 41160 34668E-01
33	6.59881 98208 97016 51843 27299E-05	1.00848 20147 58346 93354 15393E-01
34	6.26303 28775 28148 72174 88560E-05	1.00849 85746 72774 93649 83798E-01
35	5.95333 39952 36669 72314 26266E-05	1.00851 37370 33693 88865 41556E-01
36	5.66703 56240 88379 11669 53306E-05	1.00852 76547 51702 73374 78452E-01
37	5.40179 14894 07367 48303 17181E-05	1.00854 04603 87015 83099 28634E-01
38	5.15554 55043 87123 80263 84454E-05	1.00855 22693 14381 16465 85801E-01
39	4.92648 94314 06558 12647 19565E-05	1.00856 31823 28403 60397 56105E-01
40	4.71302 76146 35370 71712 84604E-05	1.00857 32877 99538 12006 09865E-01

Table 2.9 The numbers $\left\{E_n(x^{\frac{5}{4}};[0,1])\right\}_{n=1}^{40}$ and the products $\left\{(2n)^{\frac{5}{2}}E_n(x^{\frac{5}{4}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{5}{4}};[0,1])$	$(2n)^{\frac{5}{2}}E_n(x^{\frac{5}{4}};[0,1])$
1	4.09600 00000 00000 00000 00000E-02	2.31704 75005 92078 92795 66868E-01
2	8.02524 26337 78619 78009 12148E-03	2.56807 76428 09158 32962 91887E-01
3	3.01196 44802 11402 88926 97435E-03	2.65599 93959 65835 27357 22491E-01
4	1.48803 03515 13865 86732 40839E-03	2.69362 26615 47152 76446 08920E-01
5	8.57820 81604 06160 28000 38316E-04	2.71266 76029 92649 01396 32108E-01
6	5.45979 83042 07701 31043 37602E-04	2.72351 46418 46247 11986 22890E-01
7	3.72289 52467 93096 10164 22305E-04	2.73024 05060 68400 77544 79172E-01
8	2.67059 02530 17198 29007 24304E-04	2.73468 44190 89611 04903 41687E-01
9	1.99166 08993 29419 32812 19724E-04	2.73776 81075 26424 25033 10375E-01
10	1.53170 24993 36673 35655 50320E-04	2.73999 27278 58502 52914 75669E-01
11	1.20768 91129 31596 46410 45988E-04	2.74164 89993 58584 22384 86782E-01
12	9.72040 12600 31670 52294 86484E-05	2.74291 46705 87626 39645 04729E-01
13	7.96040 97604 78546 60333 24509E-05	2.74390 32460 49521 30346 82415E-01
14	6.61604 12734 65470 43829 41372E-05	2.74468 99001 37942 03187 59938E-01
15	5.56917 39757 42218 44017 28778E-05	2.74532 59918 66267 64645 58935E-01
16	4.74025 25154 09923 32022 68018E-05	2.74584 75607 51469 48297 27765E-01
17	4.07424 92968 05693 09157 30007E-05	2.74628 04915 81717 02358 27702E-01
18	3.53220 64803 67711 98660 29427E-05	2.74664 37591 33932 84078 24483E-01
19	3.08597 21632 14778 09140 91901E-05	2.74695 15262 66161 86903 04465E-01
20	2.71482 97354 96205 06433 02821E-05	2.74721 45355 90553 44226 05113E-01
21	2.40328 43447 63621 14994 14606E-05	2.74744 10537 45450 19837 48854E-01
22	2.13957 61483 68854 11172 18301E-05	2.74763 75252 43280 07291 76331E-01
23	1.91466 13376 68234 74588 61520E-05	2.74780 90335 76377 19916 53269E-01
24	1.72150 09040 34394 53416 61088E-05	2.74795 96318 27960 67366 48629E-01
25	1.55455 59207 93585 77724 26252E-05	2.74809 25833 17104 90740 03358E-01
26	1.40942 39324 65847 25679 01904E-05	2.74821 05391 86619 19146 26873E-01
27	1.28257 33589 28201 02554 93888E-05	2.74831 56711 21484 08513 27459E-01
28	1.17114 70166 88517 84638 22362E-05	2.74840 97716 93465 63968 29126E-01
29	1.07281 50604 78442 05135 93551E-05	2.74849 43310 46976 68468 65310E-01
30	9.85663 71708 61017 23072 22245E-06	2.74857 05960 89467 06207 80429E-01
31	9.08110 25074 43540 64186 25047E-06	2.74863 96166 01376 19825 79554E-01
32	8.38837 36617 01329 99424 78624E-06	2.74870 22814 66291 81251 51395E-01
33	7.76742 16347 76041 27731 51666E-06	2.74875 93473 67509 76654 54974E-01
34	7.20896 09209 31234 41355 92033E-06	2.74881 14616 88764 54549 69787E-01
35	6.70513 29307 46223 27461 19951E-06	2.74885 91809 18793 06521 07918E-01
36	6.24925 39284 32718 63341 30767E-06	2.74890 29855 40612 08609 18630E-01
37	5.83561 29794 25635 31780 96580E-06	2.74894 32921 52144 78381 21106E-01
38	5.45930 89543 34390 67178 01901E-06	2.74898 04633 91111 55794 12428E-01
39	5.11611 82491 80383 99976 33394E-06	2.74901 48161 07141 64205 22954E-01
40	4.80238 68232 86260 80710 63098E-06	2.74904 66281 26041 38170 63039E-01

