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An algorithm for determining if the inverse of a strictly diagonally dominant Stieltjes matrix is strictly ultrametric

Richard S. Varga[1,*] **and Reinhard Nabben**[2,**]

[1] Institute for Computational Mathematics, Kent State University, Kent, OH 44242, USA
[2] Fakultät für Mathematik, Universität Bielefeld, Postfach 100113, D-33501 Bielefeld, Germany

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Summary. It was recently shown that the inverse of a strictly ultrametric matrix is a strictly diagonally dominant Stieltjes matrix. On the other hand, as it is well-known that the inverse of a strictly diagonally dominant Stieltjes matrix is a real symmetric matrix with nonnegative entries, it is natural to ask, conversely, if every strictly diagonally dominant Stieltjes matrix has a strictly ultrametric inverse. Examples show, however, that the converse is not true in general, i.e., there are strictly diagonally dominant Stieltjes matrices in $\mathbb{R}^{n\times n}$ (for every $n\geqq 3$) whose inverses are not strictly ultrametric matrices. Then, the question naturally arises if one can determine which strictly diagonally dominant Stieltjes matrices, in $\mathbb{R}^{n\times n}$ ($n\geqq 3$), have inverses which are strictly ultrametric. Here, we develop an algorithm, based on graph theory, which determines if a given strictly diagonally dominant Stieltjes matrix A has a strictly ultrametric inverse, where the algorithm is applied to A and requires no computation of inverse. Moreover, if this given strictly diagonally dominant Stieltjes matrix has a strictly ultrametric inverse, our algorithm uniquely determines this inverse as a special sum of rank-one matrices.

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1. Introduction

In Martínez et al. (1994), the new concept of strictly ultrametric matrices was studied. With the notation $N:=\{1, 2, \ldots, n\}$ for any positive integer n, we begin with their following definition:

Definition 1.1. A matrix $B=[b_{i,j}]$ in $\mathbb{R}^{n\times n}$ is strictly ultrametric if

(1.1) *i)* B is symmetric with nonnegative entries;
ii) $b_{i,j}\geqq \min\{b_{i,k}; b_{i,j}\}$ for all $i, j, k\in N$;
iii) $b_{i,i}>\max\{b_{i,k}: k\in N\backslash\{i\}\}$ for all $i\in N$,

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Correspondence to: R.S. Varga

where, if $n=1$, (1.1 *iii*) is interpreted as $b_{1,1}>0$.

The result of Martínez et al. (1994) is

Theorem 1.2. *If $B=[b_{i,j}]$ in $\mathbb{R}^{n\times n}$ is strictly ultrametric, then B is nonsingular and its inverse, $A:=[a_{i,j}]$ in $\mathbb{R}^{n\times n}$, is a strictly diagonally dominant Stieltjes matrix (i.e., A is symmetric, $a_{i,j}\leqq 0$ for all i and j in N with $i\neq j$, and $a_{i,i}>\sum_{\substack{k=1\\k\neq i}}^{n}|a_{i,k}|$ for all $i\in N$), with the additional property that*

$$a_{i,j}=0 \text{ if and only if } b_{i,j}=0 \quad (i,j\in N). \tag{1.2}$$

For a shorter linear algebra proof of Theorem 1.2, see Nabben and Varga (1994).

The first question one might ask is if the converse of Theorem 1.2 is true, i.e., if A in $\mathbb{R}^{n\times n}$ is a strictly diagonally dominant Stieltjes matrix (which we abbreviate below as an s.d.d. Stieltjes matrix) and if its inverse, $B:=[b_{i,j}]$ in $\mathbb{R}^{n\times n}$, satisfies (1.2), then is B strictly ultrametric? This converse turns out to be true if $n=1$ or $n=2$, but fails in general for $n\geqq 3$. (Examples to this will be given in Sect. 4.) This being the case, the second question one might ask is if, given an s.d.d. Stieltjes matrix A in $\mathbb{R}^{n\times n}$, is there an *algorithm*, which can be directly applied to A, which determines whether A^{-1} is strictly ultrametric or not? Our main result here is to give such an algorithm, which is based on graph theory. One of the consequences of our construction is that if the given s.d.d. Stieltjes matrix does possess a strictly ultrametric inverse, then this algorithm uniquely determines this inverse as a sum of special rank-one matrices.

