SIAM J. MATRIX ANAL. APPL.
Vol. 15, No. 1, pp. 107–113, January 1994

A LINEAR ALGEBRA PROOF THAT THE INVERSE OF A STRICTLY ULTRAMETRIC MATRIX IS A STRICTLY DIAGONALLY DOMINANT STIELTJES MATRIX*

REINHARD NABBEN† AND RICHARD S. VARGA‡

Abstract. It is well known that every $n \times n$ Stieltjes matrix has an inverse that is an $n \times n$ nonsingular symmetric matrix with nonnegative entries, and it is also easily seen that the converse of this statement fails in general to be true for $n > 2$. In the preceding paper by Martínez, Michon, and San Martín [*SIAM J. Matrix Anal. Appl.*, 15 (1994), pp. 98–106], such a converse result is in fact shown to be true for the new class of *strictly ultrametric matrices*. A simpler proof of this basic result is given here, using more familiar tools from linear algebra.

Key words. Stieltjes matrices, ultrametric matrices, inverse M-matrix problem

AMS subject classifications. 15A57, 15A48

1. Introduction. It is well known (cf. [3, p. 85]) that a *Stieltjes matrix* $A = [a_{i,j}]$ in $\mathbb{R}^{n,n}$, which is defined to be a real symmetric and positive definite matrix with $a_{i,j} \leq 0$ for all $i \neq j$ $(1 \leq i,j \leq n)$, has the property that its inverse is a real nonsingular and symmetric matrix, all of whose entries are nonnegative. Now, the converse of this result is not generally true for any $n \geq 3$, as the following simple matrix below shows. For $n = 3$, define the symmetric matrix B in $\mathbb{R}^{3,3}$ by

$$B := \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 3 \\ 2 & 3 & 4 \end{bmatrix},$$

so that B possesses only nonnegative entries. As the eigenvalues of B are $(4 + \sqrt{13}, 4, 4 - \sqrt{13})$, then B is positive definite. But its inverse,

$$B^{-1} = \frac{1}{12}\begin{bmatrix} 7 & 6 & -8 \\ 6 & 12 & -12 \\ -8 & -12 & 16 \end{bmatrix},$$

fails to be a Stieltjes matrix since its off-diagonal entries are not all nonpositive. For $n > 3$, the matrix

$$\begin{bmatrix} B & O \\ O & I_{n-3} \end{bmatrix} \quad \text{and its inverse} \quad \begin{bmatrix} B^{-1} & O \\ O & I_{n-3} \end{bmatrix},$$

where I_{n-3} is the identity matrix in $\mathbb{R}^{n-3,n-3}$, similarly furnishes a counterexample in $\mathbb{R}^{n,n}$.

In the preceding paper [2, Thm. 1] by Martínez, Michon, and San Martín, it is shown that a strictly ultrametric square matrix (to be defined below) is a nonsingular

* Received by the editors March 24, 1992; accepted for publication (in revised form) April 8, 1992.

† Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, 33 501 Bielefeld, Germany (na.nabben@na-net.ornl.gov). This work was performed while the author was at the Institute for Computational Mathematics, Kent State University, Kent, Ohio 44242. This author's research was supported by the Deutsche Forschungsgemeinschaft.

‡ Institute for Computational Mathematics, Kent State University, Kent, Ohio 44242 (varga@mcs.kent.edu). The research of this author was supported by the National Science Foundation.

matrix, with nonnegative entries, whose inverse is a strictly diagonally dominant Stieltjes matrix! As can be seen from their paper, their interesting result is proved by using a variety of impressive tools from topology and real analysis, tools that may prove useful for infinite dimensional extensions. The beauty of their result gave us the stimulus to try to find a proof of their result that was fashioned solely from more familiar tools from linear algebra, as such a proof might be more accessible to numerical analysts and linear algebraists. We give such a linear algebra proof below.

With the notation that $N := \{1, 2, \ldots, n\}$ for any positive integer n, we begin with the following definition of [2].