Table 2.10 The numbers $\left\{E_n(x^{\frac{3}{2}};[0,1])\right\}_{n=1}^{40}$

and the products $\left\{(2n)^3E_n(x^{\frac{3}{2}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{3}{2}};[0,1])$	$(2n)^3E_n(x^{\frac{3}{2}};[0,1])$
1	7.40740 74074 07407 40740 74074E-02	5.92592 59259 25925 92592 59259E-01
2	8.88347 64831 84405 50105 54526E-03	5.68542 49492 38019 52067 54897E-01
3	2.67294 65836 77282 44641 24325E-03	5.77356 46207 42930 08425 08543E-01
4	1.13749 83951 66505 22830 21751E-03	5.82399 17832 52506 76890 71366E-01
5	5.85203 18044 88704 15243 17104E-04	5.85203 18044 88704 15243 17104E-01
6	3.39625 27666 49120 08524 98039E-04	5.86872 47807 69679 50731 16611E-01
7	2.14261 46606 41863 00454 88048E-04	5.87933 46288 01272 08448 19204E-01
8	1.43712 22662 89229 04230 92578E-04	5.88645 28027 20682 15729 87199E-01
9	1.01019 24793 71104 77055 55952E-04	5.89144 25396 92283 02188 02313E-01
10	7.36883 47218 05607 28201 38835E-05	5.89506 77774 44485 82561 11068E-01
11	5.53886 24493 17342 67299 85824E-05	5.89778 07360 33106 47820 88905E-01
12	4.26783 99074 57107 38942 57451E-05	5.89986 18880 68705 25514 21500E-01
13	3.35769 92629 86524 12672 84812E-05	5.90149 22246 25114 80513 79785E-01
14	2.68895 43429 69042 64373 89799E-05	5.90279 25736 85642 41153 58087E-01
15	2.18660 96281 53936 80312 99136E-05	5.90384 59960 15629 36845 07668E-01
16	1.80197 48116 40924 78261 23081E-05	5.90471 10627 84982 32766 40112E-01
17	1.50250 10197 87793 61165 22997E-05	5.90543 00081 73944 01123 81988E-01
18	1.26586 80324 20383 83292 76156E-05	5.90603 38920 60542 81090 70832E-01
19	1.07642 25761 58226 04541 59826E-05	5.90654 59598 95417 95640 65797E-01
20	9.22966 23174 53926 95488 99538E-06	5.90698 38831 70513 25112 95704E-01
21	7.97343 87300 83821 47754 21978E-06	5.90736 12863 44501 65628 14635E-01
22	6.93520 94384 90618 55193 86693E-06	5.90768 88080 83848 50728 34361E-01
23	6.06967 08844 04173 61500 86294E-06	5.90797 48520 43646 42990 47995E-01
24	5.34236 30340 70336 50809 30531E-06	5.90822 61266 39066 55103 02693E-01
25	4.72675 84310 86558 50254 98482E-06	5.90844 80388 58198 12818 73103E-01
26	4.20221 11014 15198 02059 53643E-06	5.90864 49854 77881 63279 87298E-01
27	3.75248 98205 88424 49771 10532E-06	5.90882 05710 91356 75107 57328E-01
28	3.36471 49309 93427 46486 26421E-06	5.90897 77732 13417 57669 31775E-01
29	3.02857 79801 52254 43029 94607E-06	5.90911 90686 34666 66404 58837E-01
30	2.73576 22829 02938 10013 79673E-06	5.90924 65310 70346 29629 80093E-01
31	2.47950 80340 18100 42194 23109E-06	5.90936 19073 14658 37360 66708E-01
32	2.25428 26374 57929 44504 45466E-06	5.90946 66771 37714 56441 75762E-01
33	2.05552 84597 94394 17254 26248E-06	5.90956 21007 70491 47029 31447E-01
34	1.87946 81383 67024 78071 28141E-06	5.90964 92568 30203 35849 09158E-01
35	1.72295 30824 55467 27544 67541E-06	5.90972 90728 22252 75478 23666E-01
36	1.58334 46796 32020 08638 49727E-06	5.90980 23498 32922 33203 01828E-01
37	1.45842 04742 66788 63667 18504E-06	5.90986 97826 42851 58506 71391E-01
38	1.34629 95644 87830 45088 07279E-06	5.90993 19762 06098 60005 81842E-01
39	1.24538 28999 19491 48175 13077E-06	5.90998 94592 25945 21648 04658E-01
40	1.15430 52139 44783 08506 20711E-06	5.91004 26953 97289 39551 78039E-01

Table 2.11 The numbers $\left\{E_n(x^{\frac{7}{4}};[0,1])\right\}_{n=1}^{40}$
and the products $\left\{(2n)^{\frac{7}{2}}E_n(x^{\frac{7}{4}};[0,1])\right\}_{n=1}^{40}$.

n	$E_n(x^{\frac{7}{4}};[0,1])$	$(2n)^{\frac{7}{2}}E_n(x^{\frac{7}{4}};[0,1])$
1	1.01611 41224 89277 68285 83569E-01	1.14960 18983 54538 27744 03564E 00
2	5.63623 07912 92347 11510 36143E-03	7.21437 54128 54204 30733 26262E-01
3	1.32786 42459 61522 50259 28594E-03	7.02559 40766 29139 94189 85374E-01
4	4.83302 38522 98617 57411 85654E-04	6.99896 61482 94007 92801 59680E-01
5	2.21205 99166 19973 24396 82554E-04	6.99514 76572 81269 27030 05429E-01
6	1.16867 30496 71406 71758 66473E-04	6.99563 89997 59136 20371 08707E-01
7	6.81484 99779 58745 61771 75425E-05	6.99687 99836 85161 00572 95665E-01
8	4.27129 25430 00665 83781 33779E-05	6.99808 57024 52290 90867 34383E-01
9	2.82871 03581 42012 56563 06296E-05	6.99910 08066 70952 08681 20916E-01
10	1.95653 80363 84233 30810 05859E-05	6.99992 32798 70564 03065 27544E-01
11	1.40169 94989 77925 20672 23154E-05	7.00058 44821 93830 30618 93545E-01
12	1.03377 97896 28476 23574 64135E-05	7.00111 77734 93310 86786 41727E-01
13	7.81245 53534 61452 20273 06987E-06	7.00155 11572 10406 32646 22677E-01
14	6.02786 06113 83805 09140 09854E-06	7.00190 65595 01791 07063 68211E-01
15	4.73489 29736 57098 45446 92094E-06	7.00220 07604 20709 13711 31746E-01
16	3.77767 74847 95661 34402 85460E-06	7.00244 65393 51006 69071 35285E-01
17	3.05552 91945 92718 04757 07084E-06	7.00265 36583 07337 46290 93965E-01
18	2.50158 23703 18365 71196 14876E-06	7.00282 96241 74420 23943 65098E-01
19	2.07033 35208 33333 09148 55306E-06	7.00298 02576 80883 34493 04615E-01
20	1.73013 89578 44698 22301 30284E-06	7.00311 01124 85491 94504 05757E-01
21	1.45856 37835 74176 23130 58190E-06	7.00322 27831 72980 26242 41427E-01
22	1.23942 33489 41085 51197 89707E-06	7.00332 11323 47459 46097 56726E-01
23	1.06085 73349 21207 75680 90783E-06	7.00340 74590 95964 80336 89164E-01
24	9.14049 65008 80109 58541 70479E-07	7.00348 36249 00313 99029 75295E-01
25	7.92361 36152 04385 43589 28904E-07	7.00355 11485 16345 18360 33828E-01
26	6.90733 19978 78772 93732 67229E-07	7.00361 12780 70535 96492 50248E-01
27	6.05267 29541 03888 53783 96519E-07	7.00366 50463 03814 08375 29086E-01
28	5.32930 11972 07404 95249 35537E-07	7.00371 33132 49160 75625 73391E-01
29	4.71338 67827 42933 80051 62789E-07	7.00375 67994 63194 59736 49867E-01
30	4.18604 71762 95607 56879 04577E-07	7.00379 61120 99995 45186 78004E-01
31	3.73219 90018 32778 07605 39392E-07	7.00383 17655 18377 01024 29228E-01
32	3.33970 27004 63453 24547 68298E-07	7.00386 41976 82331 90065 82245E-01
33	2.99871 90376 46803 51914 67544E-07	7.00389 37833 00234 53254 84080E-01
34	2.70122 04859 96829 17725 52041E-07	7.00392 08444 17955 97309 32810E-01
35	2.44061 69766 39379 64384 28420E-07	7.00394 56590 10741 84823 21068E-01
36	2.21146 69062 45458 43586 94868E-07	7.00396 84679 91903 25250 61039E-01
37	2.00925 22628 18333 57867 47594E-07	7.00398 94809 61273 28051 68507E-01
38	1.83020 23771 88989 46024 92260E-07	7.00400 88809 44557 25396 93949E-01
39	1.67115 48416 69500 30553 07923E-07	7.00402 68283 20086 34855 30513E-01
40	1.52944 50491 27260 76307 30793E-07	7.00404 34640 87675 83603 53259E-01