2. Background

As background for this algorithm, we need the following graph-theoretic results from Nabben and Varga (1994):

Proposition 2.1. *Let $B=[b_{i,j}]$ in $\mathbb{R}^{n\times n}$ be symmetric with all its entries nonnegative, and set*

$$\tau(B):=\min\{b_{i,j}\colon i,j\in N\}. \tag{2.1}$$

If ξ_n in $\mathbb{R}^n$ is defined by $\xi_n:=(1,1,\ldots,1)^T$ and if $n>1$, then B is strictly ultrametric if and only if $B-\tau(B)\,\xi_n\xi_n^T$ is strictly ultrametric and completely reducible, i.e., there exists a positive integer r with $1\leqq r<n$ and a permutation matrix P in $\mathbb{R}^{n\times n}$ such that

$$P[B-\tau(B)\,\xi_n\xi_n^T]\,P^T=\begin{bmatrix} C & O\\ O & D\end{bmatrix}, \tag{2.2}$$

where $C\in\mathbb{R}^{r\times r}$ and $D\in\mathbb{R}^{(n-r)\times(n-r)}$ are each strictly ultrametric.

Next, because the result of Proposition 2.1 can also be applied to the disjoint principal submatrices C and D of (2.2), a complete characterization (cf. Nabben and Varga (1994)) of strictly ultrametric matrices, via graph theory, is obtained. Their result is

Theorem 2.2. *Given any strictly ultrametric matrix B in $\mathbb{R}^{n\times n}$, there is an associated rooted tree for $N=\{1, 2, \ldots, n\}$, consisting of $(2n-1)$ vertices, such that*

$$B=\sum_{l=1}^{2n-1} \tau_l \mathbf{u}_l \mathbf{u}_l^{\mathrm{T}}, \tag{2.3}$$

where the associated partition vectors $\mathbf{u}_l$ in $\mathbb{R}^n$ of (2.3), *determined from the vertices of the tree, are nonzero vectors in $\mathbb{R}^n$ having only* 0 *and* 1 *components, and, with the notation that*

$$\chi(\mathbf{u}_l) := \textit{sum of the components of } \mathbf{u}_l, \tag{2.4}$$

where the τ_l's in (2.3) *are nonnegative with $\tau_l>0$ when $\chi(\mathbf{u}_l)=1$. Conversely, given any tree for $N=\{1, 2, \ldots, n\}$, which determines the associated partition vectors $\mathbf{u}_l$ in $\mathbb{R}^n$, and given any nonnegative constants $\{\tau_l\}_{l=1}^{2n-1}$ with $\tau_l>0$ when $\chi(\mathbf{u}_l)=1$, then $\sum_{l=1}^{2n-1} \tau_l \mathbf{u}_l \mathbf{u}_l^{\mathrm{T}}$ is strictly ultrametric in $\mathbb{R}^{n\times n}$.*

To be more specific about the rooted tree and the associated partition vectors $\mathbf{u}_l$ in $\mathbb{R}^n$ which were mentioned in Theorem 2.2, the top vertex of the tree is the set $N=\{1, 2, \ldots, n\}$, which is associated with the column vector $\mathbf{u}_1 := (1, 1, \ldots, 1)^{\mathrm{T}}$ in $\mathbb{R}^n$. The next level of the tree consists of a partition of N into two disjoint nonempty subsets S_2 and S_3, and its associated partition vector $\mathbf{u}_2$ (resp. $\mathbf{u}_3$) in $\mathbb{R}^n$ is such that its jth component is unity if $j\in S_2$ (resp. $j\in S_3$) and is zero otherwise. Each subsequent level of the tree is a partition, into two disjoint nonempty subsets of a previous subset of N, with a corresponding definition of their associated partition vectors in $\mathbb{R}^n$, and this is continued until singleton subsets of N are reached. This is illustrated below in Fig. 1 for $n=5$.

