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Optimal semi-iterative methods applied to SOR in the mixed case

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Abstract. The application of optimal semi-iterative methods to the standard successive over-relaxation (SOR) iterative method, with any real relaxation parameter ω, is completely analyzed here, under the assumptions that the associated Jacobi matrix B is consistently ordered and weakly cyclic of index 2 and that the spectrum, $\sigma(B^2)$, of B^2 satisfies $\sigma(B^2) \subset [-\alpha^2, \beta^2]$ with $0 < \alpha < \infty$ and $0 < \beta < 1$. The spectrum of B^2 is then a mixture of positive and negative eigenvalues, the so-called "mixed case". If $\kappa(\Omega_{\omega,\alpha,\beta})$ denotes the optimal asymptotic convergence factor for semi-iteration applied to $\mathcal{L}_\omega$ (the associated SOR iteration matrix), we deduce that

$$1 > \min_{\omega \in \mathbf{R}} \rho(\mathcal{L}_\omega) > \min_{\omega \in \mathbf{R}} \kappa(\Omega_{\omega,\alpha,\beta}) = [(\sqrt{1+\alpha^2} - \sqrt{1-\beta^2})^2]/[\alpha^2 + \beta^2].$$

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1 Introduction

Recently in [5], the application of optimal semi-iteration to the standard successive overrelaxation (SOR) iterative method, with any real relaxation factor ω, was analyzed under the assumptions that the associated Jacobi matrix B is consistently ordered and weakly cyclic of index 2 and that the spectrum, $\sigma(B^2)$, of B^2 satisfies $\sigma(B^2) \subset [0, \beta^2]$, where $0 < \beta = \rho(B) < 1$. (This is the so-called "nonnegative case".) It was shown in [5, Theorem 1] that *no* semi-iterative method applied to the SOR method (for any real relaxation parameter ω) is asymptotically faster than the SOR method with optimal relaxation parameter $\omega = \omega_b$. The same statement is valid in the "nonpositive case" where $\sigma(B^2) \subset [-\alpha^2, 0]$ with $0 < \alpha = \rho(B)$ (cf. Theorem 1 in Section 2.

Here, we extend the results of [5], using similar techniques, to analyze the case where the Jacobi matrix B is consistently ordered and weakly cyclic of index 2

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with $\sigma(B^2) \subset [-\alpha^2, \beta^2]$ with $0 < \alpha < \infty$ and $0 < \beta < 1$ (the so-called "mixed case"). In contrast to the nonpositive and to the nonnegative case, there always exist semi-iterative methods which *improve* the asymptotic rate of convergence of the SOR method (with optimal relaxation parameter ω).

The outline of this paper is as follows: In Section 2, we review classical results on SOR and semi-iterative methods applied to SOR. The main theorem of this paper will be stated in Section 3 and proven in Section 5. The tools necessary for this proof will be developed in Section 4. Finally, in Section 6, we briefly comment on the question as to what extent our results carry over to the p-cyclic case.

2 A review of classical results on SOR

Consider the system of linear equations

$$A\mathbf{x} = \mathbf{b}, \text{ where } A \in \mathbb{R}^{N\times N} \text{ and } \mathbf{b} \in \mathbb{R}^N \text{ are given,} \tag{1}$$

with the standard splitting of the coefficient matrix A,

$$A = D - L - U,$$

where D is a nonsingular block diagonal matrix, and where L and U denote respectively strictly lower and strictly upper triangular matrices. We further assume that the corresponding block Jacobi matrix B, defined by

$$B := D^{-1}(L + U), \tag{2}$$

is consistently ordered and weakly cyclic of index 2 (cf. [9, Definition 4.2]), and that the spectrum, $\sigma(B^2)$, of the matrix B^2 consists of real numbers satisfying

$$\sigma(B^2) \subset [-\alpha^2, \beta^2] \text{ with } 0 \le \alpha < \infty \text{ and } 0 \le \beta < 1. \tag{3}$$

We further assume that the interval $[-\alpha^2, \beta^2]$ is a sharp enclosure of $\sigma(B^2)$, i.e., that

$$-\alpha^2, \beta^2 \in \sigma(B^2).$$

The assumption (3) implies that there is a unique solution $\mathbf{x}$ to the matrix equation (1). Since the case $\alpha = \beta = 0$ is essentially trivial and since the case of $\alpha = 0$ and $0 < \beta < 1$ was treated in [5, Theorem 1],* we assume that

$$\sigma(B^2) \subset [-\alpha^2, \beta^2] \text{ with } -\alpha^2, \beta^2 \in \sigma(B^2)\,,\ 0 < \alpha < \infty \text{ and } 0 < \beta < 1\,. \tag{4}$$

We next review classical results for the SOR iterative method

$$\mathbf{x}_m = \mathcal{L}_\omega \mathbf{x}_{m-1} + \mathbf{c}_\omega \qquad (m = 1, 2, \ldots), \tag{5}$$

where $\mathcal{L}_\omega$, the SOR matrix, and $\mathbf{c}_\omega$ are defined by

$$\mathcal{L}_\omega := (D - \omega L)^{-1}[(1-\omega)D + \omega U], \text{ and } \mathbf{c}_\omega := \omega(D - \omega L)^{-1}\mathbf{b} \qquad (\omega \in \mathbb{R}). \tag{6}$$

*The case of $0 < \alpha < \infty$ and $\beta = 0$ can be analogously analyzed, and we shall briefly discuss this at the end of this section.

Here, ω is the associated *relaxation parameter.* Under the given assumptions on B and ω, the SOR method of (5) converges (for any initial vector $\mathbf{x}_0$) to the solution of (1) if and only if

$$0 < \omega < \frac{2}{1+\alpha} \tag{7}$$

holds (cf. Young [11, p. 193]). The optimal relaxation parameter ω_b which minimizes $\rho(\mathcal{L}_\omega)$, the spectral radius of $\mathcal{L}_\omega$, as a function of ω, is given by (cf. [11, p. 195])

$$\omega_b = \frac{2}{1+\sqrt{1+a^2-\beta^2}}, \tag{8}$$

and there also holds

$$1 > \rho(\mathcal{L}_\omega) > \rho(\mathcal{L}_{\omega_b}) = \left(\frac{\alpha+\beta}{1+\sqrt{1+\alpha^2-\beta^2}}\right)^2, \tag{9}$$

for all $0 < \omega < 2/(1+\alpha)$ with $\omega \neq \omega_b$.

As in [5], we now apply, for any fixed real ω, a semi-iterative method to the iterates $\{\mathbf{x}_m\}_{m=0}^{\infty}$ which are generated from the SOR iterations of (5), i.e., we consider vector sequences $\{\mathbf{y}_m\}_{m=0}^{\infty}$ of the form

$$\mathbf{y}_m := \sum_{j=0}^{m} \pi_{m,j}\mathbf{x}_j, \qquad (m = 0, 1, \ldots), \tag{10}$$

where the coefficients $\pi_{m,j}$ are (complex) constants which satisfy the constraint $\sum_{j=0}^{m} \pi_{m,j} = 1 \quad (m = 0, 1, \ldots)$.

If $1 \notin \sigma(\mathcal{L}_\omega)$, it is well-known (cf. [9, p. 134]) that the associated error vectors $\mathbf{e}_m := (I - \mathcal{L}_\omega)^{-1}\mathbf{c}_\omega - \mathbf{y}_m$, for this semi-iterative method based on the basic iterative method of (5), satisfy

$$\mathbf{e}_m = p_m(\mathcal{L}_\omega)\mathbf{e}_0 \qquad (m = 0, 1, \ldots),$$

where $p_m(z) := \sum_{j=0}^{m} \pi_{m,j}z^j \in \Pi_m$, so that $p_m(1) = 1$. (Here, Π_m denotes the collection of all complex polynomials of degree at most m.)

For a given polynomial sequence $\{p_m\}_{m=0}^{\infty}$, with $p_m \in \Pi_m$ and $p_m(1) = 1$ for each $m \geq 0$, and for $1 \notin \sigma(\mathcal{L}_\omega)$, the quantity

$$\kappa\left(\mathcal{L}_\omega, \{p_m\}_{m=0}^{\infty}\right) := \limsup_{m\to\infty} \sup_{\mathbf{e}_0 \neq \mathbf{0}} \left[\frac{\|\mathbf{e}_m\|}{\|\mathbf{e}_0\|}\right]^{1/m}$$

(which depends on the structure of the Jordan canonical form of the matrix $\mathcal{L}_\omega$, but is independent of the vector norm $\|\cdot\|$ chosen on $\mathbf{R}^N$) measures the *asymptotic decay* of the norms of the error vectors $\mathbf{e}_m$ associated with (10).

As we shall see in Section 4 the eigenvalue assumption (4) leads to sharp inclusions $\sigma(\mathcal{L}_\omega) \subseteq \Omega = \Omega_{\omega,\alpha,\beta}$ for the spectrum of the SOR matrix $\mathcal{L}_\omega$, where $\Omega \subset \mathbf{C}$ is a compact set, and we call such a set Ω a *covering domain* for $\sigma(\mathcal{L}_\omega)$. In this setting, the best, i.e., *smallest*, asymptotic convergence factor for Ω which we can hope to achieve by *any* semi-iterative method is given by

$$\kappa(\Omega) := \lim_{m\to\infty} \left[\min\left\{\max_{z\in\Omega} |p(z)| \; : \; p \in \Pi_m,\, p(1) = 1\right\}\right]^{1/m}. \tag{11}$$

The quantity $\kappa(\Omega)$ is called the *asymptotic convergence factor* of the covering domain Ω, and this has been extensively studied in [4]. Note also that the definition of $\kappa(\Omega)$ in (11) couples *complex approximation theory* to the study of such semi-iterative methods.

With respect to the information $\sigma(\mathcal{L}_\omega) \subset \Omega$, the rate of convergence of the SOR iterative method (5) can therefore be improved, by the application of a semi-iterative method of the form (10), *only if*

$$\min\{1, \rho(\mathcal{L}_\omega)\} > \kappa(\Omega)$$

holds. The exact (nonempty) set of real ω's, for which the above inequality holds, will be precisely determined in Theorem 2 below.

How does one actually determine $\kappa(\Omega)$? We first note that $\kappa(\Omega) = 1$ holds for every compact set $\Omega \subset \mathbb{C}$ with $1 \in \Omega$. (For, if $1 \in \Omega$, then $\max_{z\in\Omega} |p(z)| \geq 1$ for any polynomial $p \in \Pi_m$ with $p(1) = 1$.) The same conclusion can also be drawn for another type of compact set Ω: Suppose that the complement $\mathbb{C}_\infty \backslash \Omega$ of Ω (with respect to the extended complex plane $\mathbb{C}_\infty$) is not connected, and that the points $z = 1$ and ∞ belong to different components of $\mathbb{C}_\infty\backslash\Omega$; then $\kappa(\Omega) = 1$. (In this case, there is a component of $\mathbb{C}_\infty\backslash\Omega$ containing $z = 1$, whose closure is compact and contains points of Ω. Then by the maximum principle, $\max_{z\in\Omega} |p(z)| \geq 1$ for any $p \in \Pi_m$ with $p(1) = 1$.)

Finally, on defining the class $\mathbb{M}$ by

$$\begin{aligned} \mathbb{M} := \quad & \{\Omega \subset \mathbb{C} : \Omega \text{ is compact and consists of more than one point,} \\ & \text{does not contain the point } z = 1, \\ & \text{and its complement } \mathbb{C}_\infty\backslash\Omega \text{ is simply connected}\}, \end{aligned} \tag{12}$$

then for $\Omega \in \mathbb{M}$,

$$\kappa(\Omega) = \frac{1}{|\Phi(1)|} \tag{13}$$

(cf. [4, Theorem 11]), where Φ is a conformal map from $\mathbb{C}_\infty\backslash\Omega$ onto the exterior of the unit circle with $\Phi(\infty) = \infty$. (We note, by the Riemann Mapping Theorem, that Φ exists and is unique, up to a constant factor of modulus 1.) Thus, if $\Omega \in \mathbb{M}$, the problem of determining its asymptotic convergence factor, $\kappa(\Omega)$, is reduced, from (13), to a problem in *conformal mapping theory.*

We mention that sharp covering domains Ω for the spectrum $\sigma(\mathcal{L}_\omega)$ will be determined and analyzed in Section4 for all real ω. It turns out, for fixed α and β with $0 < \alpha < \infty$ and $0 < \beta < 1$ and for any real ω, that there are essentially only *three* different types of covering domains $\Omega = \Omega_{\omega,\alpha,\beta}$ which need to be analyzed.