Table 2.12 The numbers $\left\{E_n(x^{\frac{9}{4}};[0,1])\right\}_{n=1}^{40}$
and the products $\left\{(2n)^{\frac{9}{2}}E_n(x^{\frac{9}{4}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{9}{4}};[0,1])$	$(2n)^{\frac{9}{2}}E_n(x^{\frac{9}{4}};[0,1])$
1	1.45194 94105 24288 36121 60220E-01	3.28538 64771 88910 82721 21530E+00
2	7.01508 71176 00410 94333 14408E-03	3.59172 46042 11410 40298 56977E+00
3	8.85497 56243 15935 79310 91840E-04	2.81104 62865 80661 16859 57090E+00
4	2.28108 63185 39960 25202 60709E-04	2.64269 26765 03900 84705 70912E+00
5	8.15287 86027 10627 99019 50781E-05	2.57816 65871 41661 05909 57083E+00
6	3.54502 18457 45929 87008 37030E-05	2.54644 63053 44806 56540 33791E+00
7	1.75904 07104 34746 05339 19473E-05	2.52843 65008 75205 19051 25513E+00
8	9.60234 13406 33154 55863 52816E-06	2.51719 61683 98937 66861 88873E+00
9	5.63500 82388 81366 15972 01515E-06	2.50969 43232 20262 85157 17138E+00
10	3.50004 82060 64100 65803 76949E-06	2.50443 06282 51366 63074 20694E+00
11	2.27583 42548 97137 17829 65734E-06	2.50059 11721 51351 75300 00247E+00
12	1.53670 35771 94084 23573 42077E-06	2.49770 23930 60108 63330 23757E+00
13	1.07095 89471 66745 88990 32799E-06	2.49547 30802 52353 21916 25233E+00
14	7.66718 31999 49634 29523 87603E-07	2.49371 59407 74778 58581 92665E+00
15	5.61766 33520 18559 52380 61667E-07	2.49230 59600 80226 25382 85947E+00
16	4.19977 67423 34087 76451 95921E-07	2.49115 70441 90705 03188 97415E+00
17	3.19580 70429 11614 42502 63913E-07	2.49020 82993 55687 55775 07455E+00
18	2.47022 30066 62623 14504 05300E-07	2.48941 56513 35189 06182 82369E+00
19	1.93621 90397 25666 05504 22805E-07	2.48874 65511 49412 17077 78296E+00
20	1.53677 83319 59254 44266 25151E-07	2.48817 65313 46214 94438 55849E+00
21	1.23359 87304 57646 96904 88767E-07	2.48768 69077 13392 37184 44935E+00
22	1.00042 53824 09987 21558 19397E-07	2.48726 32101 85612 40522 27454E+00
23	8.18930 50278 96281 02899 28832E-08	2.48689 40896 62178 26767 96279E+00
24	6.76106 66346 73287 11515 24094E-08	2.48657 05424 15587 65719 61399E+00
25	5.62582 15406 67504 66158 66023E-08	2.48628 53507 19589 30855 59149E+00
26	4.71510 95316 65866 20586 14593E-08	2.48603 26733 67397 19459 91304E+00
27	3.97828 24787 52731 57920 53005E-08	2.48580 77418 19382 33201 59100E+00
28	3.37742 93840 32474 29540 63160E-08	2.48560 66319 13635 93523 79120E+00
29	2.88386 47056 84015 98427 14597E-08	2.48542 60903 78928 03846 73617E+00
30	2.47566 45486 96487 69533 01735E-08	2.48526 34015 99057 26737 25131E+00
31	2.13591 57216 97454 83511 99909E-08	2.48511 62842 88183 54208 72702E+00
32	1.85145 64712 60395 88710 31158E-08	2.48498 28106 34676 32347 52470E+00
33	1.61196 07014 52640 14239 96232E-08	2.48486 13424 93137 28382 74563E+00
34	1.40926 50325 21164 11851 96743E-08	2.48475 04806 33376 58581 74931E+00
35	1.23686 94701 09060 51153 23680E-08	2.48464 90240 77203 50758 69492E+00
36	1.08956 34971 38373 87343 66257E-08	2.48455 59372 93007 70712 93935E+00
37	9.63143 67016 33722 03406 58917E-09	2.48447 03235 58161 25012 56287E+00
38	8.54198 59451 96151 53165 62591E-09	2.48439 14031 97724 70300 05123E+00
39	7.59943 94772 41321 80491 21705E-09	2.48431 84957 04612 71369 90147E+00
40	6.78094 98538 61360 90662 69043E-09	2.48425 10049 69167 71467 67864E+00