DEFINITION 1.1. *A matrix $A = [a_{i,j}]$ in $\mathbb{R}^{n,n}$ is strictly ultrametric if*

$$(1.1) \qquad \begin{cases} \text{(i)} & A \text{ is symmetric with nonnegative entries,} \\ \text{(ii)} & a_{i,j} \geq \min\{a_{i,k}; a_{k,j}\} \text{ for all } i,j,k \in N, \\ \text{(iii)} & a_{i,i} > \max\{a_{i,k} : k \in N \backslash \{i\}\} \text{ for all } i \in N, \end{cases}$$

where, if $n = 1$, (1.1)(iii) is interpreted as $a_{1,1} > 0$.

The result of [2, Thm. 1] is stated in the following theorem.

THEOREM 1.2. *If $A = [a_{i,j}]$ in $\mathbb{R}^{n,n}$ is strictly ultrametric, then A is nonsingular and its inverse, $A^{-1} := [\alpha_{i,j}]$ in $\mathbb{R}^{n,n}$, is a strictly diagonally dominant Stieltjes matrix (i.e., $\alpha_{i,j} \leq 0$ for all $i \neq j$ and $\alpha_{i,i} > \sum_{\substack{k=1 \\ k\neq i}}^{n} |\alpha_{i,k}|$, for all $1 \leq i,j \leq n$), with the additional property that*

$$(1.2) \qquad \alpha_{i,j} = 0 \quad \text{if and only if} \quad a_{i,j} = 0.$$

Our proof of Theorem 1.2 is given in §3, after some necessary constructions are given in §2.

2. Some constructions. For notation, on setting $\xi_n := (1, 1, \ldots, 1)^T$ in $\mathbb{R}^n$, then

$$(2.1) \qquad \xi_n \xi_n^T$$

is a rank-one matrix in $\mathbb{R}^{n,n}$, all of whose entries are unity.

Our first result, which is independent of results or techniques in [2], is necessary for our complete characterization of strictly ultrametric matrices.

PROPOSITION 2.1. *Let $A = [a_{i,j}]$ in $\mathbb{R}^{n,n}$ be symmetric with all its entries nonnegative, and set*

$$(2.2) \qquad \tau(A) := \min\{a_{i,j} : i,j \in N\}.$$

If $n > 1$, then A is strictly ultrametric if and only if $A - \tau(A)\xi_n\xi_n^T$ is completely reducible, i.e., there exists a positive integer r with $1 \leq r < n$ and a permutation matrix P in $\mathbb{R}^{n,n}$ such that

$$(2.3) \qquad P(A - \tau(A)\xi_n\xi_n^T)P^T = \begin{bmatrix} C & O \\ O & D \end{bmatrix},$$

where $C \in \mathbb{R}^{r,r}$ and $D \in \mathbb{R}^{n-r,n-r}$ are each strictly ultrametric.

Proof. For $n > 1$, assume that A is strictly ultrametric. Then, from (1.1) and (2.2), it follows that $\tilde{A} = [\tilde{a}_{i,j}] := A - \tau(A)\xi_n\xi_n^T$ is strictly ultrametric with $\tau(\tilde{A}) = 0$. Moreover, as $n > 1$ and as $\tau(\tilde{A}) = 0$, some off-diagonal entry of $\tilde{A}$ is necessarily zero.

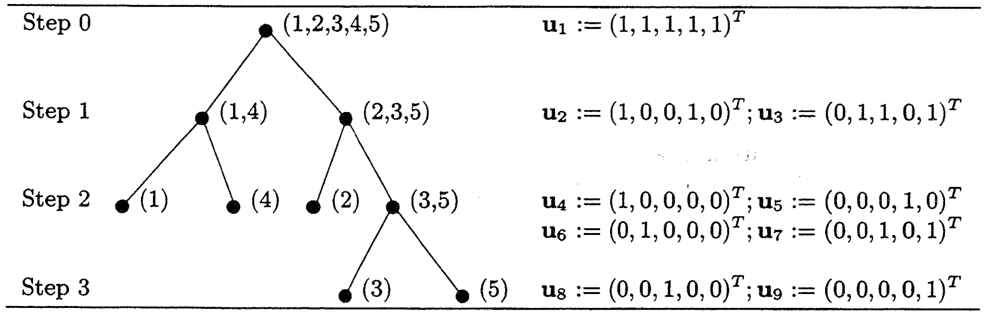

FIG. 1

By a suitable permutation of indices, we may assume, without loss of generality, that $\tilde{a}_{1,n} = 0$. Set

$$S := \{j \in N : \tilde{a}_{1,j} = 0\} \quad \text{and} \quad T := \{j \in N : \tilde{a}_{1,j} > 0\}. \tag{2.4}$$