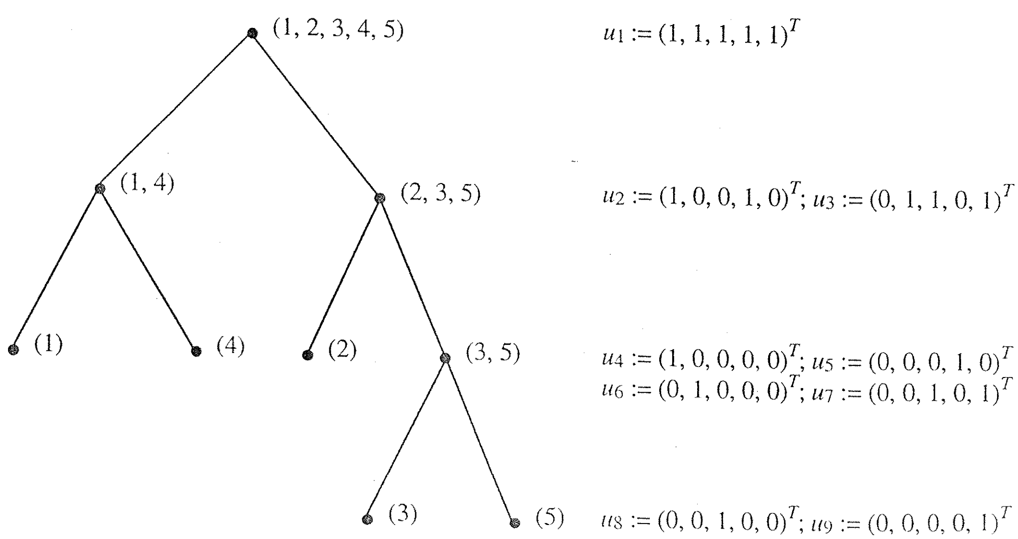


Fig. 1.

3. The Algorithm

For $n=1$, it is evident that a matrix $[\gamma]$ in $\mathbb{R}^{1\times 1}$ is both an s.d.d. Stieltjes matrix and a strictly ultrametric matrix if and only if $\gamma>0$. Thus, consider any s.d.d. Stieltjes matrix $A=[a_{i,j}]$ in $\mathbb{R}^{n\times n}$ $(n\geqq 2)$, so that

(3.1) *i)* $a_{i,j}\leqq 0$ for all $i\neq j$ $(i,j\in N)$;

ii) $a_{i,j}=a_{j,i}$ $(i,j\in N)$ (symmetry);

iii) $\sum_{j=1}^{n} a_{i,j}=:p_i>0$ $(i\in N)$ (strict diagonal dominance).

As is well-known, A is necessarily real symmetric and positive definite, the last property following from (3.1 *i*), (3.1 *iii*) and the Gerschgorin Circle Theorem. Next, set

$$\mu(A):=\min\{-a_{i,j}: i\neq j\ (i,j\in N)\}, \tag{3.2}$$

so that $\mu(A)\geqq 0$. (It is also convenient to define $\mu(A):=0$ if $n=1$.)