Table 2.13 The numbers $\left\{E_n(x^{\frac{5}{2}};[0,1])\right\}_{n=1}^{40}$
and the products $\left\{(2n)^5 E_n(x^{\frac{5}{2}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{5}{2}};[0,1])$	$(2n)^5 E_n(x^{\frac{5}{2}};[0,1])$
1	1.62865 05699 56943 94291 08537E-01	5.21168 18238 62220 61731 47318E+00
2	1.47939 44809 96337 72829 42593E-02	1.51489 99485 40249 83377 33215E+01
3	1.18214 04692 91244 23479 59960E-03	9.19232 42892 08715 16977 36651E+00
4	2.49828 08356 39482 67178 31553E-04	8.18636 66428 78816 81889 90432E+00
5	7.81682 53492 17701 19637 23277E-05	7.81682 53492 17701 19637 23277E+00
6	3.06930 57765 27008 29312 48889E-05	7.63741 49498 47685 27594 85235E+00
7	1.40120 06484 36899 81379 73831E-05	7.53599 33754 49272 05455 76376E+00
8	7.12661 26018 89121 67231 48650E-06	7.47279 49356 38487 64666 92319E+00
9	3.93245 39829 57197 51397 09442E-06	7.43063 92076 68465 79207 90490E+00
10	2.31283 25568 58472 45740 74790E-06	7.40106 41819 47111 86370 39330E+00
11	1.43190 09232 61686 21769 27725E-06	7.37949 04189 50970 46546 04383E+00
12	9.24727 39618 48993 20552 00483E-07	7.36325 65583 19387 76741 10869E+00
13	6.18676 39825 99188 09881 22754E-07	7.35072 69100 51841 10967 13797E+00
14	4.26536 47458 13107 38645 47909E-07	7.34084 96929 67003 73444 05167E+00
15	3.01766 37387 09866 35656 28211E-07	7.33292 28850 64975 24644 76553E+00
16	2.18345 61213 38618 45028 77496E-07	7.32646 29948 44042 17641 25675E+00
17	1.61132 60031 70660 52589 86643E-07	7.32112 80156 28430 53542 68795E+00
18	1.21004 35085 38399 80756 35146E-07	7.31667 03754 94038 55225 01604E+00
19	9.22937 04955 63383 01297 57585E-08	7.31290 72175 02079 07681 48041E+00
20	7.13837 99491 28881 25499 25701E-08	7.30970 10679 07974 40511 23918E+00
21	5.59100 01167 43914 46899 48648E-08	7.30694 69336 94060 10455 56469E+00
22	4.42925 70590 02669 13082 67557E-08	7.30456 34929 60653 98977 31055E+00
23	3.54553 38131 42084 38003 59566E-08	7.30248 69566 33716 09755 32061E+00
24	2.86520 91920 95143 43040 44150E-08	7.30066 67129 59167 79596 17678E+00
25	2.33569 98994 72922 44495 01155E-08	7.29906 21858 52882 64046 91110E+00
26	1.91940 11427 05985 75400 57487E-08	7.29764 05348 22231 74207 54579E+00
27	1.58905 28604 41589 90251 78967E-08	7.29637 49480 19312 78187 78768E+00
28	1.32464 54404 04562 84923 86095E-08	7.29524 33596 43070 56464 79964E+00
29	1.11132 05242 57453 80960 09435E-08	7.29422 74751 36879 82378 96261E+00
30	9.37925 92881 69741 61248 25676E-09	7.29331 20224 80791 07786 64446E+00
31	7.96007 29466 19499 05295 85864E-09	7.29248 41715 13106 49380 82677E+00
32	6.79095 56249 94476 64542 33090E-09	7.29173 30794 84629 34318 22251E+00
33	5.82197 54657 26465 40522 64928E-09	7.29104 95324 02024 13230 21776E+00
34	5.01427 70070 09186 19891 65702E-09	7.29042 56597 41227 09896 71266E+00
35	4.33739 19830 13860 82815 60303E-09	7.28985 47058 51395 89388 18401E+00
36	3.76725 64066 64874 42383 86980E-09	7.28933 08455 20827 83775 13379E+00
37	3.28473 51402 09095 70695 46648E-09	7.28884 90342 09552 11878 23642E+00
38	2.87451 46685 10051 17214 93374E-09	7.28840 48856 91462 85804 31894E+00
39	2.52426 54729 45432 18910 33143E-09	7.28799 45715 15449 27906 12179E+00
40	2.22400 35211 01882 76758 41368E-09	7.28761 47379 46649 45281 96996E+00

Table 2.14 The numbers $\left\{E_n(x^{\frac{11}{4}};[0,1])\right\}_{n=1}^{40}$ and the products $\left\{(2n)^{\frac{11}{2}}E_n(x^{\frac{11}{4}};[0,1])\right\}_{n=1}^{40}$