As $\tilde{a}_{1,n} = 0$, then $n \in S$, and similarly, since $\tilde{A}$ is strictly ultrametric, then (cf. (1.1)(iii)) $\tilde{a}_{1,1} > 0$, so that $1 \in T$. Thus, S and T form a partition of N, i.e., S and T are nonempty disjoint sets with $S \cup T = N$. Again, by a suitable permutation of indices, we may assume, without loss of generality, that

$$T = \{1, 2, \ldots, r\} \quad \text{and} \quad S = \{r+1, r+2, \ldots, n\}, \tag{2.5}$$

where $1 \leq r < n$.

Next, consider any $j \in T$ and any $k \in S$. Since $\tilde{A}$ is a nonnegative matrix, (1.1)(ii) implies that

$$0 = \tilde{a}_{1,k} \geq \min\{\tilde{a}_{1,j}; \tilde{a}_{j,k}\} \geq 0 \qquad (j \in T, k \in S). \tag{2.6}$$

But as $\tilde{a}_{1,j} > 0$ from (2.4), the inequalities of (2.6) and the symmetry of $\tilde{A}$ give that

$$\tilde{a}_{j,k} = 0 = \tilde{a}_{k,j} \qquad (j \in T, k \in S), \tag{2.7}$$

which gives the desired representation of (2.3). That the block diagonal submatrices C and D in (2.3) are each strictly ultrametric is a consequence of the fact that $\tilde{A}$ is strictly ultrametric.

Conversely, if $n > 1$, if $C \in \mathbb{R}^{r,r}$ and if $D \in \mathbb{R}^{n-r,n-r}$ (with $1 \leq r < n$) are each strictly ultrametric, and if $\tau \geq 0$, then from Definition 1.1, the matrix

$$\begin{bmatrix} C & O \\ O & D \end{bmatrix} + \tau \xi_n \xi_n^T$$

is also strictly ultrametric. $\square$

It is evident that the steps leading to the representation (2.3) can be similarly applied to each of the strictly ultrametric block submatrices C and D of (2.3), provided that their orders each exceed unity. More precisely, if $C \in \mathbb{R}^{r,r}$ and if $D \in \mathbb{R}^{n-r,n-r}$ where $1 < r < n-1$, then $C - \tau(C)\xi_r\xi_r^T$ and $D - \tau(D)\xi_{n-r}\xi_{n-r}^T$ are, from the proof of Proposition 2.1, each completely reducible strictly ultrametric matrices. This process can be continued until only 1×1 *positive* matrices remain. This entire reduction procedure can be described in terms of graph theory, as follows.

For illustration, consider an ultrametric matrix A in $\mathbb{R}^{5,5}$, and suppose that the block submatrices C and D in (2.3) are of orders 2 and 3, respectively. This is shown in (reduction) Step 1 in the rooted *reduction tree* of Fig. 1, where the top vertex of Step 0 is associated with the set (1,2,3,4,5). At Step 1, the set (1,2,3,4,5) is decomposed into the two nonempty disjoint sets (1,4) and (2,3,5), giving rise to two vertices in the tree at Step 1. This step corresponds to the complete reducibility of $A - \tau(A)\xi_5\xi_5^T$ in (2.3). In Step 2, each of the sets (1,4) and (2,3,5) is further decomposed into two disjoint nonempty sets, giving rise to four vertices in the tree at Step 2, and this procedure is continued until all remaining sets have single elements. In this way, the 5×5 ultrametric matrix A has the representation