Reduction Step 1. $\mu(A)>0$. In this case, the s.d.d. Stieltjes matrix A has no zero off-diagonal entries. With the positive vector $\mathbf{p}:=(p_1,p_2,\ldots,p_n)^{\mathrm{T}}$ in $\mathbb{R}^n$ from (3.1 *iii*), consider the real $n\times n$ matrix

$$\tilde{A}(\alpha):=A+\alpha\mathbf{p}\mathbf{p}^{\mathrm{T}}=[a_{i,j}+\alpha p_i p_j]\quad(\alpha>0), \tag{3.3}$$

which is a rank-one perturbation of the matrix A. Then, because all entries of the matrix $\alpha\mathbf{p}\mathbf{p}^{\mathrm{T}}$ are positive for $\alpha>0$ while the off-diagonal entries of A are all negative, there exists a unique $\hat{\alpha}>0$ such that

$$\mu(\tilde{A}(\hat{\alpha}))=0. \tag{3.4}$$

By construction, $\tilde{A}(\hat{\alpha})$ of (3.3) is then a real symmetric matrix with positive diagonal entries and its off-diagonal entries are nonpositve, with some zero off-diagonal entries because of (3.4). But since we have (cf. (3.1 *iii*) and (3.3))

$$\sum_{j=1}^{n}(a_{i,j}+\hat{\alpha}p_i p_j)=\sum_{j=1}^{n}a_{i,j}+\hat{\alpha}p_i\sum_{j=1}^{n}p_j=p_i\left(1+\hat{\alpha}\sum_{j=1}^{n}p_j\right)>0\quad(i\in N),$$

it is evident that $\tilde{A}(\hat{\alpha})$ is also an s.d.d. Stieltjes matrix. Thus, starting with the s.d.d. Stieltjes matrix A in $\mathbb{R}^{n\times n}$ with $\mu(A)>0$, we have constructed a rank-one perturbation of A in $\mathbb{R}^{n\times n}$, namely $\tilde{A}(\hat{\alpha})$, which is an s.d.d. Stieltjes matrix with $\mu(\tilde{A}(\hat{\alpha}))=0$.

Next, we study the effect of a rank-one perturbation of A in (3.3) on its inverse. With $B:=A^{-1}$, the Sherman-Morrison formula (cf. Golub and Van Loan (1989), p. 51), gives

$$(B-\tau\xi_n\xi_n^{\mathrm{T}})^{-1}=B^{-1}+\frac{\tau B^{-1}\xi_n\xi_n^{\mathrm{T}}B^{-1}}{1-\tau\xi_n^{\mathrm{T}}B^{-1}\xi_n}=A+\frac{\tau A\xi_n\xi_n^{\mathrm{T}}A}{1-\tau\xi_n^{\mathrm{T}}A\xi_n},$$

which, since $A\xi = \mathbf{p}$ from (3.1 *iii*), can be written as

$$(3.6)\qquad (B-\tau\xi_n\xi_n^{\mathrm{T}})^{-1} = A + \frac{\tau\mathbf{p}\mathbf{p}^{\mathrm{T}}}{(1-\tau\xi_n^{\mathrm{T}}\mathbf{p})},\ \text{provided that } 1-\tau\xi_n^{\mathrm{T}}\mathbf{p} \neq 0.$$

In particular, on setting

$$(3.7)\qquad \hat{\tau} := \frac{\hat{\alpha}}{1+\hat{\alpha}\sum_{i=1}^{n} p_i} > 0,$$

we see from (3.7) that

$$1-\hat{\tau}\xi^{\mathrm{T}}\mathbf{p} = 1-\hat{\tau}\sum_{i=1}^{n} p_i = \frac{1}{1+\hat{\alpha}\sum_{i=1}^{n} p_i} > 0,$$

so that $1-\hat{\tau}\xi_n^{\mathrm{T}}\mathbf{p} \neq 0$. It also follows from (3.7) that

$$(3.8)\qquad \hat{\alpha} = \frac{\hat{\tau}}{1-\hat{\tau}\xi_n^{\mathrm{T}}\mathbf{p}}.$$

Thus, using (3.8), (3.6) can be expressed, for $\tau = \hat{\tau}$, as

$$(3.9)\qquad (B-\hat{\tau}\xi_n\xi_n^{\mathrm{T}})^{-1} = A + \hat{\alpha}\mathbf{p}\mathbf{p}^{\mathrm{T}}.$$