n	$E_n(x^{\frac{11}{4}};[0,1])$	$(2n)^{\frac{11}{2}}E_n(x^{\frac{11}{4}};[0,1])$
1	1.78495 84271 97176 77604 05261E-01	8.07779 97312 46067 82813 46075E+00
2	2.29531 13865 66331 29850 34158E-02	4.70079 77196 87846 49933 49957E+01
3	8.73213 75344 52217 73395 38885E-04	1.66323 05156 93019 38060 91796E+01
4	1.47477 78542 25843 24210 11159E-04	1.36685 21364 25015 76416 26342E+01
5	4.00493 11874 48502 11678 76009E-05	1.26647 04424 58001 82824 24256E+01
6	1.41435 91657 69780 60681 91415E-05	1.21914 83704 71219 35975 29151E+01
7	5.92723 88070 58846 95438 22275E-06	1.19276 97639 04708 92312 54170E+01
8	2.80489 97221 84789 33398 43727E-06	1.17646 02124 35855 06426 87990E+01
9	1.45399 55666 10093 33142 00412E-06	1.16563 30703 82115 72062 39159E+01
10	8.09219 74867 51948 04158 32355E-07	1.15806 10347 34739 81973 36942E+01
11	4.76798 51587 04478 95848 44949E-07	1.15254 96400 50862 37603 78384E+01
12	2.94397 93256 35154 04975 74904E-07	1.14840 89942 99575 38214 80365E+01
13	1.89031 57477 80923 87764 58369E-07	1.14521 69472 07390 79925 21057E+01
14	1.25477 01191 63224 15027 87156E-07	1.14270 29283 60980 67422 59603E+01
15	8.57036 87928 83018 26456 56236E-08	1.14068 67883 02085 06451 66260E+01
16	6.00089 30215 89004 85683 57027E-08	1.13904 46941 79287 20695 49428E+01
17	4.29427 23596 78115 73805 07690E-08	1.13768 91842 65426 61638 56524E+01
18	3.13276 25350 59799 88040 07694E-08	1.13655 70248 66792 18055 85513E+01
19	2.32496 38178 78564 17558 09414E-08	1.13560 15587 79560 27694 63077E+01
20	1.75220 40592 54358 34264 66382E-08	1.13478 77381 39936 47997 11412E+01
21	1.33898 57956 55380 64930 02694E-08	1.13408 88147 12662 33070 13669E+01
22	1.03615 733[illegible]9 82858 51342 65663E-08	1.13348 40831 92020 60736 92633E+01
23	8.11045 97226 07716 34104 92046E-09	1.13295 73106 32464 30391 53912E+01
24	6.41519 30011 98588 88259 59444E-09	1.13249 56231 40934 80221 98436E+01
25	5.12324 86346 94376 86572 86691E-09	1.13208 87035 30348 46688 07845E+01
26	4.12784 84983 31339 85943 34433E-09	1.13172 82043 38673 29078 96242E+01
27	3.35315 24760 73564 99473 03792E-09	1.13140 73124 97929 53268 02203E+01
28	2.74457 24218 26856 64190 72267E-09	1.13112 04224 13227 24814 58376E+01
29	2.26233 01696 02348 32582 42787E-09	1.13086 28876 20620 03382 87675E+01
30	1.87710 66425 76869 19453 16864E-09	1.13063 08301 30300 41878 47711E+01
31	1.56706 10703 10308 98864 77024E-09	1.13042 09926 16945 20745 23101E+01
32	1.31576 16168 99836 85708 04919E-09	1.13023 06227 83376 04177 92277E+01
33	1.11073 08930 87327 12193 36322E-09	1.13005 73821 27563 10424 94712E+01
34	9.42411 78734 76660 25172 54428E-10	1.12989 92733 94072 57680 61518E+01
35	8.03424 16759 65144 56829 52801E-10	1.12975 45824 57186 73262 95549E+01
36	6.88025 20236 46813 24464 27064E-10	1.12962 18314 42662 74133 69997E+01
37	5.91713 77135 34796 81229 32044E-10	1.12949 97406 69101 60493 30468E+01
38	5.10937 88136 55715 63028 98748E-10	1.12938 71975 60751 13235 28720E+01
39	4.42876 42679 31851 40722 64057E-10	1.12928 32311 08459 46816 10333E+01
40	3.85275 09623 41573 01881 33378E-10	1.12918 69907 74483 62845 59615E+01

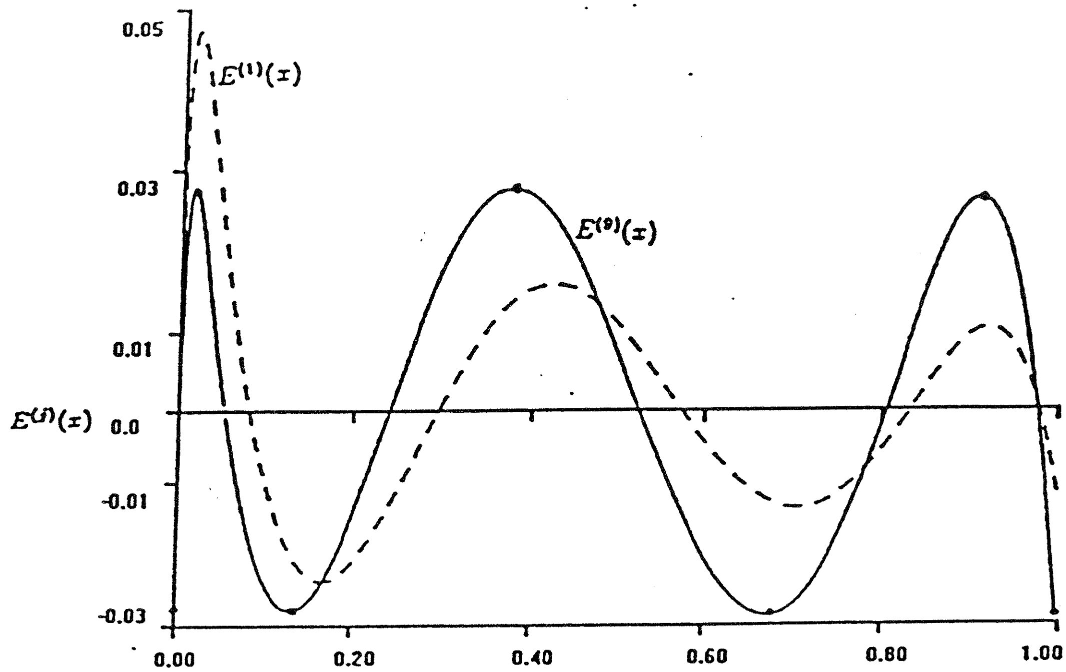

Figure 2.1 The error curves $E^{(1)}(x)$ and $E^{(9)}(x)$ for $x^{\frac{1}{2}}$ on [0,1] and $n = 5$