To conclude this section, we briefly describe the nonpositive case, where $\beta = 0$, i.e., B of (2) is consistently ordered and weakly cyclic of index 2 with

$$\sigma(B^2) \subset [-\alpha^2, 0] \text{ and } 0 < \alpha = \rho(B).$$

Matrices of this type arise for example in connection with a discretization of Theodorsen's integral equation (cf. Niethammer [8]). The optimal relaxation parameter ω_b, which minimizes $\rho(\mathcal{L}_\omega)$ as a function of ω, is then given by (cf. (8))

$$\omega_b = \frac{2}{1 + \sqrt{1 + \alpha^2}} \qquad (< 1),$$

and (cf. (9))

$$\rho(\mathcal{L}_{\omega_b}) = 1 - \omega_b = \left(\frac{\alpha}{1+\sqrt{1+\alpha^2}}\right)^2 .$$

In the nonpositive case it turns out, similar to the nonnegative case, that *no* semi-iterative method applied to the SOR method (for any real relaxation parameter ω) is asymptotically faster than the SOR method with optimal relaxation parameter $\omega = \omega_b$ (see Figure 1).

Theorem 1. *Assume that the Jacobi matrix B of (2) is a consistently ordered weakly cyclic of index 2 matrix, and that the eigenvalues of B^2 are all nonpositive and lie in $[-\alpha^2, 0]$, where $0 < \alpha = \rho(B)$. Then, there holds*

$$\min_{\omega\in\mathbf{R}} \rho(\mathcal{L}_\omega) = \min_{\omega\in\mathbf{R}} \kappa(\Omega_{\omega,\alpha,0}) = 1 - \omega_b = \left(\frac{\alpha}{1+\sqrt{1+\alpha^2}}\right)^2 .$$

More precisely, the asymptotic convergence factor $\kappa(\Omega_{\omega,\alpha,0})$ has the following properties:

(i) *For $-\infty \le \omega < \omega_1 := 2/(1-\sqrt{1+\alpha^2})$, $\kappa(\Omega_{\omega,\alpha,0}) = 1$, i.e., for these values of ω, no semi-iterative method applied to $\mathcal{L}_\omega$ converges.*

(ii) *For $\omega_1 < \omega < \omega_b$ and $\omega \neq 0$, $\kappa(\Omega_{\omega,\alpha,0})$ is a strictly decreasing function of ω which satisfies*

$$\min\{1, \rho(\mathcal{L}_\omega)\} > \kappa(\Omega_{\omega,\alpha,0}) > 1 - \omega_b .$$

(iii) *For $\omega_b \le \omega \le 1$, $\kappa(\Omega_{\omega,\alpha,0})$ is a constant function of ω which satisfies*

$$\min\{1, \rho(\mathcal{L}_\omega)\} \ge \kappa(\Omega_{\omega,\alpha,0}) = 1 - \omega_b$$

(where equality holds in the first inequality if and only if $\omega = \omega_b$),

(iv) *For $1 < \omega$, $\kappa(\Omega_{\omega,\alpha,0})$ is a strictly increasing function of ω which satisfies*

$$\min\{1, \rho(\mathcal{L}_\omega)\} > \kappa(\Omega_{\omega,\alpha,0}) > 1 - \omega_b .$$

3 Statement of the main result

To state our main result, Theorem 2 below, we define the following four specific real values of the relaxation factor ω (which are functions of α and β):

$$\omega_1 := \frac{2}{1-\sqrt{1+\alpha^2}}, \qquad \omega_2 := \frac{2}{1+\sqrt{1+\alpha^2}},$$
$$\omega_3 := \frac{2}{1+\sqrt{1-\beta^2}}, \qquad \omega_4 := \frac{2}{1-\sqrt{1-\beta^2}}. \tag{14}$$

We note that the assumptions $0 < \alpha < \infty$ and $0 < \beta < 1$ imply that

$$-\infty < \omega_1 < 0 < \omega_2 < 1 < \omega_3 < 2 < \omega_4 < \infty .$$

With the notation of (14), we come to the statement of our main result.

Theorem 2. *Assume that the Jacobi matrix B of (2) is a consistently ordered weakly cyclic of index 2 matrix, and that the eigenvalues of B^2 are (cf. (3)) all real and lie in $[-\alpha^2, \beta^2]$, where $0 < \alpha < \infty$, $0 < \beta < 1$ and $-\alpha^2, \beta^2 \in \sigma(B)$. Then, there holds*

$$\min\{1, \rho(\mathcal{L}_\omega)\} > \kappa(\Omega_{\omega,\alpha,\beta}) \quad (\omega \in \mathbb{R}) \tag{15}$$

if and only if $\omega \in (\omega_1, \omega_4) \setminus \{0\}$. More precisely, the asymptotic convergence factor $\kappa(\Omega_{\omega,\alpha,\beta})$ has the following properties:

(i) *For $\omega_1 < \omega < \omega_2$ and $\omega \neq 0$, $\kappa(\Omega_{\omega,\alpha,\beta})$ is a strictly decreasing function of ω which satisfies*

$$\min\{1, \rho(\mathcal{L}_\omega)\} > \kappa(\Omega_{\omega,\alpha,\beta}) > \frac{\left(\sqrt{1+\alpha^2} - \sqrt{1-\beta^2}\right)^2}{\alpha^2 + \beta^2}, \tag{16}$$

with $\lim_{\omega\uparrow 0} \kappa(\Omega_{\omega,\alpha,\beta}) = \lim_{\omega\downarrow 0} \kappa(\Omega_{\omega,\alpha,\beta}) = \left(\sqrt{1+\alpha^2} - \sqrt{1-\beta^2}\right)/\sqrt{\alpha^2+\beta^2}$.

(ii) *For $\omega_2 \leq \omega \leq \omega_3$, $\kappa(\Omega_{\omega,\alpha,\beta})$ is a constant function of ω which satisfies*

$$\min\{1, \rho(\mathcal{L}_\omega)\} > \kappa(\Omega_{\omega,\alpha,\beta}) = \frac{\left(\sqrt{1+\alpha^2} - \sqrt{1-\beta^2}\right)^2}{\alpha^2 + \beta^2}. \tag{17}$$

(iii) *For $\omega_3 < \omega < \omega_4$, $\kappa(\Omega_{\omega,\alpha,\beta})$ is a strictly increasing function of ω which satisfies*

$$\min\{1, \rho(\mathcal{L}_\omega)\} > \kappa(\Omega_{\omega,\alpha,\beta}) > \frac{\left(\sqrt{1+\alpha^2} - \sqrt{1-\beta^2}\right)^2}{\alpha^2 + \beta^2}. \tag{18}$$

(iv) *In all remaining cases, i.e., for $-\infty < \omega \leq \omega_1$, for $\omega_4 \leq \omega < \infty$, and for $\omega = 0$,*

$$\min\{1, \rho(\mathcal{L}_\omega)\} = \kappa(\Omega_{\omega,\alpha,\beta}) = 1, \tag{19}$$

i.e., neither the SOR method (5) nor any semi-iterative method applied to this SOR method can converge for these values of ω.

As a consequence of the above,

$$1 > \min_{\omega\in\mathbb{R}} \rho(\mathcal{L}_\omega) > \min_{\omega\in\mathbb{R}} \kappa(\Omega_{\omega,\alpha,\beta}) = \frac{\left(\sqrt{1+\alpha^2} - \sqrt{1-\beta^2}\right)^2}{\alpha^2 + \beta^2}. \tag{20}$$

The results of Theorem 2 can be seen in Figure 2 for $\alpha = 1$ and $\beta = 0.9$. For these choices of α and β, the relevant quantities of (8) and (9) are

$$\omega_b = 0.95653\ldots \text{ and } \rho(\mathcal{L}_{\omega_b}) = 0.82575\ldots,$$

the relevant quantities of (14) are

$$\omega_1 = -4.82842\ldots, \quad \omega_2 = 0.82842\ldots, \quad \omega_3 = 1.39286\ldots, \text{ and } \omega_4 = 3.54540\ldots,$$

while from (17), we have

$$\kappa(\Omega_{\omega,\alpha,\beta}) = 0.52879\ldots \text{ for all } \omega \in [\omega_2, \omega_3].$$

We again emphasize that the main result of Theorem 2 (cf. (20)),

$$\min_{\omega \in \mathbf{R}} \rho(\mathcal{L}_\omega) > \min_{\omega \in \mathbf{R}} \kappa(\Omega_{\omega,\alpha,\beta}) \tag{21}$$

crucially depends on the above assumptions that $\alpha > 0$ *and* $\beta > 0$. In [5, Theorem 1], we have shown that equality holds in (21) for the nonnegative case, i.e., for $\alpha = 0$ or, equivalently, $\sigma(B^2) \subset [0, \beta^2]$. In Theorem 1, we saw that equality also holds in (21) for the nonpositive case, i.e., for $\beta = 0$ or, equivalently, $\sigma(B^2) \subset [-\alpha^2, 0]$.

We add another remark. Instead of applying a semi-iterative method to the SOR iteration as we did in (10), we could have applied a semi-iterative method directly to the Jacobi method

$$\mathbf{x}_m = B\mathbf{x}_{m-1} + D^{-1}\mathbf{b} \qquad (m = 1, 2, \ldots). \tag{22}$$

From $\sigma(B) \subset [-\alpha^2, \beta^2]$, we conclude that

$$\sigma(B) \subset [-\beta, \beta] \cup [-i\alpha, i\alpha],$$

and one might ask what is the smallest asymptotic convergence factor $\kappa([-\beta, \beta] \cup [-i\alpha, i\alpha])$ of any semi-iterative method applied to (22) with respect to this information. It turns out (cf. [2, eq. (4.6)]) that

$$\kappa([-\beta, \beta] \cup [-i\alpha, i\alpha]) = \frac{\sqrt{1+\alpha^2} - \sqrt{1-\beta^2}}{\sqrt{\alpha^2+\beta^2}}, \tag{23}$$

and consequently (cf. (17)), the optimal semi-iterative method applied to the SOR iterative method with $\omega = 1$ is exactly *twice* as fast as the optimal semi-iterative method applied to the Jacobi iteration!