This can be used as follows. Given our s.d.d. Stieltjes matrix A with $\mu(A) > 0$, then $\tilde{A}(\hat{\alpha}) := A + \hat{\alpha}\mathbf{p}\mathbf{p}^{\mathrm{T}}$, with $\mu(\tilde{A}(\alpha)) = 0$, is also an s.d.d. Stieltjes matrix but now with some zero off-diagonal entries. Since $\tilde{A}(\hat{\alpha})$ is a Stieltjes matrix, the inverse of $\tilde{A}(\hat{\alpha})$, namely (cf. (3.9)) $B-\hat{\tau}\xi_n\xi_n^{\mathrm{T}}$, certainly has only nonnegative entries (cf. Varga (1962), Cor. 3, p. 85), but from (3.7) and the definition of (2.1), this implies that

$$(3.10)\qquad 0 < \hat{\tau} \leqq \tau(B).$$

The last inequality of (3.10) insures that B is strictly ultrametric if and only if $B-\hat{\tau}\xi_n\xi_n^{\mathrm{T}}$ is also strictly ultrametric.

We summarize the above results in

Proposition 3.1. *(Reduction Step 1). If A is an s.d.d. Stieltjes matrix in $\mathbb{R}^{n\times n}$ ($n \geqq 2$) with $\mu(A) > 0$, then there exists a unique $\hat{\alpha} > 0$ such that $A + \hat{\alpha}\mathbf{p}\mathbf{p}^{\mathrm{T}}$ is an s.d.d. Stieltjes with $\mu(A+\hat{\alpha}\mathbf{p}\mathbf{p}^{\mathrm{T}}) = 0$. Moreover, the inverse of $A+\hat{\alpha}\mathbf{p}\mathbf{p}^{\mathrm{T}}$ is strictly ultrametric if and only if the inverse of A is strictly ultrametric.*

We remark that Proposition 3.1 is called "reduction step 1" because it takes the question, of whether the s.d.d. Stieltjes matrix A with $\mu(A) > 0$ has a strictly ultrametric inverse, and *reduces* it to the equivalent question, of whether the s.d.d. Stieltjes matrix $A+\hat{\alpha}\mathbf{p}\mathbf{p}^{\mathrm{T}}$ (with $\mu(A+\hat{\alpha}\mathbf{p}\mathbf{p}^{\mathrm{T}}) = 0$) has a strictly ultrametric inverse.

This brings us to

Reduction Step 2. $\mu(A) = 0$. In this case, A in $\mathbb{R}^{n\times n}$ (with $n \geqq 2$) necessarily has some zero off-diagonal entries. First, assume that there is *no* permutation matrix

P in $\mathbb{R}^{n\times n}$ such that PAP^{T} is completely reducible. Then, the algorithm *fails*, and the result is that while A is an s.d.d. Stieltjes matrix, its inverse is *not* strictly ultrametric. For, if its inverse B were strictly ultrametric, then from (1.2) of Theorem 1.2, A and B would have the same zero entries, and from (2.2) of Proposition 2.1 (with $\tau(B)=0$), B would be necessarily completely reducible, as would be A, because of their common zero structure, which is a contradiction.

Next, assume (in this case $\mu(A)=0$) that there is a permutation matrix P in $\mathbb{R}^{n\times n}$ for which PAP^{T} is completely reducible, i.e., for some positive integer r with $1\leqq r<n$,

$$PAP^{\mathrm{T}}=\begin{bmatrix} C & O \\ O & D \end{bmatrix}, \quad \text{with } C\in\mathbb{R}^{r\times r} \text{ and } D\in\mathbb{R}^{(n-r)\times(n-r)},$$

where C and D are each an s.d.d. Stieltjes matrix. If the inverse, B, of A is strictly ultrametric, then the common zero structures of A and B, from (1.2) of Theorem 1.2, implies that

$$PBP^{\mathrm{T}}=\begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix}, \quad \text{with } B_1\in\mathbb{R}^{r\times r} \text{ and } B_2\in\mathbb{R}^{(n-r)\times(n-r)},$$

where B_1 and B_2 are each strictly ultrametric.