$$A = \sum_{\ell=1}^{9} \tau_\ell \mathbf{u}_\ell \mathbf{u}_\ell^T, \tag{2.8}$$

where the sum over nine terms in (2.8) comes from the fact that there are nine vertices in the tree of Fig. 1. The associated vectors $\mathbf{u}_\ell$ are also explicitly given in Fig. 1. The scalars $\{\tau_\ell\}_{\ell=1}^{9}$ are nonnegative, with (cf. (1.1)(iii)) $\tau_4, \tau_5, \tau_6, \tau_8$, and τ_9 necessarily positive numbers. In fact, if the constants $\{\tau_1, \tau_2, \ldots, \tau_9\}$ in (2.8) are chosen to be $\{1, 0, 0, 1, 1, 1, 2, 1, 1\}$, then A can be computed from (2.8) to be

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 3 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 4 \end{bmatrix}.$$

But, it is easy to verify (by induction) that for $N = (1, 2, \ldots, n)$, the reduction steps, as indicated in Fig. 1 for $n = 5$, give exactly $2n - 1$ vertices for its associated reduction tree. Hence, Proposition 2.1 gives the following representation for strictly ultrametric matrices in $\mathbb{R}^{n,n}$ for *all* $n \geq 1$, which goes beyond the results of [2].

THEOREM 2.2. *Given any strictly ultrametric matrix A in $\mathbb{R}^{n,n}$ ($n \geq 1$), there is an associated rooted tree for $N = \{1, 2, \ldots, n\}$, consisting of $2n - 1$ vertices, such that*

$$A = \sum_{\ell=1}^{2n-1} \tau_\ell \mathbf{u}_\ell \mathbf{u}_\ell^T, \tag{2.9}$$

where the vectors $\mathbf{u}_\ell$ in (2.9), determined from the vertices of the tree, are nonzero vectors in $\mathbb{R}^n$ having only 0 and 1 components, and, with the notation that

$$\chi(\mathbf{u}_\ell) := \textit{ sum of the components of } \mathbf{u}_\ell, \tag{2.10}$$

where the τ_ℓ's in (2.9) are nonnegative with $\tau_\ell > 0$ when $\chi(\mathbf{u}_\ell) = 1$. Conversely, given any tree for $N = \{1, 2, \ldots, n\}$, which determines the vectors $\mathbf{u}_\ell$ in $\mathbb{R}^n$, and given any nonnegative constants $\{\tau_\ell\}_{\ell=1}^{2n-1}$ with $\tau_\ell > 0$ when $\chi(\mathbf{u}_\ell) = 1$, then $\sum_{\ell=1}^{2n-1} \tau_\ell \mathbf{u}_\ell \mathbf{u}_\ell^T$ is strictly ultrametric in $\mathbb{R}^{n,n}$.

COROLLARY 2.3. *Any strictly ultrametric matrix in $\mathbb{R}^{n,n}$ is a real symmetric and positive definite matrix.*

Proof. From Theorem 2.2, any strictly ultrametric matrix admits a representation (2.9) as a sum of rank-one nonnegative definite symmetric matrices. But, as the condition that τ_ℓ be positive whenever $\chi(\mathbf{u}_\ell) = 1$ implies that the sum in (2.9) contains a positive diagonal matrix, the sum (2.9) is necessarily positive definite. □

3. Proof of Theorem 1.2. With the constructions of §2, we come to the proof of Theorem 1.2. The proof is an induction on n. If A is an $n \times n$ strictly ultrametric matrix, then from Corollary 2.3, A is nonsingular and A^{-1} exists. That A^{-1} is a strictly diagonally dominant Stieltjes matrix that also satisfies (1.2) of Theorem 1.2 is obvious for $n = 1$. Thus, by the inductive hypothesis, assume that Theorem 1.2 is valid for all ultrametric matrices in $\mathbb{R}^{j,j}$ with $1 \leq j \leq n-1$ where $n \geq 2$, and consider any strictly ultrametric matrix $A = [a_{i,j}]$ in $\mathbb{R}^{n,n}$. Up to a suitable permutation, we have from (2.2) and (2.3) that