4 Covering domains $\Omega_{\omega,\alpha,\beta}$ and their asymptotic convergence factors

From our initial assumption that the Jacobi matrix B of (2) is consistently ordered and weakly cyclic of index 2, then Young's fundamental relationship

$$(\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2 \tag{24}$$

holds between the eigenvalues λ of $\mathcal{L}_\omega$ and the eigenvalues of μ of B (cf. [10] or [11, Theorem 5-2.2]). The relation (24) will be used to determine sharp covering domains $\Omega_{\omega,\alpha,\beta}$ for the eigenvalues of $\mathcal{L}_\omega$. In analogy with [11, pp. 203-206], it is necessary to distinguish between several cases. To this end, we first define certain "extremal" eigenvalues of $\mathcal{L}_\omega$, namely

$$\begin{aligned} \lambda_1 &= \lambda_1(\omega, \beta) := \left[\frac{\omega\beta - \sqrt{\omega^2\beta^2 - 4(\omega-1)}}{2}\right]^2, \\ \lambda_2 &= \lambda_2(\omega, \beta) := \left[\frac{\omega\beta + \sqrt{\omega^2\beta^2 - 4(\omega-1)}}{2}\right]^2 \end{aligned} \tag{25}$$

(for given real ω and $\mu^2 = \beta^2$, λ_1 and λ_2 are just the two roots of (24)), and

$$\begin{aligned} \lambda_3 &= \lambda_3(\omega,\alpha) := -\left[\frac{\omega\alpha - \sqrt{\omega^2\alpha^2 + 4(\omega-1)}}{2}\right]^2, \\ \lambda_4 &= \lambda_4(\omega,\alpha) := -\left[\frac{\omega\alpha + \sqrt{\omega^2\alpha^2 + 4(\omega-1)}}{2}\right]^2 \end{aligned} \tag{26}$$

(for given real ω and $\mu^2 = -\alpha^2$, λ_3 and λ_4 are again just the two roots of (24)). For these extremal eigenvalues of $\mathcal{L}_\omega$, it can be verified that

$$\begin{aligned} &\lambda_1 \text{ and } \lambda_2 \text{ are both nonreal if and only if } \omega \in (\omega_3, \omega_4)\,, \text{ and} \\ &\lambda_3 \text{ and } \lambda_4 \text{ are both nonreal if and only if } \omega \in (\omega_1, \omega_2) \setminus \{0\}\,. \end{aligned}$$

For any real ω, it follows from (14) that ω necessarily lies in one of the five disjoint real intervals: $-\infty < \omega \le \omega_1$, $\omega_1 < \omega < \omega_2$, $\omega_2 \le \omega \le \omega_3$, $\omega_3 < \omega < \omega_4$, and $\omega_4 \le \omega < \infty$.

Our immediate goal below is to determine sharp *covering domains* $\Omega_{\omega,\alpha,\beta}$ for the spectrum, $\sigma(\mathcal{L}_\omega)$, of the SOR iterative matrix $\mathcal{L}_\omega$, as a function of ω in these five intervals.

Case 1: $\omega \in (-\infty, \omega_1]$. Fixing any ω in $(-\infty, \omega_1]$, the image of the two roots λ of (24), for this fixed ω and for a variable μ^2 satisfying $0 \le \mu^2 \le \beta^2$, can be verified to be the real interval $[\lambda_2, \lambda_1]$, where λ_1 and λ_2 are given in (25). In the same fashion, the image of the two roots λ of (24), for a fixed ω and a variable μ^2 satisfying $-\alpha^2 \le \mu^2 \le 0$, can be verified, with (26), to be the union of the real interval $[\lambda_4, \lambda_3]$ and the circle $\partial\mathbb{D}(0; 1-\omega)$, where $\partial\mathbb{D}(a; b) := \{z \in \mathbb{C} : |z-a| = b\}$. More geometrically, for $\mu^2 = \beta^2$ these images are just λ_1 and λ_2 from (25), and decreasing μ^2 from β^2 moves these λ's toward one another until these images meet in the common point $1-\omega$ when $\mu^2 = 0$. On decreasing μ^2 further for $-\alpha^2 \le \mu^2 \le 0$, these images move, as conjugate complex pairs, along the circle $\partial\mathbb{D}(0; 1-\omega)$ until they meet in the point $\omega - 1$, where they separate and trace out the real interval $[\lambda_3, \lambda_4]$, where λ_3 and λ_4 are defined in (26). (This movement of the λ's, as μ^2 decreases from β^2 to $-\alpha^2$, is indicated by the arrows in Figure 3a.) Thus, from Young's fundamental relationship (24), the spectrum $\sigma(\mathcal{L}_\omega)$, for any ω in $(-\infty, \omega_1]$, satisfies

$$\begin{aligned} &\sigma(\mathcal{L}_\omega) \subset \Omega_{\omega,\alpha,\beta} := [\lambda_3, \lambda_4] \cup \partial\mathbb{D}(0; 1-\omega) \cup [\lambda_2, \lambda_1] \\ &\text{with } \lambda_3 \le \omega - 1 \le \lambda_4 < 1 < \lambda_2 < 1-\omega < \lambda_1\,, \end{aligned}$$

and this $\Omega_{\omega,\alpha,\beta}$ (cf. Figure 3a) is the associated *covering domain* for $\sigma(\mathcal{L}_\omega)$.

We note in this case that $1 \notin \Omega_{\omega,\alpha,\beta}$ since $\omega - 1 \le \lambda_4 < 1 < \lambda_2 < 1-\omega$. However (cf. (12)), $\Omega_{\omega,\alpha,\beta} \notin \mathbb{M}$ because the complement of $\Omega_{\omega,\alpha,\beta}$ is clearly not connected. Moreover, since 1 and ∞ are contained in different components of $C_\infty \setminus \Omega_{\omega,\alpha,\beta}$, we conclude from the discussion following (11) that

$$\kappa(\Omega_{\omega,\alpha,\beta}) = 1 \qquad (-\infty < \omega \le \omega_1)\,. \tag{27}$$

Consequently, *no* semi-iterative method applied to the SOR iterative method converges for *any* ω in this interval $(-\infty, \omega_1]$.

We further remark that the above covering domain $\Omega_{\omega,\alpha,\beta}$ is also the limiting case of the covering domain of Case 2 when $\omega \downarrow \omega_1$. (This continuity of the covering domains, in passing from one interval in ω to the next, is valid in all cases considered below.)

Case 2: $\omega \in (\omega_1, \omega_2)$ **and** $\omega \neq 0$. Fixing any $\omega \neq 0$ in (ω_1, ω_2), the use of (24), as in Case 1, similarly determines a covering domain $\Omega_{\omega,\alpha,\beta}$ for $\sigma(\mathcal{L}_\omega)$. Specifically, it can be verified that if $\omega \in (\omega_1, \omega_2)$ with $\omega < 0$, then

$$\begin{aligned}&\sigma(\mathcal{L}_\omega) \subset \Omega_{\omega,\alpha,\beta} := \{z : |z| = 1-\omega \text{ and } |\arg z| \leq \arg \lambda_4\} \cup [\lambda_2, \lambda_1] \\ &\text{with } 1 < \lambda_2 < 1-\omega < \lambda_1\,,\end{aligned} \tag{28}$$

while if $\omega \in (\omega_1, \omega_2)$ with $\omega > 0$, then

$$\begin{aligned}&\sigma(\mathcal{L}_\omega) \subset \Omega_{\omega,\alpha,\beta} := \{z : |z| = 1-\omega \text{ and } |\arg z| \leq \arg \lambda_3\} \cup [\lambda_1, \lambda_2] \\ &\text{with } \lambda_1 < 1-\omega < \lambda_2 < 1\,.\end{aligned} \tag{29}$$

For either $\omega < 0$ or $\omega > 0$ of this case, $\Omega_{\omega,\alpha,\beta}$ has the form

$$D_{\tau,\theta,\zeta,\eta} := \left\{\tau e^{i\varphi} : -\theta \leq \varphi \leq \theta\right\} \cup [\zeta, \eta], \tag{30}$$

with $\tau > 0$ and $0 < \theta < \pi$, and with either $\zeta < \tau < \eta < 1$ or $1 < \zeta < \tau < \eta$. In either situation, we see that $1 \notin D_{\tau,\theta,\zeta,\eta}$ and that (cf. Figure 3b) $\mathbb{C}_\infty \setminus D_{\tau,\theta,\zeta,\eta}$ is simply connected. Consequently,

$$D_{\tau,\theta,\zeta,\eta} \in \mathbb{M}.$$

(Its asymptotic convergence factor $\kappa(D_{\tau,\theta,\zeta,\eta})$ will be obtained later in closed form in Proposition 5.)

Case 3: $\omega \in [\omega_2, \omega_3]$. For any $\omega \in [\omega_2, \omega_3]$, it can be similarly verified that

$$\begin{aligned}&\sigma(\mathcal{L}_\omega) \subset \Omega_{\omega,\alpha,\beta} := [\lambda_4, \lambda_3] \cup \partial\mathbb{D}(0; |1-\omega|) \cup [\lambda_1, \lambda_2] \\ &\text{with } \lambda_4 \leq -|\omega - 1| \leq \lambda_3 \leq 0 \leq \lambda_1 \leq |\omega - 1| \leq \lambda_2 < 1\,,\end{aligned} \tag{31}$$

where the associated covering domain $\Omega_{\omega,\alpha,\beta}$ is shown in Figure 3c.

Since $|1-\omega| \leq \lambda_2 < 1$ from (31), the critical point $z = 1$ is now not in $\Omega_{\omega,\alpha,\beta}$. In addition (cf. Figure 3c), the intervals $(-|1-\omega|, \lambda_3]$ and $[\lambda_1, |1-\omega|)$, which are *interior* to the closed disk $\overline{\mathbb{D}}(0; |1-\omega|)$, have no effect on the determination of the asymptotic convergence factor of $\Omega_{\omega,\alpha,\beta}$, as can be seen from applying the maximum principle to the expression in (11). Hence, the new covering domain in this case, with the identical asymptotic convergence factor, is of the form

$$B_{\tau,\zeta,\eta} := \overline{\mathbb{D}}(0; \tau) \cup [\zeta, \eta]\,, \quad \text{where } \zeta \leq -\tau < \tau \leq \eta < 1, \tag{32}$$

with the choices $\tau = |1-\omega|, \zeta = \lambda_4$ and $\eta = \lambda_2$ (cf. Figure 3d). Now, $B_{\tau,\zeta,\eta} \in \mathbb{M}$ so that

$$\kappa(\Omega_{\omega,\alpha,\beta}) = \kappa(B_{\tau,\zeta,\eta})\,, \quad \text{where } \tau := |1-\omega|, \zeta := \lambda_4 \text{ and } \eta := \lambda_2. \tag{33}$$

(The asymptotic convergence factor $\kappa\left(B_{\tau,\zeta,\eta}\right)$ will be obtained later in closed form in Proposition 3.)

We remark that if $\omega = 1$ (i.e., SOR is the Gauss–Seidel method), then from (25) and (26),

$$\lambda_1 = \lambda_3 = 0\,, \quad \lambda_4 = -\alpha^2\,, \text{ and } \lambda_2 = \beta^2,$$

so that $\Omega_{1,\alpha,\beta}$ reduces in this case to the interval $[-\alpha^2, \beta^2]$. For the extremal values of ω for Case 3 (i.e., ω_2 and ω_3), we also remark that a similar reduction takes place, in that it can be verified that

$$\omega = \omega_2 \text{ implies } \lambda_3 = \lambda_4 = \omega_2 - 1 \text{ and } \Omega_{\omega_2,\alpha,\beta} = \partial\mathbb{D}(0; 1-\omega_2) \cup [\lambda_1, \lambda_2]\,,$$

and

$$\omega = \omega_3 \text{ implies } \lambda_1 = \lambda_2 = \omega_3 - 1 \text{ and } \Omega_{\omega_3,\alpha,\beta} = [\lambda_4, \lambda_3] \cup \partial\mathbb{D}(0; \omega_3 - 1)\,.$$

Case 4: $\omega_3 < \omega < \omega_4$. Fixing any ω in (ω_3, ω_4), it can be verified that

$$\begin{aligned}&\sigma\left(\mathcal{L}_\omega\right) \subset \Omega_{\omega,\alpha,\beta} := \{z : |z| = \omega - 1 \text{ and } \arg\lambda_2 \le \arg z \le \arg\lambda_1\} \cup [\lambda_4, \lambda_3]\\ &\text{with } \operatorname{Im}\lambda_2 > 0 \text{ and } \lambda_4 < 1 - \omega < \lambda_3 < 0\,.\end{aligned}$$

With the choice $\tau = |\omega - 1|$, $\theta = \arg\lambda_2$, $\zeta = \lambda_4$, and $\eta = \lambda_3$, the associated covering domain has the form

$$C_{\tau,\theta,\zeta,\eta} := \left\{\tau e^{i\varphi} : \theta \le \varphi \le 2\pi - \theta\right\} \cup [\zeta, \eta], \tag{34}$$

where $\tau > 0, 0 < \theta < \pi$ and $\zeta < -\tau < \eta < 1$. This is shown in Figure 3e. Again, $1 \notin C_{\tau,\theta,\zeta,\eta}$, and $C_\infty \setminus C_{\tau,\theta,\zeta,\eta}$ is simply connected, so that

$$C_{\tau,\theta,\zeta,\eta} \in \mathbf{M}.$$

(Its convergence factor $\kappa(C_{\tau,\theta,\zeta,\eta})$ will be obtained later in closed form in Proposition 4.)