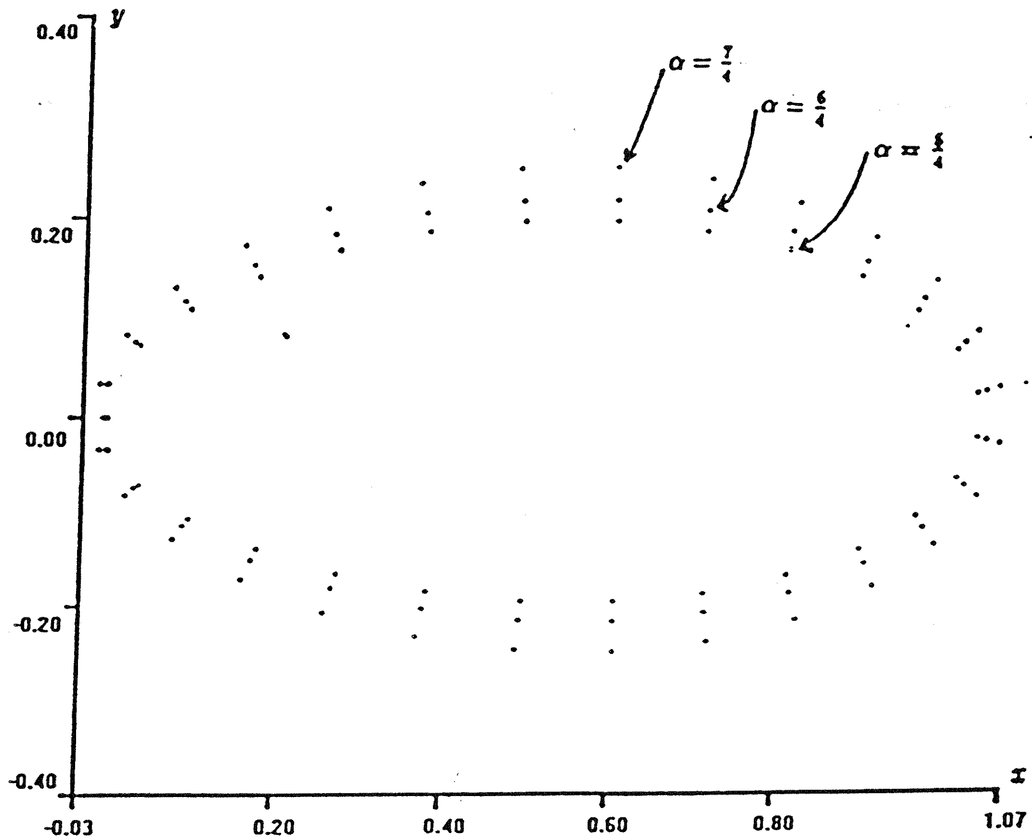

Figure 2.2 Zeros of the best uniform polynomial approximants to x^α on [0,1] for $n = 30$ and $\alpha = \frac{5}{4}$, $\frac{6}{4}$, and $\frac{7}{4}$

3. Extrapolation of the Products $\left\{(2n)^{2\alpha}E_n(x^\alpha;[0,1])\right\}_{n=1}^{40}$

As can be seen, the convergence of the products $\left\{(2n)^{2\alpha}E_n(x^\alpha;[0,1])\right\}_{n=1}^{40}$ in Tables 2.2 – 2.14 is quite slow. Thus, an algorithm is needed to accelerate the convergence of these products. The following scalar convergence accelerators were used on these products (cf. Brezinski [6]): the Richardson extrapolation method, Aitken's Δ^2 method, Wynn's ε algorithm, Brezinski's θ algorithm, and Wynn's ρ algorithm. The best candidate to emerge from the above list of convergence accelerators was the Richardson extrapolation method (cf. Brezinski [6, p. 6]), which can be described as follows: Let $\{S_i\}_{i=1}^n$, for $n \geq 2$, be a given finite sequence of real numbers. Then the 0^{th} and $(k+1)^{th}$ columns of the Richardson extrapolation table are defined by

$$T_0^{(i)} := S_i \ (1 \leq i \leq n), \text{ and} \tag{3.1}$$

$$T_{k+1}^{(i)} := \frac{x_i T_k^{(i+1)} - x_{i+k+1} T_k^{(i)}}{x_i - x_{i+k+1}} \qquad (1 \leq i \leq n-k-1), \tag{3.2}$$

for each $k = 0, 1, \cdots, n-2$, where $\{x_i\}_{i=1}^n$ are given constants. In this way, a triangular table, consisting of $\frac{1}{2}n(n+1)$ entries, is created. Preliminary calculations indicated that $(2i)^{2\alpha}E_i(x^\alpha;[0,1]) \approx \beta(\alpha) + \frac{k(\alpha)}{i^2} +$ lower-order terms, and so $x_i := 1/i^2$, for $1 \leq i \leq n$, were chosen as the required constants in (3.2).

The Richardson extrapolation of the products $\left\{(2i)^{2\alpha}E_i(x^\alpha;[0,1])\right\}_{i=1}^{40}$ produced unexpectedly beautiful results, the calculations having been done to 200 decimal digits with Brent's MP package [5]. The resulting extrapolation table had 40 columns consisting of 820 entries. Column 1 had 40 entries (the products $\left\{(2i)^{2\alpha}E_i(x^\alpha;[0,1])\right\}_{i=1}^{40}$), while column 40 had 1 entry (i.e., the final estimate of $\beta(\alpha)$). Rather than present all the extrapolation tables here, we give in Table 3.1 the last eight columns of the extrapolation table for $\alpha = \frac{3}{8}$, rounded to 80 decimal digits. On examining the entries of Table 3.1, it can be seen that *every* entry in this table has the same first 42 digits, and that the latter entries all agree to at least 50 significant digits! More importantly, what Table 3.1 shows, for the Richardson extrapolation for the particular case $\alpha = \frac{3}{8}$, actually holds for *all* values of α for which $\beta(\alpha)$ was calculated.

The success of the Richardson extrapolation method (with $x_i := 1/i^2$) applied to the products $\left\{(2i)^{2\alpha}E_i(x^\alpha;[0,1])\right\}_{i=1}^{40}$ gives strong numerical evidence for the following new conjecture (which extends the corresponding conjecture in [18] for the case $\alpha = \frac{1}{2}$):

CONJECTURE: $(2i)^{2\alpha}E_i(x^\alpha;[0,1])$ *admits an asymptotic series expansion of the form*

$$(2i)^{2\alpha}E_i(x^\alpha;[0,1]) \approx \beta(\alpha) - \frac{K_1(\alpha)}{i^2} + \frac{K_2(\alpha)}{i^4} - \frac{K_3(\alpha)}{i^6} + \cdots, \qquad (3.3)$$

as $i \to \infty$, *where the constants* $K_j(\alpha)$ $(j = 1, 2, \cdots)$ *are independent of* i.