We summarize the above results in

Proposition 3.2. *(Reduction Step 2). If A is an s.d.d. Stieltjes matrix is $\mathbb{R}^{n\times n}$ $(n\geqq 2)$ with $\mu(A)=0$, then the inverse of A is strictly ultrametric if and only if A is completely reducible, i.e., there is a permutation matrix P in $\mathbb{R}^{n\times n}$ and a positive integer r with $1\leqq r<n$ with*

$$PAP^{\mathrm{T}}=\begin{bmatrix} C & O \\ O & D \end{bmatrix}, \quad \textit{with } C\in\mathbb{R}^{r\times r} \textit{ and } D\in\mathbb{R}^{(n-r)\times(n-r)}, \tag{3.10}$$

and C and D are each an s.d.d. Stieltjes matrix with a strictly ultrametric inverse.

We note that if A is an s.d.d. Stieltjes matrix in $\mathbb{R}^{2\times 2}$ with $\mu(A)=0$, then A must have the form

$$A=\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \quad \text{with } \alpha>0 \text{ and } \beta>0.$$

As such, A is then completely reducible and its inverse is strictly ultrametrix. This shows (along with the first remark in this section) that any s.d.d. Stieltjes matrix in $\mathbb{R}^{n\times n}$, for $n=1$ or $n=2$, must have a strictly ultrametric inverse.

The reason for calling Proposition 3.2 the "reduction step 2" is clear: if the s.d.d. Stieltjes matrix A in $\mathbb{R}^{n\times n}$ $(n\geqq 2)$, with $\mu(A)=0$, is completely reducible, then A has a strictly ultrametric inverse if and only if *each* of the two s.d.d. Stieltjes matrices C and D of (3.10) has a strictly ultrametric inverse. Because the matrices C and D are square s.d.d. Stieltjes matrices of *reduced* order, we can then apply reduction steps 1 and/or 2 to both C and D to determine if each has a strictly ultrametric inverse. Of course, this algorithm can be continued until it either *fails* at some reduction step 2 (in which case, the original s.d.d. Stieltjes matrix A in $\mathbb{R}^{n\times n}$ does not have a strictly ultrametric inverse), or else the algorithm successfully continues to *termination*, in which case A has a strictly ultrametric inverse.

It is important to remark that *if* this algorithm does successfully continue to termination, then the totality of reduction steps 1 *determines* all the τ_l's in (2.3) of Theorem 2.2, as well as the associated column vectors $\mathbf{u}_l$ in $\mathbb{R}^n$. (For example, $\hat{\tau}$ of (3.7) *is* the correct multiplier in (2.3) for the associated vector $\mathbf{u}_1 := (1, 1, \ldots, 1)^T$ in $\mathbb{R}^n$, if A has a strictly ultrametric inverse.) This will be made clear in the first example of the next section.

4. Examples

In this section, we give examples showing how the algorithm of Sect. 2 can be applied in sample cases.

Example 1. Consider

$$A_1 := \tfrac{1}{38}\begin{bmatrix} 27 & -22 & -2 \\ -22 & 32 & -4 \\ -2 & -4 & 10 \end{bmatrix}, \tag{4.1}$$

so that A_1 is, by inspection, a 3×3 s.d.d. Stieltjes matrix. For this example (cf. (3.1 *iii*)), we have $\mathbf{p} = \frac{1}{38}(3, 6, 4)^T$ with (cf. (3.2)) $\mu(A_1) = +\frac{1}{19} > 0$. As $\mu(A_1) > 0$, we apply reduction step 1 to A_1 of (4.1).