Case 5: $\omega_4 \le \omega < \infty$. Fixing any ω in $[\omega_4, \infty)$, it can be verified that

$$\begin{aligned}&\sigma\left(\mathcal{L}_\omega\right) \subset \Omega_{\omega,\alpha,\beta} := [\lambda_4, \lambda_3] \cup \partial\mathbb{D}(0; \omega - 1) \cup [\lambda_1, \lambda_2]\\ &\text{with } \lambda_4 < 1 - \omega < \lambda_3 < 1 < \lambda_1 \le \omega - 1 \le \lambda_2\,,\end{aligned}$$

which is shown in Figure 3f. Now, $1 \notin \Omega_{\omega,\alpha,\beta}$, but as in Case 1, since the complement of $\Omega_{\omega,\alpha,\beta}$ is not connected with 1 and ∞ in different components, we again have, from the discussion following (11), that

$$\kappa\left(\Omega_{\omega,\alpha,\beta}\right) = 1 \qquad (\omega_4 \le \omega < \infty)\,, \tag{35}$$

i.e., no semi-iterative method applied to the SOR iterative method converges for any ω in $[\omega_4, \infty)$.

With the above five cases for the determination of the associated covering domains $\Omega_{\omega,\alpha,\beta}$, only Cases 2-4 have any interest for us since the remaining two (cf. (27) and (35)) result in no convergence via semi-iterative methods for the SOR iterative method. But in Cases 2-4 (i.e., $D_{\tau,\theta,\zeta,\eta}$ for Case 2, $B_{\tau,\zeta,\eta}$ for Case 3, and

$C_{\tau,\theta,\zeta,\eta}$ for Case 4), each has an associated covering domain in $\mathbb{M}$ from which, using (13), its asymptotic convergence factor can be determined. We now explicitly determine the asymptotic convergence factors for the three cases. We begin with the simplest case of

Proposition 3. *The asymptotic convergence factor* $\kappa(B_{\tau,\zeta,\eta})$ *of the set* $B_{\tau,\zeta,\eta}$ *of* (32) *is given by*

$$\kappa\left(B_{\tau,\zeta,\eta}\right) = t - \sqrt{t^2-1}\,,$$

where

$$t := \frac{(\zeta+\eta)(\tau^2+\zeta\eta) - 2\zeta\eta(\tau^2+1)}{(\eta-\zeta)(\tau^2-\zeta\eta)} \ (>1)\,, \tag{36}$$

provided that $\zeta \leq -\tau < \tau \leq \eta < 1$.

Proof. Since, as noted above, $B_{\tau,\zeta,\eta} \in \mathbb{M}$, then (13) can be applied to determine $\kappa\left(B_{\tau,\zeta,\eta}\right)$. To this end, we explicitly construct a conformal mapping function Φ which maps $\mathbb{C}_\infty \setminus B_{\tau,\zeta,\eta}$ onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(0;1)$, where Φ is normalized by $\Phi(\infty) = \infty$. The mapping Φ can be expressed as the composition of three elementary mappings. The Joukowski-like transformation

$$u = \Phi_1(z) := \frac{z}{\tau} + \frac{\tau}{z}$$

maps $\mathbb{C}_\infty \setminus B_{\tau,\zeta,\eta}$ in the z-plane conformally onto $\mathbb{C}_\infty \setminus [\Phi_1(\zeta), \Phi_1(\eta)]$ in the u-plane, the linear transformation

$$v = \Phi_2(u) := \frac{2u - \Phi_1(\zeta) - \Phi_1(\eta)}{\Phi_1(\eta) - \Phi_1(\zeta)}$$

maps $\mathbb{C}_\infty \setminus [\Phi_1(\zeta), \Phi_1(\eta)]$ in the u-plane conformally onto $\mathbb{C}_\infty \setminus [-1,1]$ in the v-plane, and, finally, the inverse Joukowski transformation

$$w = \Phi_3(v) := v + \sqrt{v^2-1},$$

maps $\mathbb{C}_\infty \setminus [-1,1]$ in the v-plane conformally onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(0;1)$ in the w-plane, where the branch of the square root is chosen so that $\Phi_3(v) > 1$ for all $v > 1$. (This choice guarantees that $|\Phi_3(v)| > 1$ for all $v \notin [-1,+1]$.)

The composition of these mappings, namely $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$, then maps $\mathbb{C}_\infty \setminus B_{\tau,\zeta,\eta}$ conformally onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(0;1)$ with $\Phi(\infty) = \infty$, and as $B_{\tau,\zeta,\eta} \in \mathbb{M}$, it follows from (13) that

$$\kappa\left(B_{\tau,\zeta,\eta}\right) = \frac{1}{|\Phi(1)|} = \frac{1}{|(\Phi_3 \circ \Phi_2 \circ \Phi_1)(1)|}$$

has the desired form given in Proposition 3. □

Next, we determine the asymptotic convergence factor $\kappa\left(C_{\tau,\theta,\zeta,\eta}\right)$ for the set $C_{\tau,\theta,\zeta,\eta}$ of (34), which is also an element of $\mathbb{M}$. This determination again makes use of (13), as well as Proposition 3.

Proposition 4. *The asymptotic convergence factor* $\kappa\left(C_{\tau,\theta,\zeta,\eta}\right)$ *of the set* $C_{\tau,\theta,\zeta,\eta}$ *of* (34) *with* $\tau > 0, 0 < \theta < \pi$ *and* $\zeta < -\tau < \eta < 1$*, satisfies*

$$\kappa\left(C_{\tau,\theta,\zeta,\eta}\right) = \kappa(B_{\hat{\tau},\hat{\zeta},\hat{\eta}}), \tag{37}$$

where $B_{\hat{\tau},\hat{\zeta},\hat{\eta}}$ *is defined from* (32) *with*

$$\hat{\tau} := 2\cos\left(\frac{\theta}{2}\right)\frac{\tau}{1+\tau+\sqrt{1-2\tau\cos(\theta)+\tau^2}},$$

$$\hat{\zeta} := \frac{\zeta+\tau-\sqrt{\zeta^2-2\zeta\tau\cos(\theta)+\tau^2}}{1+\tau+\sqrt{1-2\tau\cos(\theta)+\tau^2}} \quad \text{and} \quad \hat{\eta} := \frac{\eta+\tau+\sqrt{\eta^2-2\eta\tau\cos(\theta)+\tau^2}}{1+\tau+\sqrt{1-2\tau\cos(\theta)+\tau^2}}.$$

Proof. We explicitly construct a conformal map Υ of $\mathbb{C}_\infty \backslash C_{\tau,\theta,\zeta,\eta}$ onto $\mathbb{C}_\infty \backslash B_{\hat{\tau},\hat{\zeta},\hat{\eta}}$ with $\Upsilon(\infty) = \infty$ and $\Upsilon(1) = 1$. In view of (13), this will give the assertion of Proposition 4.

As similarly in [5, Proposition 4], we consider the three elementary conformal mappings

$$\begin{aligned} u = \Upsilon_1(z) &:= i\left[\frac{z-\tau\cos(\theta)}{\tau\sin(\theta)}\right], \\ v = \Upsilon_2(u) &:= u+\sqrt{u^2-1}, \text{ and} \\ w = \Upsilon_3(v) &:= \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)v. \end{aligned}$$

A straight-forward computation shows that the composition of these three maps can be expressed as

$$\tilde{\Upsilon}(z) := \left(\Upsilon_3 \circ \Upsilon_2 \circ \Upsilon_1\right)(z) = \frac{z+\tau+\sqrt{z^2-2\tau z\cos(\theta)+\tau^2}}{2\tau\cos(\theta/2)}. \tag{38}$$

In [5, Proposition 4], we showed that $\tilde{\Upsilon}$ maps $\mathbb{C}_\infty \setminus \left\{\tau e^{i\varphi} : \theta \leq \varphi \leq 2\pi-\theta\right\}$ conformally onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(0;1)$ with $\tilde{\Upsilon}(\infty) = \infty$, provided that we choose the branch of the square root in (38) for which $\tilde{\Upsilon}(1) > 0$. Now, with $\hat{\tau} := 1/\tilde{\Upsilon}(1) > 0$ (which, from(38), agrees with the definition of $\hat{\tau}$ in Proposition 4),

$$\Upsilon(z) := \hat{\tau}\tilde{\Upsilon}(z)$$

is then the conformal map from $\mathbb{C}_\infty \setminus \left\{\tau e^{i\varphi} : \theta \leq \varphi \leq 2\pi-\theta\right\}$ onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(0;\hat{\tau})$ with $\Upsilon(\infty) = \infty$ and $\Upsilon(1) = 1$. Moreover, $[\zeta, -\tau) \cup (\tau, \eta]$ is mapped by Υ onto $[\hat{\zeta}, -\hat{\tau}) \cup (\hat{\tau}, \hat{\eta}]$ (whose definitions are given in Proposition 4), and it can be verified, from the assumptions on τ, θ, ζ, and η in Proposition 4, that $\hat{\zeta} < -\hat{\tau} < \hat{\tau} < \hat{\eta} < 1$. Consequently, Υ is a conformal map of $\mathbb{C}_\infty \backslash C_{\tau,\theta,\zeta,\eta}$ onto $\mathbb{C}_\infty \backslash B_{\hat{\tau},\hat{\zeta},\hat{\eta}}$ with $\Upsilon(1) = 1$. Appealing again to (13), we conclude that $\kappa(C_{\tau,\theta,\zeta,\eta}) = \kappa(B_{\hat{\tau},\hat{\zeta},\hat{\eta}})$, which is the desired result (37) of Proposition 4. □

Next, the sets $C_{\tau,\theta,\zeta,\eta}$ and $D_{\tau,\theta,\zeta,\eta}$, from Figures 3b and 3e, appear to be simply "reflections" of one another. This simplifies the proof of a

Proposition 5. *The asymptotic convergence factor* $\kappa(D_{\tau,\theta,\zeta,\eta})$ *of the set* $D_{\tau,\theta,\zeta,\eta}$ *of* (30)*, with* $\tau > 0, 0 < \theta < \pi$ *and* $\zeta < \tau < \eta < 1$ *or* $1 < \zeta < \tau < \eta$*, satisfies*

$$\kappa(D_{\tau,\theta,\zeta,\eta}) = \kappa(B_{\check{\tau},\check{\zeta},\check{\eta}}), \tag{39}$$

where $B_{\check{\tau},\check{\zeta},\check{\eta}}$ *is defined from* (32) *with*

$$\check{\tau} := 2\sin\left(\frac{\theta}{2}\right)\frac{\tau}{1-\tau+\sqrt{1-2\tau\cos(\theta)+\tau^2}},$$

$$\check{\zeta} := \frac{\zeta-\tau-\sqrt{\zeta^2-2\zeta\cos(\theta)+\tau^2}}{1-\tau+\sqrt{1-2\tau\cos(\theta)+\tau^2}} \quad \textit{and} \quad \check{\eta} := \frac{\eta-\tau+\sqrt{\eta^2-2\eta\tau\cos(\theta)+\tau^2}}{1-\tau+\sqrt{1-2\tau\cos(\theta)+\tau^2}},$$

provided that $\zeta < \tau < \eta < 1$*, and with*

$$\check{\tau} := 2\sin\left(\frac{\theta}{2}\right)\frac{\tau}{-1+\tau+\sqrt{1-2\tau\cos(\theta)+\tau^2}},$$

$$\check{\zeta} := \frac{-\eta+\tau-\sqrt{\eta^2-2\eta\tau\cos(\theta)+\tau^2}}{-1+\tau+\sqrt{1-2\tau\cos(\theta)+\tau^2}} \quad \textit{and}\; \check{\eta} := \frac{-\zeta+\tau+\sqrt{\zeta^2-2\zeta\tau\cos(\theta)+\tau^2}}{-1+\tau+\sqrt{1-2\tau\cos(\theta)+\tau^2}},$$

provided that $1 < \zeta < \tau < \eta$*.*

Proof. This is a direct consequence of Proposition 4 because $z \mapsto -z$ maps $\mathbb{C}_\infty \setminus D_{\tau,\theta,\zeta,\eta}$ onto $\mathbb{C}_\infty \setminus C_{\tilde{\tau},\tilde{\theta},\tilde{\zeta},\tilde{\eta}}$, where $\tilde{\tau} := \tau$, $\tilde{\theta} := \pi - \theta$, $\tilde{\zeta} := -\eta$, and $\tilde{\eta} := -\zeta$. □

As mentioned at the end of Section 2, there are essentially only *three* different types of covering domain $\Omega_{\omega,\alpha,\beta}$ which needed to be analyzed. The asymptotic convergence factors of these three different covering domains now have been determined in the three Propositions 3-5.