If we assume that (3.3) is valid, it follows that

$$i^2\left((2i)^{2\alpha}E_i(x^\alpha;[0,1]) - \beta(\alpha)\right) \approx -K_1(\alpha) + \frac{K_2(\alpha)}{i^2} - \frac{K_3(\alpha)}{i^4} + \cdots, \text{ as } i \to \infty. \qquad (3.4)$$

We can apply Richardson extrapolation to the sequence

$$\left\{i^2\left((2i)^{2\alpha}E_i(x^\alpha;[0,1]) - \beta(\alpha)\right)\right\}_{i=1}^{40},$$

with $x_i = 1/i^2$, to obtain an extrapolated estimate for $K_1(\alpha)$ of (3.4). This boot-strapping procedure can be continued to give, via Richardson extrapolations, extrapolated estimates for the successive $K_j(\alpha)$'s of (3.3). In Table 3.2, extrapolated estimates of $\{K_j(\alpha)\}_{j=1}^{40}$ are given, rounded to 25 decimal digits, for the two cases $\alpha = \frac{1}{4}$ and $\alpha = \frac{1}{2}$. These estimates were computed to 200 decimal digits with Brent's MP package [5].

It turns out, for the cases we considered, that the constants $\{K_j(\alpha)\}_{j=1}^{40}$ in (3.3) are all *positive* for $\alpha \in (0, 2)$ with $\alpha \neq 1$. There are differences when $\alpha > 2$. For example, when $\alpha = \frac{9}{4}$, the constant $K_1(\frac{9}{4})$ is negative, while the constants $\{K_j(\frac{9}{4})\}_{j=2}^{40}$ are all positive.

With the values of $\{K_j(\frac{1}{2})\}_{j=1}^{40}$ from the right column of Table 3.2, we wish to show now how these numbers can be used in (3.3) for $\alpha = \frac{1}{2}$, to *estimate numerically* the numbers $\left\{2nE_n(x^{\frac{1}{2}};[0,1])\right\}_{n=1}^{10}$ of Table 2.5. Because the numbers $\{K_j(\frac{1}{2})\}_{j=1}^{40}$ are all positive in Table 3.2, then what is conjectured to be an *asymptotic series* in (3.3) for $\alpha = \frac{1}{2}$, namely

$$2nE_n(x^{\frac{1}{2}};[0,1]) \approx \beta(\tfrac{1}{2}) - \frac{K_1(\frac{1}{2})}{n^2} + \frac{K_2(\frac{1}{2})}{n^4} - \frac{K_3(\frac{1}{2})}{n^6} + \cdots, \text{ as } n \to \infty, \qquad (3.5)$$

is an *alternating* asymptotic series. The general rule used in summing such an alternating asymptotic series is (cf. Gradshteyn and Ryzhik [9, p. 18]) to add the terms, up to and including the *smallest* term in modulus, and discarding all subsequent terms in (3.5). In this case, the error (for a true alternating asymptotic series) is bounded above by the modulus of the first discarded term. This method of adding the alternating series in (3.5) was applied for each n with $n = 1, 2, \cdots, 10$, and in Table 3.3, we give, for each

n, the numerical estimates of $\left\{2nE_n(x^{\frac{1}{2}};[0,1])\right\}_{n=1}^{10}$, the number of terms added in (3.5), and the resulting number of digits of $\left\{2nE_n(x^{\frac{1}{2}};[0,1])\right\}_{n=1}^{10}$ from Table 2.5, which are correctly given by this sum.

The accuracy of the conjectured asymptotic series in (3.5), as judged by the last column of Table 3.3, is surprisingly good, even for very low values of n. We believe that this tends to give further *credence* to our conjecture in (3.3).

Acknowledgments. A very special thanks goes to Dr. Ewing L. Lusk of the Mathematics and Computer Science Division, Argonne National Laboratory, for introducing the first author to parallel processing and for teaching him how to use the p4 package. The Advanced Computing Research Facility at Argonne provided the ideal environment in which these computations were performed.

Table 3.1 The last eight columns of the extrapolation table for $\alpha = \frac{3}{8}$

Column 33
0.31524127414611071876466738566548249932993826824939969531741611252395671501432981
0.31524127414611071876466738566548249932993833001364646776197893809006183093591420
0.31524127414611071876466738566548249932993833298138747703642507364407574134051357
0.31524127414611071876466738566548249932993833313042559930641009750459240994018703
0.31524127414611071876466738566548249932993833313839926635481111181268243245197816
0.31524127414611071876466738566548249932993833313885902275467890696848713932699485
0.31524127414611071876466738566548249932993833313888776694702269803498655613444183
0.31524127414611071876466738566548249932993833313888972018764541714738893161164126
Column 34
0.31524127414611071876466738566548249932993833006712200609309977170527057995402813
0.31524127414611071876466738566548249932993833299110980630513742789650494334020104
0.31524127414611071876466738566548249932993833313146782393766873403508553349682811
0.31524127414611071876466738566548249932993833313849355953498436844899391830577613
0.31524127414611071876466738566548249932993833313886712276172379412768031598088873
0.31524127414611071876466738566548249932993833313888846377592800206084108745098600
0.31524127414611071876466738566548249932993833313888978189544071904068855533722654
Column 35
0.31524127414611071876466733566548249932993833299349868522687928872672654004231393
0.31524127414611071876466738566548249932993833313190236888389948111291395637656999
0.31524127414611071876466738566548249932993833313854005337349601602967419438171770
0.31524127414611071876466738566548249932993833313887130834409622466749753052066590
0.3152412741461107187646673856654824993299383331388882041052366061587385435389538
0.31524127414611071876466738566548249932993833313888981223578755907347276815302236
Column 36
0.31524127414611071876466738566548249932993833313200924431529486350085695824246516
0.31524127414611071876466738566548249932993833313355950446357541979499803390700751
0.31524127414611071876466738566548249932993833313887338590140312493073126572989972
0.31524127414611071876466738566548249932993833313888900658531624465253433354414898
0.31524127414611071876466738566548249932993833313888982797904571619184735408634183
Column 37
0.31524127414611071876466738566548249932993833313856429266543819797942357562898452
0.31524127414611071876466738566548249932993833313887425779428597966721941359607442
0.31524127414611071876466738566548249932993833313888909956557763226992601847161475
0.31524127414611071876466738566548249932993833313888983627595207449022425328373772
Column 38
0.31524127414611071876466738566548249932993833313887447260033507722902661791898994
0.31524127414611071876466738566548249932993833313888913870010971309419155994788512
0.31524127414611071876466738566548249932993833313888984044338976775231267824496280
Column 39
0.31524127414611071876466738566548249932993833313888914834885956482831337898869361
0.31524127414611071876466738563548249932993833313888984220214485560859869608530636
Column 40
0.31524127414611071876466738566548249932993833313888984263607436485861763624609223

