Constr. Approx. (1994) 10: 107–129

CONSTRUCTIVE
APPROXIMATION

Lehmer Pairs of Zeros, the de Bruijn–Newman Constant Λ, and the Riemann Hypothesis

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Dedicated to the memory of Professor Derrick H. Lehmer

Abstract. We give here a rigorous formulation for a pair of consecutive simple positive zeros of the function H_0 (which is closely related to the Riemann ξ-function) to be a "Lehmer pair" of zeros of H_0. With this formulation, we establish that each such pair of zeros gives a lower bound for the de Bruijn–Newman constant Λ (where the Riemann Hypothesis is equivalent to the assertion that $\Lambda \le 0$). We also numerically obtain the following new lower bound for Λ:

$$-4.379 \cdot 10^{-6} < \Lambda.$$

1. Introduction

The purpose of this paper is fourfold:

(i) To give a rigorous formulation for a pair of consecutive simple positive zeros of the function H_0 (which is closely related to the Riemann ξ-function) to be a "Lehmer pair" of zeros of H_0.
(ii) To establish theoretically that each such Lehmer pair of zeros of H_0 determines a lower bound for the de Bruijn–Newman constant Λ.
(iii) To show that such Lehmer pairs of zeros of H_0 do indeed exist.
(iv) To determine numerically a new improved lower bound for Λ.

To begin, it is known (see p. 255 of [T]) that the Riemann ξ-function can be expressed in the form

$$\frac{\xi(x/2)}{8} = \int_0^\infty \Phi(u)\cos(xu)\,du \qquad (x \in \mathbf{C}), \tag{1.1}$$

where

$$\Phi(u) := \sum_{n=1}^\infty (2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}) \exp(-\pi n^2 e^{4u}) \qquad (0 \le u < \infty), \tag{1.2}$$

Date received: November 2, 1992. Date revised: June 30, 1993. Communicated by Dietrich Braess.
AMS classification: 30D10, 30D15.
Key words and phrases: Riemann Hypothesis, de Bruijn–Newman constant, Lehmer pairs of zeros.

and the *Riemann Hypothesis is the statement that all zeros of* ξ *are real.* Next, on defining

$$H_t(x) := \int_0^\infty e^{tu^2}\Phi(u)\cos(xu)\,du \qquad (t \in \mathbf{R};\ x \in \mathbf{C}), \tag{1.3}$$

then H_0 and the Riemann ξ-function are related through

$$H_0(x) = \frac{\xi(x/2)}{8}, \tag{1.3'}$$

so that the Riemann Hypothesis is also equivalent to the statement that all zeros of H_0 are real.

In 1950 de Bruijn [B] established that:

(i) $H_t(x)$ has only real zeros for $t \geq \frac{1}{2}$.
(ii) If $H_t(x)$ has only real zeros for some real t, then $H_{t'}(x)$ also has only real zeros for any $t' \geq t$.

Subsequently, Newman [N] showed in 1976 that there is a real constant Λ, which satisfies $-\infty < \Lambda \leq \frac{1}{2}$, such that

(1.4)
(i) $H_t(x)$ has only real zeros if and only if $t \geq \Lambda$.
(ii) H_t has some nonreal zeros if and only if $t < \Lambda$.

(This constant Λ is now called the de Bruijn–Newman constant in the literature.) Because the Riemann Hypothesis is then equivalent to the statement $\Lambda \leq 0$, there have been recent results giving lower bounds for Λ. These results are summarized below:

$$\begin{cases} -50 < \Lambda & \text{(1988, see [CNV2])}, \\ -5 < \Lambda & \text{(1991, see [R2])}, \\ -0.385 < \Lambda & \text{(1992, see [NRV])}, \\ -0.0991 < \Lambda & \text{(1991, see [CRV])}. \end{cases} \tag{1.5}$$

It is perhaps of interest to mention that, while each of the known lower bounds for Λ of (1.5) depended on numerical calculations, the underlying mathematical analysis for *each* lower bound above was different. Similarly, the new lower bounds to be derived in this paper depend again on numerical calculations, but the underlying mathematical analysis here (depending now on a differential equation approach) is completely different from the techniques used in deriving the result of (1.5).