Also mentioned in Section 2 is that *sharp* covering domains will be derived in this section. To demonstrate this point, fix any α and β with $0 < \alpha < \infty$ and $0 < \beta < 1$, and fix any real ω with $\omega \neq 0$. Next, let λ be any real or complex number such that if (cf. (24))

$$(\lambda+\omega-1)^2 = \lambda\omega^2\mu^2, \quad \text{then } \mu^2 \text{ satisfies } -\alpha^2 \le \mu^2 \le \beta^2. \tag{40}$$

By definition, λ is then an arbitrary point of the covering domain $\Omega_{\omega,\alpha,\beta}$. Next, with a value of μ (possibly complex), determined by (40) from the given values of λ and ω, we define the matrix $\tilde{B}$ by

$$\tilde{B} := \left[\begin{array}{c|c|c} \begin{matrix}0 & \beta\\ \beta & 0\end{matrix} & O & O \\ \hline O & \begin{matrix}0 & \mu\\ \mu & 0\end{matrix} & O \\ \hline O & O & \begin{matrix}0 & \alpha\\ -\alpha & 0\end{matrix}\end{array}\right], \tag{41}$$

whose eigenvalues are $\pm\beta$, $\pm\mu$, and $\pm i\alpha$. It can be verified, from the case $p = 2$ of [9, Definition 4.2 and Theorem 4.1] that $\tilde{B}$ is a consistently ordered weakly cyclic of index 2 matrix. Moreover, since the diagonal entries of $\tilde{B}$ of (41) are all zero, then on setting $\tilde{B} := \tilde{L} + \tilde{U}$, where $\tilde{L}$ and $\tilde{U}$ are respectively the strictly lower and strictly upper triangular matrix determined from $\tilde{B}$, an associated 6×6 SOR matrix $\tilde{\mathcal{L}}_\omega$ for $\tilde{B}$ can be defined by

$$\tilde{\mathcal{L}}_\omega := (I - \omega\tilde{L})^{-1}[(1-\omega)I + \omega\tilde{U}].$$

But as μ is an eigenvalue of $\tilde{B}$ and as (24) is satisfied, it follows (cf. [9, Theorem 4.3]) that λ is an eigenvalue of $\tilde{\mathcal{L}}_\omega$. In other words, for any $\omega \neq 0$ and for any α and β with $0 < \alpha < \infty$ and $0 < \beta < 1$, each point of the covering domain $\Omega_{\omega,\alpha,\beta}$ *is* an eigenvalue of some SOR matrix derived from a consistently ordered weakly cyclic of index 2 Jacobi matrix B whose spectrum satisfies $\sigma(B^2) \subset [-\alpha^2, \beta^2]$. It is in this sense that the covering domains $\Omega_{\omega,\alpha,\beta}$, for $\omega \neq 0$, are *sharp*.

5 Proof of Theorem 2

With the results of Propositions 3–5, we are in position to establish Theorem 2.
Proof. As seen from the discussion in Section 3, there were five real intervals in ω to be separately considered, namely Cases 1–5. As we have seen (cf. (27) and (35)), Cases 1 and 5 both give an asymptotic convergence factor $\kappa(\Omega_{\omega,\alpha,\beta}) = 1$, so that no semi-iterative method applied to the SOR iterative method is convergent when $-\infty < \omega \leq \omega_1$ (Case 1) and $\omega_4 \leq \omega < \infty$ (Case 5).

For Case 2 (i.e., $\omega \in (\omega_1, \omega_2)$ and $\omega \neq 0$), we distinguish between the two subcases $\omega_1 < \omega < 0$ and $0 < \omega < \omega_2$. Since the treatment of each of these subcases is similar, we consider only the subcase $\omega_1 < \omega < 0$. Then (cf. (28) and the following inequalities),

$$1 < \lambda_2 < \lambda_1\,,\ \operatorname{Im}(\lambda_4) > 0,$$

and $\sigma(\mathcal{L}_\omega) \subset D_{\tau,\theta,\zeta,\eta}$ with $\tau := 1-\omega$, $\theta := \arg(\lambda_4)$, $\zeta := \lambda_2$ and $\eta := \lambda_1$. Inserting these definitions into the expression for $\check{\tau}$, $\check{\zeta}$, and $\check{\eta}$ of Proposition 5, we obtain (after some algebraic manipulations and simplifications) that

$$\check{\tau} := \sqrt{(1-\omega_2)(1-\omega)},$$

$$\check{\zeta} := -\frac{\beta + \sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\alpha^2}}\sqrt{\lambda_1} \quad \text{and} \quad \check{\eta} := \frac{\beta + \sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\alpha^2}}\sqrt{\lambda_2}\,.$$

Now, these values $\check{\tau}$, $\check{\zeta}$, $\check{\eta}$ determine an associated $\check{t}$ from (36) of Proposition 3, which, after simplifications, can be expressed in terms of ω, α, and β as

$$\check{t} := \frac{(\check{\zeta}+\check{\eta})(\check{\tau}^2+\check{\zeta}\check{\eta}) - 2\check{\zeta}\check{\eta}(\check{\tau}^2+1)}{(\check{\eta}-\check{\zeta})(\check{\tau}^2-\check{\zeta}\check{\eta})} = \frac{(1-\beta^2-\sqrt{1+\alpha^2})\omega + 2\sqrt{1+\alpha^2}}{\sqrt{\alpha^2+\beta^2}\sqrt{\beta^2\omega^2-4(\omega-1)}}. \tag{42}$$

Hence, from Propositions 3 and 5, it follows that the asymptotic convergence factor $\kappa(\Omega_{\omega,\alpha,\beta})$ for the covering domain $\Omega_{\omega,\alpha,\beta}$ when $\omega_1 < \omega < 0$ is given by

$$\kappa(\Omega_{\omega,\alpha,\beta}) = \kappa(D_{\tau,\theta,\zeta,\eta}) = \kappa\left(B_{\check{\tau},\check{\zeta},\check{\eta}}\right) = \check{t} - \sqrt{\check{t}^2-1}\,, \tag{43}$$

where $\check{t} > 1$ is explicitly given, as a function of ω, α, and β, in the final expression of (42). With α and β fixed with $\alpha > 0$ and $0 < \beta < 1$, differentiation of $\check{t} = \check{t}(\omega)$ with respect to ω yields, after some manipulations, that

$$\frac{d\check{t}}{d\omega} = \frac{2(1-\beta^2)(\sqrt{1+\alpha^2}-1)}{\beta^3\sqrt{\alpha^2+\beta^2}} \cdot \frac{(\omega-\omega_1)}{[(\omega_3-\omega)(\omega_4-\omega)]^{3/2}} > 0 \qquad (\omega_1 < \omega < 0)\,, \tag{44}$$

so that for $\omega_1 < \omega < 0$ $(< \omega_3 < \omega_4)$, $\check{t}$ is a strictly increasing function of ω in the interval $(\omega_1, 0)$. On the other hand, as

$$\frac{d}{d\check{t}}\left\{\check{t} - \sqrt{\check{t}^2-1}\right\} = \frac{(\check{t}^2-1)^{1/2} - \check{t}}{\sqrt{\check{t}^2-1}} < 0 \qquad (\text{for all } \check{t} > 1)\,, \tag{45}$$

then $\check{t} - \sqrt{\check{t}^2-1}$ is a strictly decreasing function of $\check{t}$ in the interval $[1, \infty)$. Hence (cf. (43)), it follows from (44) and (45) that $\kappa(\Omega_{\omega,\alpha,\beta})$ is a *strictly decreasing* function of ω in the interval $(\omega_1, 0)$, as claimed in (i) of Theorem 2. (This can also be seen in Figure 2.) As mentioned above, the treatment of the remaining subcase of (i) of Theorem 2, namely, $0 < \omega < \omega_2$, is similar and is omitted.

We now turn to case (ii) of Theorem 2, i.e., $\omega \in [\omega_2, \omega_3]$. From our discussion of Case 3 in Section 4, we know from (33) and (13) that

$$\kappa(\Omega_{\omega,\alpha,\beta}) = \kappa(B_{\tau,\zeta,\eta}) \text{ with } \tau := |1-\omega|\,,\ \zeta := \lambda_4\,,\ \text{and } \eta := \lambda_2\,, \tag{46}$$

where λ_2 and λ_4, defined in (25) and (26), are both real and nonzero from (31). We specifically treat now the subcase when $1 < \omega \le \omega_3$, so that $\tau = \omega - 1 > 0$ and λ_2 and λ_4 are both real and nonzero. From (36) of Proposition 3, the associated value for t in (36) (for the values $\tau = \omega - 1$, $\zeta = \lambda_4$ and $\eta = \lambda_2$) is given by

$$t = \frac{2(\frac{1}{\omega-1} + \omega - 1) - (\frac{\zeta}{\omega-1} + \frac{\omega-1}{\zeta} + \frac{\eta}{\omega-1} + \frac{\omega-1}{\eta})}{\frac{\eta}{\omega-1} + \frac{\omega-1}{\eta} - \frac{\zeta}{\omega-1} - \frac{\omega-1}{\zeta}}\,. \tag{47}$$

Since $\zeta = \lambda_4$ and $\eta = \lambda_2$ are particular roots of (24), then

$$\frac{\eta}{\omega-1} + \frac{\omega-1}{\eta} = \frac{\omega^2\beta^2}{\omega-1} - 2 \text{ and } \frac{\zeta}{\omega-1} + \frac{\omega-1}{\zeta} = -\frac{\omega^2\alpha^2}{\omega-1} - 2\,,$$

and substituting these expressions in (47) yields

$$t = \frac{2-(\beta^2-\alpha^2)}{\alpha^2+\beta^2}.$$

Thus by Proposition 3,

$$\kappa(\Omega_{\omega,\alpha,\beta}) = t - \sqrt{t^2-1} = \frac{(\sqrt{1+\alpha^2} - \sqrt{1-\beta^2})^2}{\alpha^2+\beta^2}\,, \tag{48}$$

for all $1 < \omega \le \omega_3$. The case $\omega_2 \le \omega < 1$ is similar, and gives the same result in (48). For $\omega = 1$, the covering domain $\Omega_{\omega,\alpha,\beta}$ reduces to the interval $[-\alpha^2, \beta^2]$ whose asymptotic convergence factor is known to be

$$\kappa([-\alpha^2, \beta^2]) = \frac{(\sqrt{1+\alpha^2} - \sqrt{1-\beta^2})^2}{\alpha^2+\beta^2}$$

(cf. [2, §6]). This, together with (9), establishes the assertion (17) of Theorem 2.