1. Reductions Step 1 for A_1 of (4.1). In this case, we have

$$\mathbf{p}\mathbf{p}^T = \frac{1}{(38)^2}\begin{bmatrix} 9 & 18 & 12 \\ 18 & 36 & 24 \\ 12 & 24 & 16 \end{bmatrix},$$

from which it follows (cf. (3.4)) that $\hat{\alpha}_1 = 19/3$. Similarly from (3.7), $\hat{\tau}_1 = 2$. Thus, (cf. (3.3)), the result of the reduction step 1 for A_1 of (4.1) gives

$$\tilde{A}_1(\hat{\alpha}_1) = \tfrac{1}{38}\begin{bmatrix} 28.5 & -19 & 0 \\ -19 & 38 & 0 \\ 0 & 0 & 38/3 \end{bmatrix}, \quad \text{with } \mu(\tilde{A}_1(\hat{\alpha}_1)) = 0. \tag{4.2}$$

(At this point, we set $\tau_1 := \hat{\tau}_1 = 2$, and we associate with τ_1 the vector $\mathbf{u}_1 := (1, 1, 1)^T$, since A_1 is a 3×3 matrix.)

2. Reduction Step 2 for $\tilde{A}_1(\hat{\alpha}_1)$ of (4.2). It is already clear from (4.2) that $\tilde{A}_1(\hat{\alpha}_1)$ is completely reducible, so that the algorithm can be continued by applying the reduction steps to the two principal submatrices of $\tilde{A}_1(\hat{\alpha}_1)$ of (4.2) which can be expressed as

$$C_{1,1} := \tfrac{1}{4}\begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}, \quad \text{and } C_{2,2} := [1/3]. \tag{4.3}$$

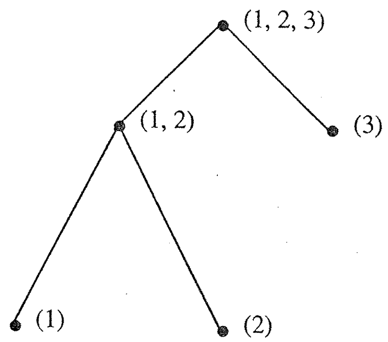
We see that $\mu(C_{1,1}) = \frac{1}{2}$ and, by definition, that $\mu(C_{2,2}) = 0$.

3. *Reduction Step 1 for* $C_{1,1}$ *of* (4.3). In this case, $\mathbf{p}=\frac{1}{4}(1, 2)^{\mathrm{T}}$, and we similarly determine that $\hat{\alpha}_2=4$ and $\hat{\tau}_2=1$. (Here, we set $\tau_2:=\hat{\tau}_2=1$ and associate with τ_2 the vector of $\mathbf{u}_2:=(1, 1, 0)^{\mathrm{T}}$, since $C_{1,1}$ is the 2×2 upper principal submatrix of $\tilde{A}_1(\hat{\alpha}_1)$.) In this case, reduction step 1, applied to $C_{1,1}$, gives

(4.4) $$\tilde{C}_{1,1}(\hat{\alpha}_2)=\left[\begin{array}{c|c}1 & 0\\ \hline 0 & 2\end{array}\right], \quad \text{with } \mu(\tilde{C}_{1,1}(\hat{\alpha}_2))=0.$$

4. *Reduction Step 2 for* $\tilde{C}_{1,1}(\hat{\alpha}_2)$ *of* (4.4). It is again clear from (4.4) that $\tilde{C}_{1,1}(\hat{\alpha}_2)$ is completely reducible, so that the algorithm can be continued. But as the two principal submatrices of (4.4), as well as $C_{2,2}$ of (4.3), are trivially 1×1 s.d.d. Stieltjes matrices, the algorithm has then *successfully terminated*, and we conclude that the s.d.d. Stieltjes matrix A_1 of (4.1) has a strictly ultrametric inverse!