Next, the *Laguerre–Pólya class* is defined as the collection of all real entire functions $f(x)$ which can be expressed as

$$f(x) = Ce^{-\alpha x^2 + \beta x}x^n \prod_{j=1}^{\infty}\left(1 - \frac{x}{x_j}\right)e^{x/x_j} \qquad (x \in \mathbf{C}),$$

where $\alpha \geq 0$, β and C are real numbers, n is a nonnegative integer, and the x_j's are real and nonzero and satisfy $0 < |x_1| \leq |x_2| \leq \cdots$ with $\sum_{j=1}^{\infty} x_j^{-2} < \infty$. (For any such real entire function f, we write $f \in \mathscr{L} - \mathscr{P}$.) It is well known that the Riemann Hypothesis is equivalent to the statement that $H_0 \in \mathscr{L} - \mathscr{P}$. It is also known (see [CRV]) that the de Bruijn–Newman constant Λ has the following characterization in terms of the Laguerre–Pólya class:

$$(1.6) \qquad H_t \in \mathscr{L} - \mathscr{P} \quad \text{if and only if} \quad t \geq \Lambda.$$

It is known (see [CNV2]) that H_t is an even real entire function of order 1 and of maximal type, for each real t. Thus, from the Hadamard factorization theorem, $H_t(x)$ can be represented as

$$(1.7) \qquad H_t(x) = H_t(0) \prod_{j=1}^{\infty} \left(1 - \frac{x^2}{x_j^2(t)}\right) \qquad (x \in \mathbf{C}),$$

where, from (1.3) and the well-known fact that $\Phi(u) > 0$ for all u, $H_t(0) > 0$ and where its zeros, $\{x_j(t)\}_{j=1}^{\infty}$, which are numbered according to increasing moduli, i.e.,

$$(1.8) \qquad 0 < |x_1(t)| \leq |x_2(t)| \leq \cdots,$$

satisfy

$$(1.8') \qquad \sum_{j=1}^{\infty} |x_j(t)|^{-2} < \infty.$$

In 1956 Lehmer [L], from his calculations of zeros of the Riemann ζ-function in the critical strip, found two consecutive simple zeros on the critical line which were exceptionally close to one another, and this has been referred to in the literature as "*Lehmer's near counterexample*" to the Riemann Hypothesis (see p. 177 of [E]). In our formulation here, those zeros, found by Lehmer, correspond to the following two consecutive simple positive zeros of H_0:

$$(1.9) \qquad \begin{aligned} x_{6709}(0) &= 14{,}010.125\,732\,349\,841\ldots, \\ x_{6710}(0) &= 14{,}010.201\,129\,345\,293\ldots. \end{aligned}$$

Here, and below, we find it convenient to use the above numbering system (which differs from that of (1.8)) where the zeros of H_0 in $\operatorname{Re} z > 0$ are numbered according to increasing modulus, and where, from the evenness of H_0, $x_{-j}(0) := -x_j(0)$ $(j = 1, 2, \ldots)$.

We begin this paper with a rigorous formulation of Lehmer's notion of a "close pair" of consecutive zeros of H_0.

Definition 1. With k a positive integer, let $x_k(0)$ and $x_{k+1}(0)$ (with $0 < x_k(0) < x_{k+1}(0)$) be two consecutive simple positive zeros of H_0, and set

$$(1.10) \qquad \Delta_k := x_{k+1}(0) - x_k(0).$$

Then $\{x_k(0); x_{k+1}(0)\}$ is a *Lehmer pair of zeros of* H_0 if

$$(1.11) \qquad \Delta_k^2 \cdot g_k(0) < \tfrac{4}{5},$$

where

$$g_k(0) := \sum_{j \neq k, k+1}' \left\{ \frac{1}{(x_k(0) - x_j(0))^2} + \frac{1}{(x_{k+1}(0) - x_j(0))^2} \right\}; \tag{1.12}$$

here (and in what follows) the prime in the above summation means that $j \neq 0$, and the summation extends over all positive and negative integers with $j \neq k$, $k + 1$, 0.