$$A = \begin{bmatrix} C & O \\ O & D \end{bmatrix} + \tau(A)\xi_n\xi_n^T \quad \text{with } \xi_n := (1,1,\ldots,1)^T \in \mathbb{R}^n, \tag{3.1}$$

where, from Proposition 2.1, C in $\mathbb{R}^{r,r}$ and D in $\mathbb{R}^{n-r,n-r}$ (with $1 \leq r < n$) are both strictly ultrametric and nonsingular. But as r and $n-r$ are both less than n, the inductive hypothesis, applied to C and D, gives that C^{-1} and D^{-1} are strictly diagonally dominant Stieltjes matrices. Hence, if

$$M := \begin{bmatrix} C & O \\ O & D \end{bmatrix} \quad \text{so that } M^{-1} = \begin{bmatrix} C^{-1} & O \\ O & D^{-1} \end{bmatrix}, \tag{3.2}$$

then M^{-1} is also a strictly diagonally dominant Stieltjes matrix. Next, the Sherman–Morrison formula (cf. Golub and Van Loan [1, p. 51]), applied to (3.1), gives the following representation for A^{-1} of (3.1):

$$\left(M + \tau(A)\xi_n \cdot \xi_n^T\right)^{-1} = A^{-1} = M^{-1} - \frac{\tau(A)M^{-1}\xi_n\xi_n^T M^{-1}}{[1+\tau(A)\xi_n^T M^{-1}\xi_n]}. \tag{3.3}$$

We first claim that the term in brackets in the denominator above is *positive.* To see this, M^{-1}, as previously noted, is a strictly diagonally dominant Stieltjes matrix, so that $M^{-1}\xi_n$ is a positive vector in $\mathbb{R}^n$. On writing $M^{-1}\xi_n := \mathbf{p} > 0$, this denominator is just

$$[1+\tau(A)\xi_n^T M^{-1}\xi_n] = 1 + \tau(A)\xi_n^T\mathbf{p} \geq 1. \tag{3.4}$$

Moreover, since $M\mathbf{p} = \xi_n$ and since M is real symmetric, then the last term in (3.3) can be expressed as the matrix

$$-\frac{\tau(A)}{[1+\tau(A)\xi_n^T\mathbf{p}]}\mathbf{p}\mathbf{p}^T, \tag{3.5}$$

which is obviously a real nonpositive definite symmetric matrix in $\mathbb{R}^{n,n}$, all of whose terms are zero if $\tau(A) = 0$, or negative if $\tau(A) > 0$. But, as the matrix of (3.5) is *added* in (3.3) to M^{-1}, which as noted above is a Stieltjes matrix, then all off-diagonal entries of A^{-1} are necessarily nonpositive.

To show that A^{-1} is strictly diagonally dominant, let

$$M^{-1}\xi_n = \mathbf{p} =: (p_1, p_2, \ldots, p_n)^T > \mathbf{0}.$$

For the ith row sum of A^{-1}, it follows from the second part of (3.3) and (3.5) that

$$(A^{-1}\xi_n)_i = p_i - \frac{\tau(A)p_i\sum_{j=1}^n p_j}{\left[1+\tau(A)\sum_{j=1}^n p_j\right]} = \frac{p_i}{\left[1+\tau(A)\sum_{j=1}^n p_j\right]} > 0 \quad (i \in N). \tag{3.6}$$

But, as all off-diagonal entries $\alpha_{i,j}$ of A^{-1} are nonpositive, (3.6) succinctly and precisely gives that A^{-1} is strictly diagonally dominant!