Next, let ω satisfy $\omega_3 < \omega < \omega_4$ (cf. (14)), which corresponds to Case 4 of Section 4. Note that in this case,

$$\lambda_4 < \lambda_3 < 0 \text{ and } \mathrm{Im}(\lambda_2) > 0$$

(cf. (25) and (26)). Thus, $\sigma(\mathcal{L}_\omega) \subset C_{\tau,\theta,\zeta,\eta}$ (cf. (34)) with $\tau := \omega - 1$, $\theta := \arg(\lambda_2)$, $\zeta := \lambda_4$, and $\eta := \lambda_3$. Inserting into the expressions for $\hat{\tau}$, $\hat{\zeta}$, and $\hat{\eta}$ which are defined in Proposition 4, we obtain similarly

$$\hat{\tau} := \sqrt{(1-\omega_3)(1-\omega)},$$

$$\hat{\zeta} := -\frac{\alpha + \sqrt{\alpha^2+\beta^2}}{1+\sqrt{1-\beta^2}}\sqrt{-\lambda_4} \text{ and } \hat{\eta} := \frac{\alpha + \sqrt{\alpha^2+\beta^2}}{1+\sqrt{1-\beta^2}}\sqrt{-\lambda_3}\,.$$

A further calculation shows that

$$\hat{t} := \frac{(\hat{\zeta}+\hat{\eta})(\hat{\tau}^2+\hat{\zeta}\hat{\eta}) - 2\hat{\zeta}\hat{\eta}(\hat{\tau}^2+1)}{(\hat{\eta}-\hat{\zeta})(\hat{\tau}^2-\hat{\zeta}\hat{\eta})} = \frac{(\alpha^2+1-\sqrt{1-\beta^2})\omega + 2\sqrt{1-\beta^2}}{\sqrt{\alpha^2+\beta^2}\sqrt{\alpha^2\omega^2+4(\omega-1)}}\,. \tag{49}$$

Hence, from Propositions 3 and 4, we have

$$\kappa(\Omega_{\omega,\alpha,\beta}) = \kappa(\mathbb{C}_{\tau,\zeta,\eta,\theta}) = \kappa\left(B_{\hat{\tau},\hat{\zeta},\hat{\eta}}\right) = \hat{t} - \sqrt{\hat{t}^2-1} \qquad (\text{where } \hat{t} > 1). \tag{50}$$

Then, using (49), we deduce that

$$\frac{d\hat{t}}{d\omega} = \frac{2(\alpha^2+1)(1-\sqrt{1-\beta^2})}{\sqrt{\alpha^2+\beta^2}}\,\frac{(\omega-\omega_4)}{[(\omega-\omega_1)(\omega-\omega_2)]^{3/2}} < 0 \qquad (\omega_3 < \omega < \omega_4), \tag{51}$$

so that $\hat{t}$ is a strictly decreasing function of $\omega \in (\omega_2, \omega_4)$. But as $\hat{t} - \sqrt{\hat{t}^2-1}$ is also a strictly decreasing function of $\hat{t} > 1$ from (45), it follows that $\kappa(\Omega_{\omega,\alpha,\beta})$ of (50) is a then strictly increasing function of ω in (ω_3, ω_4), as claimed in (iii) of Theorem 2. (This can also be seen in Figure 2.)

To complete the proof of Theorem 2, consider the omitted value $\omega = 0$. In this case (cf. (6)), $\mathcal{L}_0 = I_N$, and its associated covering domain is $\Omega_{\omega,\alpha,\beta} = \{1\} \not\in \mathbb{M}$. We use (11) to deduce that $\kappa(\Omega_{\omega,\alpha,\beta}) = 1$. Hence, $\kappa(\Omega_{\omega,\alpha,\beta}) = 1 = \rho(\mathcal{L}_0)$, and neither the SOR method for $\omega = 0$, nor any semi-iterative method applied to $\mathcal{L}_0$, converges in this case. □

6 Remarks and conclusions

Theorem 2, the main result of this paper, has a lengthy statement and an even lengthier proof, since this proof depends on treating, via conformal mapping theory, five disjoint real intervals in which the real parameter ω (the relaxation parameter of the SOR iteration method) can traverse. While the proof is somewhat condensed, we hope that the reader has a complete picture of the arguments involved in this proof.

For remarks and conclusions related to Theorem 2, we give the following:

1. In Theorem 2, it is assumed that the eigenvalues of B^2 (where B is the associated Jacobi matrix of (2)) are all real and lie in $[-\alpha^2, \beta^2]$ with $\alpha > 0$ and $0 < \beta < 1$. On letting $\alpha \downarrow 0$, it can be verified that the results of Theorem 1 reduce exactly to the results of Theorem 1 of [5]. As such, Theorem 2 of this paper is obviously a generalization of Theorem 1 of [5]. We remark however again, that, on letting $\alpha \downarrow 0$, equality then holds in (20).

2. Similarly, one can let $\beta \downarrow 0$ (i.e., the eigenvalues of B^2 are all real and lie in $[-\alpha^2, 0]$), so that the result of Theorem 2 can be used to deduce Theorem 1. We further remark that letting $\beta \downarrow 0$ also gives equality in (20).

3. In essence, Theorem 2 of this paper assumes knowledge of the extreme eigenvalues, namely $-\alpha^2$ and β^2, of B^2, whose eigenvalues are assumed to lie in the interval $[-\alpha^2, \beta^2]$ with $\alpha > 0$ and with $0 < \beta < 1$. It is natural to ask here, as in [5, Theorem 5] (see also Dancis [1]), if *improvements* of the asymptotic convergent rates, for optimal semi-iteration applied to the SOR iteration method, are *possible* if one assumes further *explicit* knowledge of the spectrum of B^2, i.e., say, that the k largest eigenvalues and the ℓ smallest eigenvalues of B^2 are assumed known (with $k + \ell > 2$), and the spectrum of B^2 is contained in

$$\bigcup_{j=1}^{\ell-1} \{-\alpha_j^2\} \bigcup_{j=1}^{k-1} \{\beta_j^2\} \cup [-\alpha_\ell^2, \beta_k^2],$$

with $-\alpha_1^2 < -\alpha_2^2 < \cdots < -\alpha_\ell^2 < 0$ and $0 < \beta_k^2 < \beta_{k+1}^2 < \cdots < \beta_1^2 < 1$. The answer is *yes*; one need only apply Theorem 2 of this paper with α^2 and β^2 replaced, respectively, by α_ℓ^2 and β_k^2. (The details of this are easy, and are left to the reader.)

4. We wish to reiterative what is in (20) and what is also clear from Figure 2. Under the assumptions of Theorem 2, the best (for any real ω) asymptotic rate of convergence for semi-iterative methods applied to the SOR iterative method is actually *strictly better* than the best (for any real ω) asymptotic rate of convergence for the associated SOR iterative method.

5. We note that the covering domains $\Omega_{\omega,\alpha,\beta}$ studied in Section 4 were composed only from the "building blocks" of circles (with centers at the origin), their circular arcs (which are symmetric with respect to the real axis), and real intervals. It is worth mentioning that this is a consequence of the fact that Young's relationship (24) is a *quadratic* in λ, with ω and μ^2 real.

6. It is natural to ask how much of this analysis, of optimal semi-iterative methods applied to the SOR matrix, can be extended to the case where B is a consistently ordered weakly cyclic of index p ($p > 2$) matrix. In analogy to (4), we assume that

$$\sigma(B^p) \subset [-\alpha^p, \beta^p] \text{ with } \alpha \geq 0, \; 0 \leq \beta < 1 \text{ and } -\alpha^p, \beta^p \in \sigma(B^p).$$

There is, of course, the known p-cyclic analogue (cf. [9, p. 106]) of (24), namely

$$(\lambda+\omega-1)^p = \lambda^{p-1}\omega^p\mu^p. \tag{52}$$

Since (52) is now a polynomial equation of degree p (> 2) in λ, it is more difficult to find an explicit description of the associated covering domains for $\sigma(\mathcal{L}_\omega)$,

$$\Omega^{(p)}_{\omega,\alpha,\beta} := \left\{\lambda \in \mathbb{C} : (\lambda+\omega-1)^p = \lambda^{p-1}\omega^p\mu^p \text{ for some } \mu^p \in [-\alpha^p,\beta^p]\right\}.$$

For the special case $\omega = 1$, we have

$$\Omega^{(p)}_{1,\alpha,\beta} = [-\alpha^p,\beta^p] \text{ and } \kappa\left(\Omega^{(p)}_{1,\alpha,\beta}\right) = \frac{\left(\sqrt{1+\alpha^p}+\sqrt{1-\beta^p}\right)^2}{\alpha^p+\beta^p}$$

(cf. [2, (4.2)]). This is actually sufficient to show that

$$\min_{\omega\in\mathbf{R}} \rho(\mathcal{L}_\omega) > \min_{\omega\in\mathbf{R}} \kappa\left(\Omega^{(p)}_{\omega,\alpha,\beta}\right) = \kappa\left(\Omega^{(p)}_{1,\alpha,\beta}\right) = \frac{\left(\sqrt{1+\alpha^p}+\sqrt{1-\beta^p}\right)^2}{\alpha^p+\beta^p} \tag{53}$$

holds true in the p-cyclic case for any value of α and β. We briefly sketch the proof of (53): It is well-known that, under the given assumptions, one SOR step is equivalent to p steps of a certain semi-iterative method, the so-called p-step relaxation method, applied to the Jacobi method (see Gutknecht, Niethammer and Varga [6]). Consequently, any semi-iterative method applied to SOR is asymptotically at most p-times faster than an optimal semi-iterative method applied to the Jacobi method. Since the requirement that the eigenvalues of B^p, the pth power of the Jacobi matrix B, are contained in $[-\alpha^p,\beta^p]$ is equivalent to

$$\sigma(B) \subset \Sigma_{p,\alpha,\beta} := \Big\{\mu \in \mathbb{C} : \mu = te^{2k\pi i/p}\ (0 \le t \le \beta) \text{ or } \mu = se^{(2k+1)\pi i/p}\ (0 \le s \le \alpha) \text{ for } k = 0,1,\ldots,p-1\Big\},$$

we conclude that

$$\min_{\omega\in\mathbf{R}} \kappa\left(\Omega^{(p)}_{\omega,\alpha,\beta}\right) \ge \left[\kappa\left(\Sigma_{p,\alpha,\beta}\right)\right]^p.$$

But

$$\left[\kappa\left(\Sigma_{p,\alpha,\beta}\right)\right]^p = \kappa\left([-\alpha^p,\beta^p]\right) = \kappa\left(\Omega^{(p)}_{1,\alpha,\beta}\right)$$

(cf. [2, Theorem 6]) which proves the last two relations of (53). Using the expressions for $\rho(\mathcal{L}_{\omega_b})$ given in [3, §3], an easy calculation finally shows that the first inequality in (53) is also valid.

An interesting open question, which appears to be quite difficult, is whether there exists — as in the two-cyclic case — a whole interval $[\omega_2,\omega_3]$, $\omega_2 < 1 < \omega_3$, of relaxation parameters ω with

$$\kappa\left(\Omega^{(p)}_{\omega,\alpha,\beta}\right) = \min_{\omega\in\mathbf{R}} \kappa\left(\Omega^{(p)}_{\omega,\alpha,\beta}\right) \text{ for all } \omega \in [\omega_2,\omega_3].$$

These problems are of recent interest since they have applications to Markov chains (cf. Kontovasilis, Plemmons, and Stewart [7]).