We remark that the convergence of the sum in (1.12) is guaranteed by the convergence of the sum $\sum_{j=1}^{\infty} |x_j(0)|^{-2}$ (see (1.8′)).

Though no nonreal zeros of H_0 have to date been found, the sum in the definition of $g_k(0)$ in (1.12) is over *all* zeros (real or nonreal) of H_0, where it is known (see [B]) that $|\text{Im } x_j(0)| \leq 1$ if nonreal zeros exist. However, because H_0 is an even real entire function, then $\alpha + i\beta$ is a zero of H_0 if and only if $\pm\alpha \pm i\beta$ are all zeros of H_0. Consequently, $g_k(0)$ of (1.12) is always a real number. We note, moreover, that it can be shown (as on p. 315 of [CRV]) that $g_k(0) > 0$, provided that the consecutive simple positive zeros $x_k(0)$ and $x_{k+1}(0)$ of Definition 1 satisfy

$$0 < x_k(0) < x_{k+1}(0) < 1{,}090{,}879{,}645.50.$$

This above result makes use of the numerical results of van de Lune, te Riele, and Winter [LRW] on the zeros of the Riemann ζ-function in the upper-half critical strip.

From density considerations, it is known (see p. 214 of [T]) that consecutive pairs $\{x_{k_i}(0); x_{k_i+1}(0)\}$ of real zeros of H_0 can be found for which $\lim_{i\to\infty} \Delta_{k_i} = 0$, i.e., "extremely close pairs" of consecutive zeros of H_0 certainly do exist. We remark, however, that the inequality in (1.11) is not solely dependent on Δ_k, so that a Lehmer pair of zeros of H_0, from Definition 1, requires more than just close pairs of zeros.

With the above notations, our main result (proved in Section 3) can be stated as

Theorem 1. *Let $\{x_k(0); x_{k+1}(0)\}$ be a Lehmer pair of zeros of H_0. If (see (1.12)) $g_k(0) \leq 0$, then $\Lambda > 0$. If $g_k(0) > 0$, set*

$$\lambda_k := \frac{(1 - \frac{5}{4}\Delta_k^2 \cdot g_k(0))^{4/5} - 1}{8 g_k(0)}, \tag{1.13}$$

so that $-1/[8g_k(0)] < \lambda_k < 0$. Then the de Bruijn–Newman constant Λ satisfies

$$\lambda_k \leq \Lambda. \tag{1.14}$$

As a consequence of Theorem 1, we have the following corollary (whose proof is also given in Section 3).

Corollary 1. *Suppose that H_0 has infinitely many Lehmer pairs $\{x_{k_i}(0); x_{k_i+1}(0)\}_{i=1}^{\infty}$ with $g_{k_i}(0) > 0$ for all $i \geq 1$ and with $\lim_{i\to\infty} \Delta_{k_i}^2 = 0$. Then*

$$0 \leq \Lambda. \tag{1.14′}$$

The result of (1.14′) of Corollary 1 (that is, the reverse of the inequality $\Lambda \le 0$, which is *equivalent* to the Riemann Hypothesis) gives credence to the conjecture of Newman [N] that $0 \le \Lambda$. Note that the existence of infinitely many Lehmer pairs, as in Corollary 1, would *not* disprove the Riemann Hypothesis, but would, in paraphrasing the words of Newman [N], yield a "quantitative version of the dictum that the Riemann Hypothesis, if true, is only barely so."

In Section 4 it is shown that the pair of zeros of H_0 in (1.9), which were originally determined by Lehmer, is a "Lehmer pair" of zeros of H_0 in the sense of Definition 1, and, on applying Theorem 1 to this pair of zeros, a new lower bound for Λ is established, namely,

$$-7.113 \cdot 10^{-4} < \Lambda, \tag{1.15}$$

which improves the previous lower bounds for Λ given in (1.5).