Table 3.2 The numbers $\{K_j(\alpha)\}_{j=1}^{40}$ for the products $\left\{(2n)^{\frac{1}{2}}E_n(x^{\frac{1}{4}};[0,1])\right\}_{n=1}^{40}$ and $\left\{2nE_n(x^{\frac{1}{2}};[0,1])\right\}_{n=1}^{40}$

j	$K_j(\frac{1}{4})$ for $(2n)^{\frac{1}{2}}E_n(x^{\frac{1}{4}};[0,1])$	$K_j(\frac{1}{2})$ for $2nE_n(x^{\frac{1}{2}};[0,1])$
1	1.88287 39793 84817 89228 09318E-02	4.39675 28880 37559 56907 22422E-02
2	7.00185 85948 43410 47450 01816E-03	2.64071 68775 63975 40815 39953E-02
3	5.73516 97516 32046 08075 86193E-03	3.12534 26468 88366 21604 92812E-02
4	8.06896 10339 10935 34854 06498E-03	5.88900 16571 94464 07344 76045E-02
5	1.74579 86898 22971 29737 14933E-02	1.60106 99716 56970 12802 60793E-01
6	5.43809 24778 10764 14890 08586E-02	5.95435 31510 38536 37996 70917E-01
7	2.32493 06747 75236 92465 30680E-01	2.92591 54709 09529 94602 95871E+00
8	1.31310 12172 46725 96283 60417E+00	1.84941 40338 08461 76530 65579E+01
9	9.49486 35907 59470 93940 69982E+00	1.46943 01234 81390 08073 36135E+02
10	8.56541 78912 67836 39867 72242E+01	1.43803 27177 19218 36274 50095E+03
11	9.43599 97850 93765 79285 94514E+02	1.70262 52451 48179 70302 60874E+04
12	1.24700 85029 18428 51920 24258E+04	2.40118 63088 02885 85761 84991E+05
13	1.94752 59148 98658 72577 90066E+05	3.97938 55356 34528 46153 52389E+06
14	3.54888 23773 90288 38001 09713E+06	7.65939 47118 46820 15817 55294E+07
15	7.46332 98488 63627 69015 04457E+07	1.69481 50880 32764 21388 41074E+09
16	1.79420 05295 60613 89130 60764E+09	4.27276 88119 78247 93977 76088E+10
17	4.88971 13775 38487 47692 20864E+10	1.21765 42900 93742 38977 76490E+12
18	1.49957 83078 82119 28648 24578E+12	3.89510 79873 27815 60235 55055E+13
19	5.14140 87071 60691 14851 54086E+13	1.38987 13708 56504 27722 89745E+15
20	1.95917 79147 22797 36052 51848E+15	5.50104 16293 72886 50088 08098E+16
21	8.25372 63078 93983 97215 36782E+16	2.40283 18915 18267 43096 78599E+18
22	3.82590 58790 26863 79946 66492E+18	1.15294 25357 75728 36192 51101E+20
23	1.94284 65029 20831 32751 20729E+20	6.05164 23525 26015 18598 53359E+21
24	1.07656 07728 79016 03680 36217E+22	3.46139 35603 27124 64102 04503E+23
25	6.48565 36937 36696 87497 40010E+23	2.14984 92585 89692 81716 58944E+25
26	4.23378 83030 85676 73979 22890E+25	1.44521 08550 86315 35148 11714E+27
27	2.98550 41518 97459 60246 62921E+27	1.04835 78587 43688 17461 00059E+29
28	2.26752 07339 22410 12868 00710E+29	8.18281 06018 33942 82217 61899E+30
29	1.84953 46875 82409 12337 49776E+31	6.85254 13926 52250 61076 32825E+32
30	1.61458 33856 41541 53327 34624E+33	6.13524 03190 25207 75193 57221E+34
31	1.50073 51996 39063 86688 88165E+35	5.84094 79665 76950 20135 61201E+36
32	1.47195 35912 45356 18949 33492E+37	5.85710 21598 72912 73130 62501E+38
33	1.50062 87502 36148 80295 52069E+39	6.08952 24914 19377 29195 01726E+40
34	1.55533 93658 09662 18046 90055E+41	6.41699 96149 92711 87493 01466E+42
35	1.59427 33971 86612 52530 68539E+43	6.66604 10147 94249 78486 15605E+44
36	1.56879 83657 37115 21278 64428E+45	6.62788 48678 47571 75585 59365E+46
37	1.43989 06906 27776 24195 36193E+47	6.13133 94490 51024 25546 54637E+48
38	1.20074 39921 82110 70341 69152E+49	5.14335 38961 72948 37983 87026E+50
39	8.88579 04196 78705 74474 36614E+50	3.82318 47580 49471 00098 49618E+52
40	5.71045 56570 95263 56043 17223E+52	2.46526 33006 55214 86369 90342E+54

Table 3.3 The numerical estimates of $\left\{2nE_n(x^{\frac{1}{2}};[0,1])\right\}_{n=1}^{10}$

n	Numerical estimate of $2nE_n(x^{\frac{1}{2}};[0,1])$	Number of terms used	Number of correct digits
1	2.62609 13891 91332 91427 25402E-01	2	1
2	2.70558 78465 14674 71053 48747E-01	6	3
3	2.75574 18419 18367 11887 55676E-01	9	6
4	2.77517 82509 28654 28588 84202E-01	12	8
5	2.78451 18553 46228 66724 98914E-01	15	11
6	2.78967 91746 49567 94439 09082E-01	19	14
7	2.79282 94495 85180 28421 41933E-01	22	17
8	2.79488 83759 45074 47709 85560E-01	25	19
9	2.79630 65741 01282 01252 92946E-01	28	22
10	2.79732 43377 19738 29689 89387E-01	31	25

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