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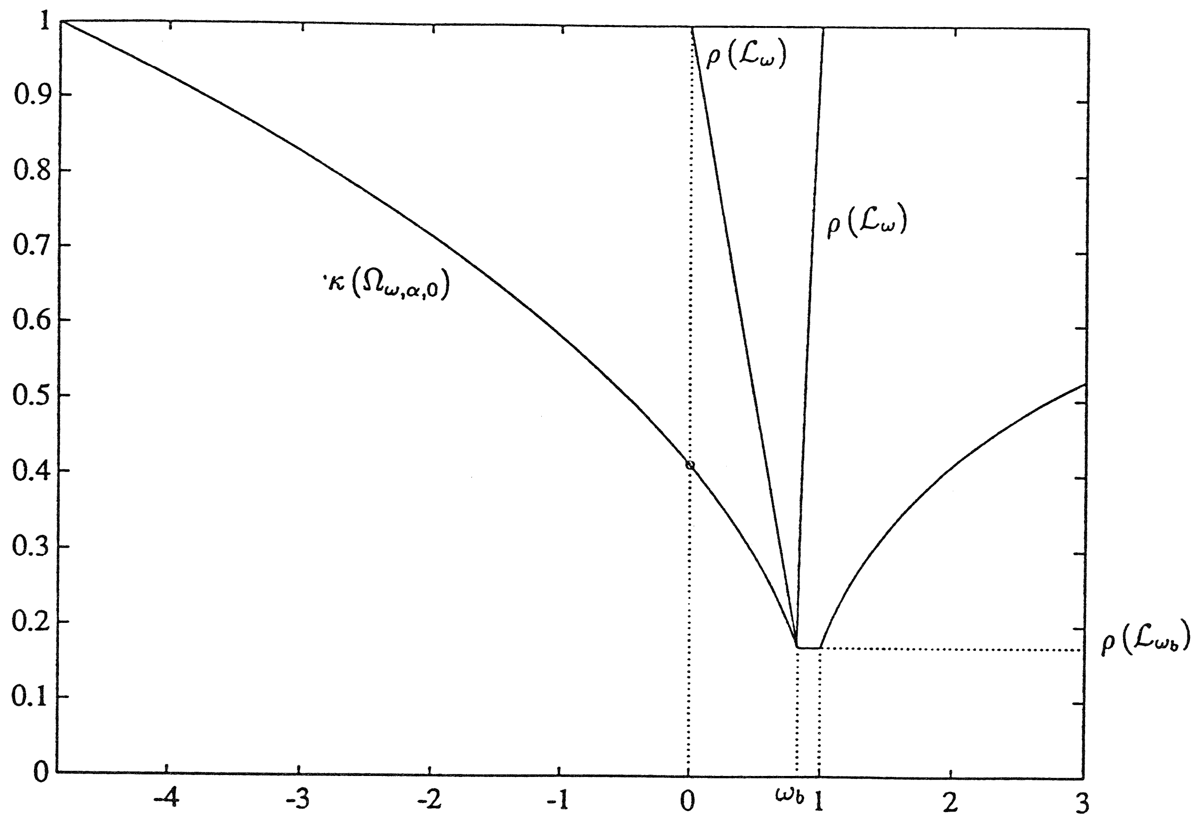

Figure 1: The nonpositive case: $\rho(\mathcal{L}_\omega)$ and $\kappa(\Omega_{\omega,\alpha,0})$ as functions of $\omega \in [\omega_1, 3]$ for $\alpha = 1$.

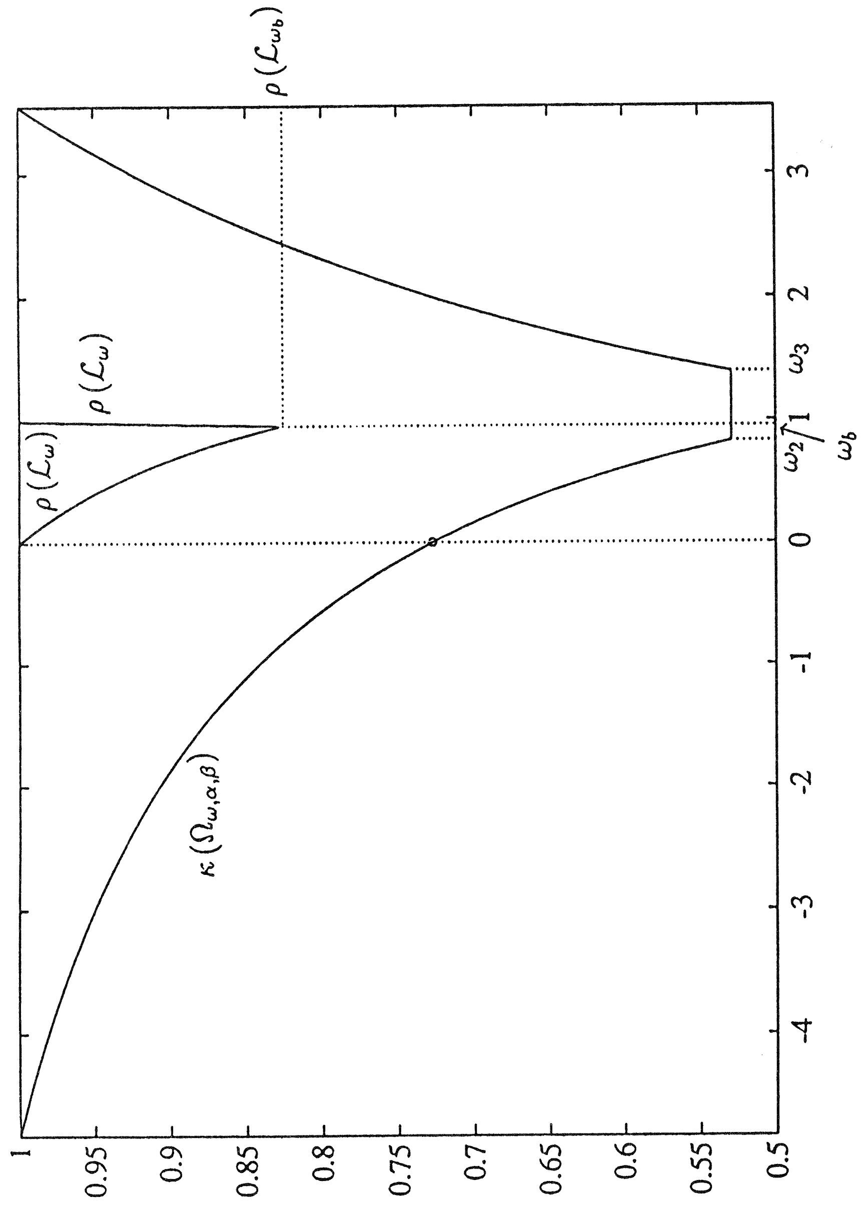

Figure 2: The mixed case: $\rho(\mathcal{L}_\omega)$ and $\kappa(\Omega_{\omega,\alpha,\beta})$ as functions of $\omega \in [\omega_1, \omega_4]$ for $\alpha = 1$ and $\beta = 0.9$.

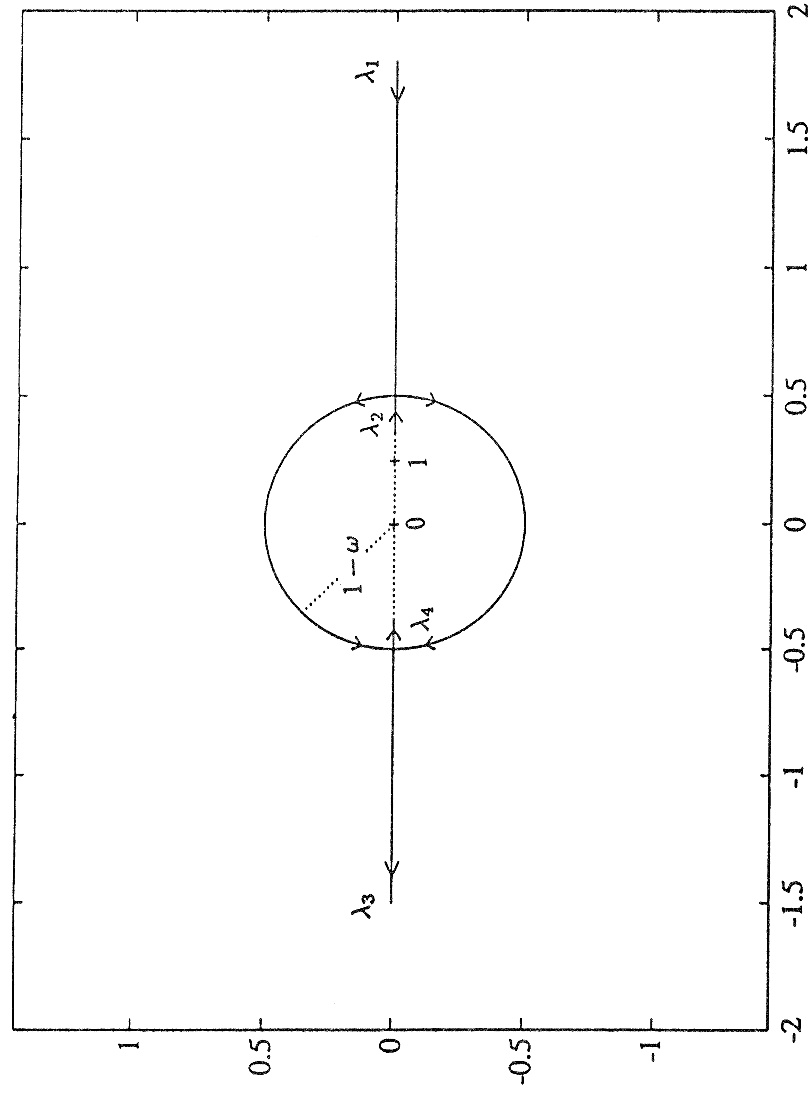

Figure 3a: $\Omega_{\omega,\alpha,\beta}$ for Case 1.

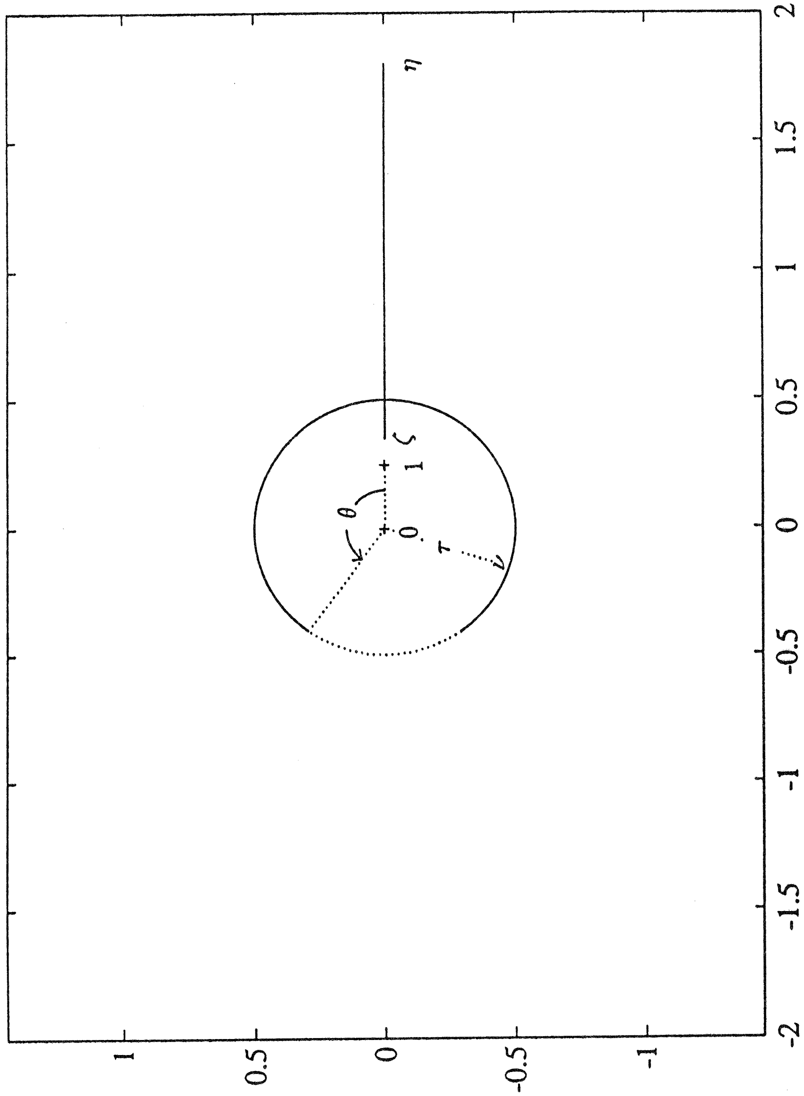

Figure 3b: $D_{\tau,\theta,\zeta,\eta}$.