However, in Section 5, an even more remarkable Lehmer pair of zeros of H_0 is determined, and this Lehmer pair generates the following new lower bound for Λ:

$$-4.379 \cdot 10^{-6} < \Lambda, \tag{1.16}$$

which was given in the Abstract.

2. The Movement of the Zeros of H_t, as a Function of t, and the de Bruijn–Newman Constant Λ

In this section, in preparation to proving Theorem 1 in the next section, we investigate both the *movement* of the zeros of H_t, as a function of t, and how this movement is related to the de Bruijn–Newman constant Λ. To this end, we begin with Lemma 2.1, which shows that the movement of a simple real zero of H_t is governed by local conditions only.

Lemma 2.1. *Suppose x_0 is a simple real zero of H_{t_0}, where t_0 is real. Then, in some open real interval I containing t_0, there is a real differentiable function $x(t)$ defined on I, satisfying $x(t_0) = x_0$, such that $x(t)$ is a simple real zero of H_t and $H_t(x(t)) \equiv 0$ for $t \in I$. Moreover,*

$$x'(t) = \frac{H_t''(x(t))}{H_t'(x(t))} \qquad (t \in I). \tag{2.1}$$

Proof. Since x_0 is by hypothesis a simple real zero of H_{t_0}, then, by the implicit function theorem, there is an open interval I containing t_0, and a real differentiable function $x(t)$ defined on I, such that $x(t)$ is a simple real zero of H_t, with $x(t_0) = x_0$, and $H_t(x(t)) \equiv 0$ for $t \in I$. From (1.3), differentiating $H_t(x(t)) \equiv 0$ with respect to t directly gives (2.1). ■

The significance of Lemma 2.1 is that the *movement* of the simple real zero $x(t)$ of H_t is locally determined solely by the ratio $H_t''(x(t))/H_t'(x(t))$ of (2.1).

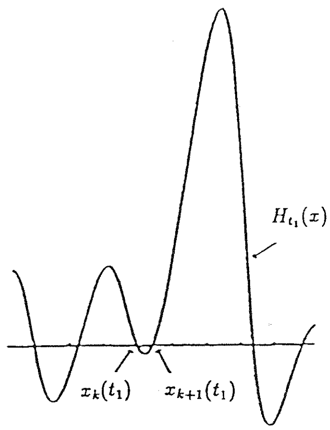

Fig. 1. Graph of $H_{t_1}(x)$.

To illustrate the result of Lemma 2.1, consider the graph of $H_t(x)$ in Fig. 1, where H_{t_1} has two close consecutive simple positive zeros $x_k(t_1)$ and $x_{k+1}(t_1)$, and the remaining zeros of H_t are widely separated from $x_k(t_1)$ and $x_{k+1}(t_1)$. From the graph we see that $H''_{t_1}(x) > 0$ on an interval containing $[x_k(t_1), x_{k+1}(t_1)]$, and $H'_{t_1}(x_k(t_1)) < 0$, while $H'_{t_1}(x_{k+1}(t_1)) > 0$. Using (2.1), we conclude from Fig. 1 that

$$x'_k(t_1) < 0 \quad \text{and} \quad x'_{k+1}(t_1) > 0, \tag{2.1'}$$

and this indicates that, on *decreasing* t, $x_k(t)$ increases, while $x_{k+1}(t)$ decreases, i.e., these two zeros move *toward* one another, as t decreases from t_1, and, similarly, these two zeros move *away* from one another, if t increases from t_1. (We note that (2.1′) also remains valid if the curve in Fig. 1 is inverted, i.e., if H_{t_1} has a positive local maximum on $[x_k(t_1), x_{k+1}(t_1)]$.)

What is of interest to us is the situation where H_t has two consecutive simple positive zeros $x_k(t)$ and $x_{k+1}(t)$, and where these zeros move toward one another as t decreases, and *collide* (i.e., they coalesce) when $t = t_0$. We show, in what follows, that this process leads to a *lower bound* for the de Bruijn–Newman constant Λ.

We begin with the coalesced case where $x_k(t_0) = x_{k+1}(t_0)$, which is equivalent to $H_{t_0}(x_k(t_0)) = H'_{t_0}(x_k(t_0)) = 0$.