Finally, we establish (cf. (1.2)) that $\alpha_{i,j} = 0$ if and only if $a_{i,j} = 0$. First, if $\tau(A) > 0$, then the strictly ultrametric matrix $A = [a_{i,j}]$ is, up to a permutation matrix P, given from (3.1) by the sum

$$A = \begin{bmatrix} C & O \\ O & D \end{bmatrix} + \tau(A)\xi_n\xi_n^T, \tag{3.7}$$

which has only positive entries, i.e., $a_{i,j} > 0$ for all i, j in N. On the other hand, from (3.3) and (3.4),

$$A^{-1} = \begin{bmatrix} C^{-1} & O \\ O & D^{-1} \end{bmatrix} - \frac{\tau(A)\mathbf{p}\mathbf{p}^T}{[1 + \tau(A)\xi_n^T\mathbf{p}]}, \tag{3.8}$$

where every entry of the last matrix is negative. As the matrices C and D in (3.7) are strictly ultrametric from Proposition 2.1, then C^{-1} and D^{-1} are Stieltjes matrices. Thus, from (3.8), the entries $\alpha_{i,j}$ of A^{-1} satisfy $\alpha_{i,j} < 0$ for all $i \neq j$. Moreover, since A^{-1} is a strictly diagonal dominant matrix, then $\alpha_{i,i} > 0$ for all $1 \leq i \leq n$. Hence, in this case where $\tau(A) > 0$, (1.2) of Theorem 1.2 vacuously holds.

If $\tau(A) = 0$, then from (3.7) we have that

$$A = \begin{bmatrix} C & O \\ O & D \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} C^{-1} & O \\ O & D^{-1} \end{bmatrix},$$

so that A and A^{-1} have the same off-diagonal blocks of zeros. But we can evidently apply the inductive hypothesis to the block submatrices C and D, and we thus establish (1.2), namely, that the zero entries of A and A^{-1} are the same. □

Having established Theorem 1.2, we deduce from it the following corollary, which appears in [2, Lemma 1] as a step in establishing proof of Theorem 1.2.

COROLLARY 3.1. *Let A in $\mathbb{R}^{n,n}$ be strictly ultrametric. If $\xi_n := (1, 1, \ldots, 1)^T$ in $\mathbb{R}^n$, then there exists a vector $\mathbf{p}$ in $\mathbb{R}^n$, with all positive components, such that*

$$A\mathbf{p} = \xi_n. \tag{3.9}$$

Proof. From Theorem 1.2, A^{-1} is a strictly diagonally dominant Stieltjes matrix in $\mathbb{R}^{n,n}$. Hence $A^{-1}\xi_n =: \mathbf{p} > \mathbf{0}$, from which (3.9) directly follows. □

In conclusion, we note that the more general problem of determining which nonsingular matrices in $\mathbb{R}^{n,n}$, with nonnegative coefficients, have inverses that are M-matrices, has been studied by a number of authors over the years. Although we know of no overlap between the results of this paper and results from these more general investigations, we have nonetheless listed, for the benefit of interested readers, a number of papers [4]–[8] that deal with this more general problem.

Acknowledgment. We thank Professor C. R. Johnson for stimulating discussions related to this research.

REFERENCES

[1] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, 2nd ed., The Johns Hopkins University Press, Baltimore, MD, 1989.

[2] S. MARTÍNEZ, G. MICHON, AND J. SAN MARTÍN, *Inverses of ultrametric matrices are of Stieltjes type*, SIAM J. Matrix Anal. Appl., 15 (1994), pp. 98–106.

[3] R. S. VARGA, *Matrix Iterative Analysis*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1962.
[4] C. R. JOHNSON, *Inverse M-matrices*, Linear Algebra Appl., 47 (1982), pp. 195–216.
[5] I. KOLTRACHT AND M. NEUMANN, *On the inverse M-matrix problems for real symmetric positive-definite Toeplitz matrices*, SIAM J. Matrix Anal. Appl., 12 (1991), pp. 310–320.
[6] T. L. MARKHAM, *Nonnegative matrices whose inverses are M-matrices*, Proc. Amer. Math. Soc., 36 (1972), pp. 326–330.
[7] M. NEUMANN AND R. J. PLEMMONS, *Generalized inverse-positivity and splittings of M-matrices*, Linear Algebra Appl., 23 (1979), pp. 21–35.
[8] R. A. WILLOUGHBY, *The inverse M-matrix problem*, Linear Algebra Appl., 18 (1977), pp. 75–94.