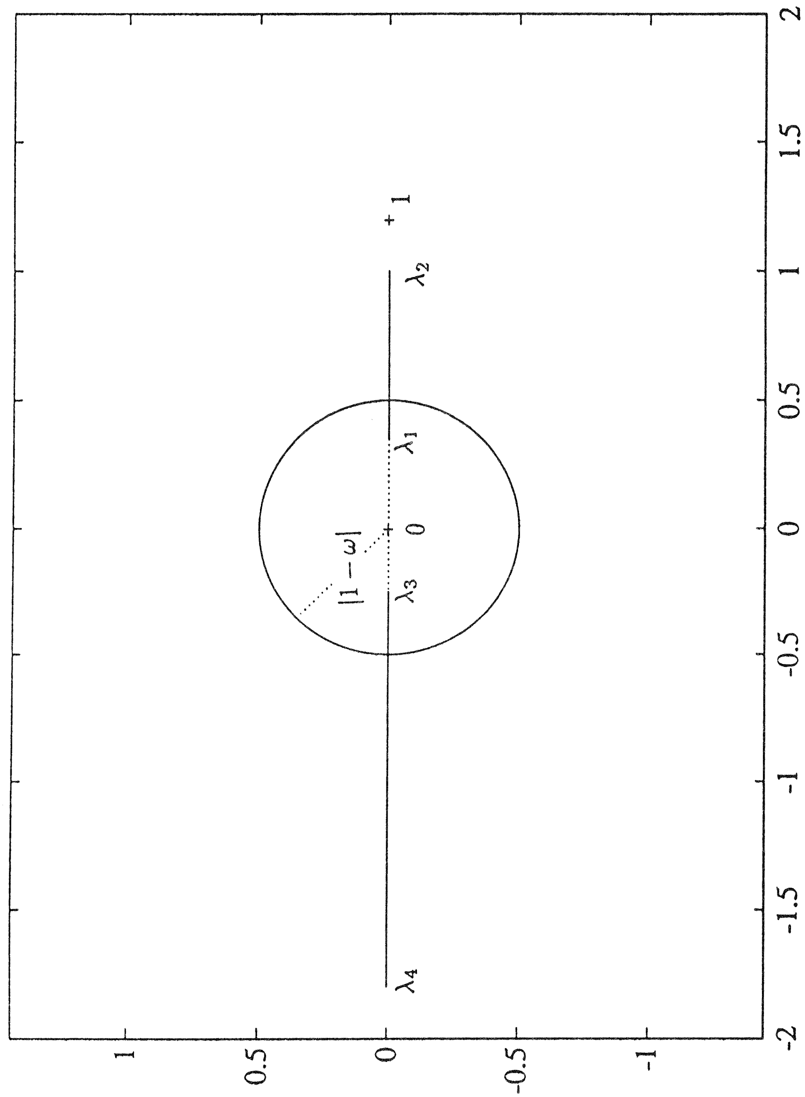

Figure 3c: $\Omega_{\omega,\alpha,\beta}$ for Case 3.

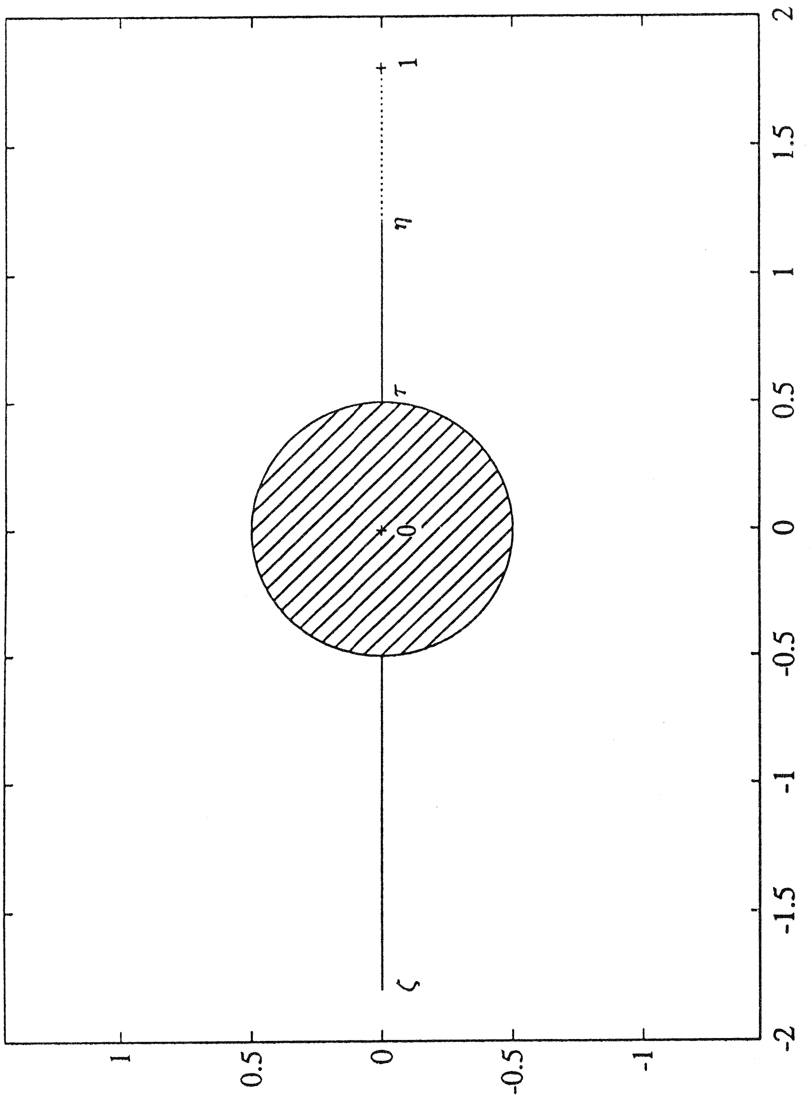

Figure 3d: $B_{\tau,\zeta,\eta}$.

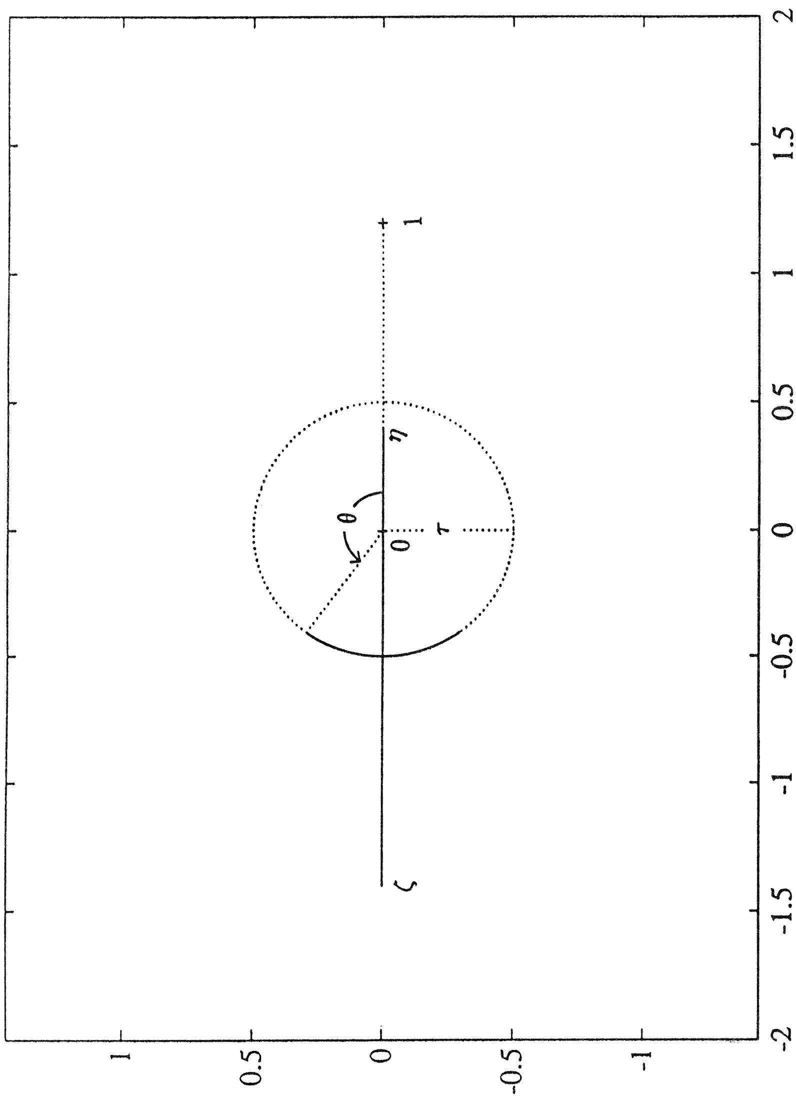

Figure 3e: $C_{\tau,\theta,\zeta,\eta}$.

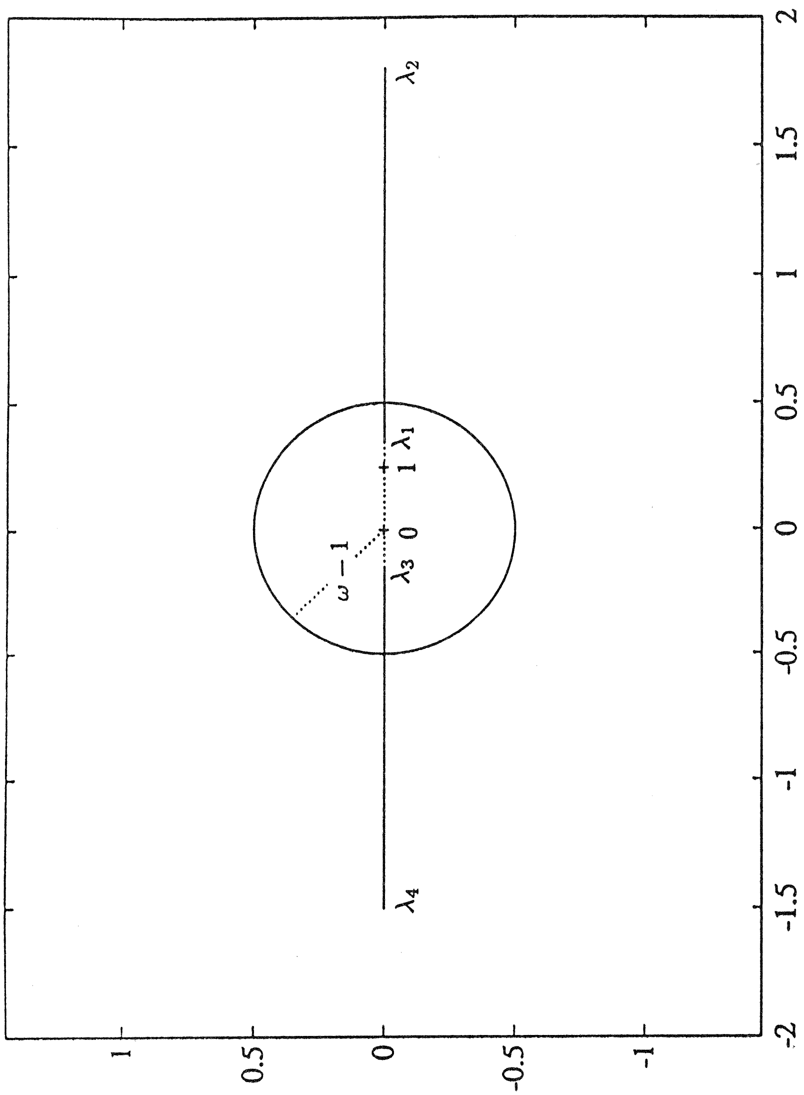

Figure 3f: $\Omega_{\omega,\alpha,\beta}$ for Case 5.