Lemma 2.2. *Suppose, for some real t_0 and x_0, that*

$$H_{t_0}(x_0) = H'_{t_0}(x_0) = 0. \tag{2.2}$$

Then

$$t_0 \leq \Lambda. \tag{2.3}$$

Proof. First assume that $H''_{t_0}(x_0) \neq 0$. Then, for any $\delta > 0$, it follows from the definition of $H_t(x)$ in (1.3) that

$$H_{t_0-\delta}(x) = \int_0^\infty e^{t_0u^2}e^{-\delta u^2}\Phi(u)\cos(xu)\,du = \int_0^\infty e^{t_0u^2}\left(\sum_{k=0}^\infty \frac{(-\delta u^2)^k}{k!}\right)\Phi(u)\cos(xu)\,du$$

$$= \sum_{k=0}^\infty \frac{\delta^k}{k!} H_{t_0}^{(2k)}(x) \qquad (x \in \mathbf{R}),$$

where the termwise integration is justified from the known rapidly decreasing property of $\Phi(t)$ as $t \to \infty$ (see [CNV1]). On differentiating with respect to x, the above expression yields

$$H_{t_0-\delta}^{(j)}(x) = \sum_{k=0}^\infty \frac{\delta^k}{k!} H_{t_0}^{(2k+j)}(x) \qquad (j = 0, 1, \ldots; x \in \mathbf{R}).$$

If, for any real entire function g, we set

$$L_1(g(x)) := (g'(x))^2 - g(x)g''(x) \qquad (x \in \mathbf{R}), \tag{2.4}$$

then the above expressions for $\{H_{t_0-\delta}^{(j)}(x_0)\}_{j=0}^2$, when substituted in (2.4), yield, with the hypothesis of (2.2),

$$L_1(H_{t_0-\delta}(x_0)) = -\delta(H''_{t_0}(x_0))^2 + \delta^2\{(H_{t_0}^{(3)}(x_0))^2 - \tfrac{3}{2}H_{t_0}^{(2)}(x_0)H_{t_0}^{(4)}(x_0)\} + O(\delta^3),$$

as $\delta \downarrow 0$. Hence, since $H''_{t_0}(x_0) \neq 0$, it is evident that

$$L_1(H_{t_0-\delta}(x_0)) < 0 \tag{2.5}$$

for all $\delta > 0$ sufficiently small. For a real entire function f in the Laguerre–Pólya class (written $f \in \mathscr{L} - \mathscr{P}$), it is well known (see, for example, [CV]) that f must satisfy the following Laguerre inequalities:

$$L_m(f(x)) := (f^{(m)}(x))^2 - f^{(m-1)}(x)\cdot f^{(m+1)}(x) \geq 0 \qquad (m = 1, 2, \ldots) \tag{2.6}$$

for *all* real x. Thus, (2.5) shows that $H_{t_0-\delta} \notin \mathscr{L} - \mathscr{P}$ for all $\delta > 0$ sufficiently small. Hence, from (1.6), we conclude that $t_0 - \delta < \Lambda$ for all $\delta > 0$ sufficiently small, which gives that $t_0 \leq \Lambda$, the desired result of (2.3), is valid whenever (2.2) holds.

Finally, suppose that x_0 is a zero of H_{t_0} of multiplicity $k + 1$ with $k \geq 1$, i.e.,

$$H_{t_0}^{(j)}(x_0) = 0 \quad (j = 0, 1, \ldots, k) \qquad \text{with} \quad H_{t_0}^{(k+1)}(x_0) \neq 0, \quad \text{where} \quad k \geq 1. \tag{2.7}$$

Then, from (1.3), consider the function

$$h_t(x) := H_t^{(k-1)}(x),$$

where (2.7) implies

$$h_{t_0}(x_0) = h'_{t_0}(x_0) = 0 \quad \text{and} \quad h''_{t_0}(x_0) \neq 0. \tag{2.8}$$