On gathering the results of the reduction steps 1 for the matrix of (4.1), we specifically have the following values for τ_i and the partition vectors $\mathbf{u}_i$ for the associated rooted tree:



$\tau_1=2$ and $u_1=(1, 1, 1)^T$

$\tau_2=1$ and $u_2=(1, 1, 0)^T$

$\tau_3=3$ and $u_3=(0, 0, 1)^T$

$\tau_4=1$ and $u_4=(1, 0, 0)^T$

$\tau_5=1/2$ and $u_5=(0, 1, 0)^T$.

Fig. 2.

Thus (cf. (2.3)),

(4.5) $$B_1=\sum_{l=1}^{5}\tau_l\mathbf{u}_l\mathbf{u}_l^{\mathrm{T}}.$$

As all the τ_l's are positive from Fig. 2, it follows directly from Theorem 2.2 that B_1, the inverse of A_1 is strictly ultrametric. Moreover, the explicit representation of (4.5) allows us to directly express B_1 by

(4.6) $$B_1=:A_1^{-1}=\begin{bmatrix}4 & 3 & 2\\ 3 & 3.5 & 2\\ 2 & 2 & 5\end{bmatrix}.$$

Example 2. Consider the matrix

$$A_2 := \begin{bmatrix} 5 & -2 & -2 \\ -2 & 4 & -1 \\ -2 & -1 & 4 \end{bmatrix}, \tag{4.7}$$

which, by inspection, is a 3×3 s.d.d. Stieltjes matrix. In this case (cf. (3.2)), $\mu(A)=1$, and the reduction step 1 of Sect. 3 can be directly applied. As $\mathbf{p} = (1, 1, 1)^T$ from (3.1 *iii*), it follows that $\mu(A_2+\mathbf{p}\mathbf{p}^T)=0$, i.e., $\hat{\alpha}_2 := 1$. Then,

$$A_2+\mathbf{p}\mathbf{p}^T = \begin{bmatrix} 6 & -1 & -1 \\ -1 & 5 & 0 \\ -1 & 0 & 5 \end{bmatrix}, \tag{4.8}$$

but as the above matrix is *not* completely reducible, the algorithm *fails* at this step. Consequently, its inverse, $\tilde{B}_2 := (A_2+\mathbf{p}\mathbf{p}^T)^{-1}$, given explicitly by

$$\tilde{B}_2 = (A_2+\mathbf{p}\mathbf{p}^T)^{-1} = \tfrac{1}{140}\begin{bmatrix} 25 & 5 & 5 \\ 5 & 29 & 1 \\ 5 & 1 & 29 \end{bmatrix} =: [\tilde{b}_{i,j}] \tag{4.9}$$

is then *not* strictly ultrametric. (This can also be seen directly from Definition 1.1: $\tilde{b}_{2,3}=1$ while the min $\{\tilde{b}_{2,1}; \tilde{b}_{1,3}\}=5$, thus contradicting (1.1 *ii*).)

Consider again the matrix A_2 of (4.7), whose inverse is given by

$$B_2 = \tfrac{1}{35}\begin{bmatrix} 15 & 10 & 10 \\ 10 & 16 & 9 \\ 10 & 9 & 16 \end{bmatrix}, \tag{4.10}$$

(where B_2 again fails to satisfy (1.1 *ii*) for $i=2$, $j=3$). We note that, for A_2 and its inverse B_2, the condition (1.2) vacuously holds. For any $n>3$, the $n\times n$ matrix A, defined from the matrix A_2 in (4.7) by

$$A := \left[\begin{array}{c|c} A_2 & 0 \\ \hline 0 & I_{n-3} \end{array}\right] \quad \text{(where } I_j \text{ denotes the identity matrix in } \mathbb{R}^{j\times j}\text{)}, \tag{4.11}$$

is then an s.d.d. Stieltjes matrix satisfying (1.2), but the inverse of A is clearly not strictly ultrametric. This shows, for any $n\geq 3$, that there are s.d.d. Stieltjes matrices in $\mathbb{R}^{n\times n}$, satisfying (1.2) of Theorem 1.2, whose inverses are not strictly ultrametric.

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