We note from (1.3) that $h_t(x)$ can be expressed as

$$h_t(x) = \int_0^\infty u^{k-1}e^{tu^2}\Phi(u)\cos\left[\frac{(k-1)\pi}{2} + xu\right] du.$$

In the same manner as in the first part of this proof, it can be verified that

$$L_1(h_{t_0-\delta}(x_0)) = -\delta(h''_{t_0}(x_0))^2 + O(\delta^2),$$

as $\delta \downarrow 0$, so that $L_1(h_{t_0-\delta}(x_0)) < 0$ for all $\delta > 0$ sufficiently small. Consequently, $h_{t_0-\delta} = H^{(k-1)}_{t_0-\delta}$ is not in $\mathscr{L}-\mathscr{P}$ for all $\delta > 0$ sufficiently small. However, since the Laguerre–Pólya class is closed under differentiation (see [Ob]), it follows that $H_t \notin \mathscr{L}-\mathscr{P}$ for all $t < t_0$, and (2.3) again is valid. ■

Remarks. (1) The function $H_t(x)$, defined in (1.3), can be shown, by differentiation, to satisfy the *backward heat equation*:

$$\frac{\partial(H_t(x))}{\partial t} = -\frac{\partial^2 H_t(x)}{\partial x^2} \qquad (x, t \text{ real}).$$

This can be used to give an alternate proof of Lemma 2.2.

(2) For the special case $t_0 = 0$, we see from Lemma 2.2 that if H_0 has a multiple real zero (at some x_0), then $0 \leq \Lambda$, which is the reverse of the inequality equivalent to the Riemann Hypothesis.

(3) The argument used in the proof of Lemma 2.2 also shows that if

$$H_{t_0}(x_0) = H'_{t_0}(x_0) = 0 \qquad \text{but} \quad H''_{t_0}(x_0) \neq 0,$$

then, for sufficiently small $\varepsilon > 0$, $H_{t_0+\varepsilon}$ has two *simple* real zeros near x_0. To see this, it can be verified, with the above assumptions, that

$$H_{t_0+s^2}(x_0) = -s^2 H^{(2)}_{t_0}(x_0) + \frac{s^4}{2} H^{(4)}_{t_0}(x_0) + O(s^6)$$

and

$$H_{t_0+s^2}(x_0 \pm 2s) = s^2 H^{(2)}_{t_0}(x_0) \mp \tfrac{2}{3}s^3 H^{(3)}_{t_0}(x_0) + O(s^4),$$

as $s \to 0$. Since $H^{(2)}_{t_0}(x_0) \neq 0$, the above two expressions show that $H_{t_0+s^2}$ has two sign changes, and hence two simple zeros, in $[x_0 - 2s, x_0 + 2s]$, for $s > 0$ sufficiently small.

(4) Coupling the result of Remark (3) above with the characterization of Λ in (1.6) then gives the apparently new result of

Corollary 2. *For any $t > \Lambda$, the zeros of H_t are real and simple.*

Our next goal is to obtain an alternate and useful formula for the velocity, $x'(t)$, of a simple real zero of H_t, which bypasses the direct calculation of $H''_t(x)$ and $H'_t(x)$ in (2.1) of Lemma 2.1. To this end, we include the following elementary result.

Lemma 2.3. *Let q be analytic in a domain D of $\mathbf{C}$, and set*

$$f(z) := (z - w)q(z), \quad \text{where} \quad w \in D \quad \text{and} \quad q(w) \neq 0. \tag{2.9}$$

Then $f'(w) \neq 0$ and

$$\frac{f''(w)}{f'(w)} = 2\,\frac{q'(w)}{q(w)}. \tag{2.10}$$

Proof. Since $f'(z) = (z - w)q'(z) + q(z)$ for all $z \in D$, then

$$\frac{d}{dz} \log f'(z) = \frac{f''(z)}{f'(z)} = \frac{(z - w)q''(z) + 2q'(z)}{(z - w)q'(z) + q(z)} \qquad (z \in D),$$

and (2.10) follows on setting $z = w$. ■

This brings us to

Lemma 2.4. *For a positive integer k, let $x_k(t)$ and $x_{k+1}(t)$ be two consecutive simple positive zeros of H_t, where it is assumed that $t > \Lambda$. Then the following convergent series representation for $x_k'(t)$ is valid:*

$$x_k'(t) = 2 \sum_{j \neq k}' \frac{1}{(x_k(t) - x_j(t))}. \tag{2.11}$$

Moreover, from (2.11) *and the analogous formula for $x_{k+1}(t)$, it holds that*

$$x_{k+1}'(t) - x_k'(t) = + \frac{4}{[x_{k+1}(t) - x_k(t)]} - f_k(t) \cdot [x_{k+1}(t) - x_k(t)], \tag{2.12}$$

where

$$f_k(t) := \sum_{j \neq k, k+1}' \frac{2}{[x_k(t) - x_j(t)][x_{k+1}(t) - x_j(t)]}. \tag{2.13}$$

Proof. From Corollary 1, we know that, for $t > \Lambda$, all the zeros of H_t are real *and* simple. Then, for $t > \Lambda$, write (1.7) in the form

$$H_t(x) = (x - x_k(t))q_t(x), \tag{2.14}$$

where

$$q_t(x) := H_t(0)\left(1 + \frac{x}{x_k(t)}\right)\left(\frac{1}{-x_k(t)}\right) \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \left(1 - \frac{x^2}{x_j^2(t)}\right). \tag{2.15}$$

By Lemmas 2.1 and 2.3, a calculation yields (2.11), where the convergence of the series follows from the fact (see (1.8′)) that $\sum_{j=1}^{\infty} 1/|x_j(t)|^2 < \infty$. Finally, a direct verification shows that (2.12) is a straightforward consequence of (2.11). ■

As noted above, the assumption $t > \Lambda$ implies that all zeros $x_j(t)$ of H_t are real and simple. Consequently, as each summand in (2.13) is positive for any $j \neq k$, $k+1$,

$$0 < f_k(t) \qquad (t > \Lambda). \tag{2.16}$$

In Lemma 2.4 we have described, for $t > \Lambda$, the "law of motion" of a zero $x_k(t)$ of H_t in terms of the first-order nonlinear ordinary differential equation (2.11), and in Lemma 2.2 we have demonstrated that *if* there is a coalescence of two real zeros of H_t, so that H_t has a zero of multiplicity greater than one, then as t is decreased, a pair of complex conjugate nonreal zeros of H_t is produced. Now we expect the dominant term on the right-hand side of (2.12) to be

$$+\frac{4}{(x_{k+1}(t) - x_k(t))}$$

for t near the time $t = t_0$ when the coalescence occurs. Whether or not a coalescence will occur depends on the nature of the function $f_k(t)$ defined in (2.13). In order to make the analysis of the movement of the zeros of H_t more tractable, we introduce the following auxiliary function:

$$g_k(t) := \sum_{j \neq k, k+1}' \left\{ \frac{1}{[x_k(t) - x_j(t)]^2} + \frac{1}{[x_{k+1}(t) - x_j(t)]^2} \right\} \qquad (t > \Lambda). \tag{2.17}$$

As each (positive) summand in the definition of $f_k(t)$ in (2.13) is strictly less than the corresponding summand in (2.17), it follows that

$$0 < f_k(t) < g_k(t) \qquad (t > \Lambda). \tag{2.18}$$

As the final result of this section, in preparation for the proof of Theorem 1 in Section 3, we determine a lower bound for $g_k'(t)$ in (2.19), and an upper bound for $g_k(t)$ in (2.20).

Lemma 2.5. *For $t > \Lambda$ and for any positive integer k, then* (*see* (2.17))

$$g_k'(t) > -8(g_k(t))^2 \qquad (t > \Lambda). \tag{2.19}$$

In addition, if $\Lambda < 0$, then

$$g_k(t) < \frac{g_k(0)}{1 + 8g_k(0) \cdot t} \qquad \text{if} \quad t \in (\Lambda, 0] \cap \left(-\frac{1}{8g_k(0)}, 0 \right). \tag{2.20}$$

Proof. Since $t > \Lambda$, we know from Corollary 1 that all the zeros of H_t are real and simple. Hence, it follows from Lemma 2.1 that $x_j'(t)$ is defined (and is locally analytic) for $t > \Lambda$. Also, on any compact subset S of $(\Lambda, 0)$, the series for $g_k(t)$ in (2.17) converges uniformly and absolutely, so that the termwise differentiation of this series (for $t \in S$) is justified. Thus, from (2.17),

$$g_k'(t) = -2 \sum_{j \neq k, k+1}' \left[\frac{x_k'(t) - x_j'(t)}{(x_k(t) - x_j(t))^3} + \frac{x_{k+1}'(t) - x_j'(t)}{(x_{k+1}(t) - x_j(t))^3} \right] \qquad (t > \Lambda). \tag{2.21}$$

Now, if $j \neq k$ and if $t > \Lambda$, it follows, as in (2.12) of Lemma 2.4, that

$$(2.22)\quad x_k'(t) - x_j'(t) = \frac{4}{x_k(t) - x_j(t)} - [x_k(t) - x_j(t)] \sum_{i \neq k,j}' \frac{2}{(x_k(t) - x_i(t))(x_j(t) - x_i(t))},$$

where a similar expression is valid for $x_{k+1}'(t) - x_j'(t)$ when $j \neq k+1$. On substituting these expressions for $x_k'(t) - x_j'(t)$ and $x_{k+1}'(t) - x_j'(t)$ into the sum in (2.21), a calculation yields that

$$(2.23)\qquad g_k'(t) = A_k(t) + B_k(t) \qquad (t > \Lambda),$$

where

$$(2.24)\qquad A_k(t) := -8 \sum_{j \neq k,k+1}' \left[\frac{1}{(x_k(t) - x_j(t))^4} + \frac{1}{(x_{k+1}(t) - x_j(t))^4}\right],$$

and where, on suppressing for convenience the t-dependence of the $x_l(t)$'s,

$$\begin{aligned}(2.25)\quad B_k(t) := 4 \sum_{j \neq k,k+1}' &\left\{\frac{1}{(x_k - x_j)^2(x_k - x_{k+1})(x_j - x_{k+1})} + \frac{1}{(x_{k+1} - x_j)^2(x_{k+1} - x_k)(x_j - x_k)}\right\} \\ + 4 \sum_{j \neq k,k+1}' \sum_{i \neq j,k,k+1}' &\left\{\frac{1}{(x_k - x_j)^2(x_k - x_i)(x_j - x_i)} + \frac{1}{(x_{k+1} - x_j)^2(x_{k+1} - x_i)(x_j - x_i)}\right\}.\end{aligned}$$

The first sum of (2.25) then reduces to

$$4 \sum_{j \neq k,k+1}' \frac{1}{(x_k - x_j)^2(x_{k+1} - x_j)^2},$$

and the double sum in (2.25) can be expressed, on interchanging the roles of the summation indices i and j, as

$$\begin{aligned}2 \sum_{j \neq k,k+1}' \sum_{i \neq j,k,k+1}' &\left\{\frac{1}{(x_k - x_j)^2(x_k - x_i)(x_j - x_i)} + \frac{1}{(x_k - x_i)^2(x_k - x_j)(x_i - x_j)}\right\} \\ + 2 \sum_{j \neq k,k+1} \sum_{i \neq j,k,k+1} &\left\{\frac{1}{(x_{k+1} - x_j)^2(x_{k+1} - x_i)(x_j - x_i)} + \frac{1}{(x_{k+1} - x_i)^2(x_{k+1} - x_j)(x_i - x_j)}\right\},\end{aligned}$$

and combining the above double sums yields that

(2.26)

$$B_k(t) = 4 \sum_{j\neq k,k+1}' \frac{1}{(x_k - x_j)^2(x_{k+1} - x_j)^2} + 2 \sum_{j\neq k,k+1}' \sum_{i\neq j,k,k+1}' \left\{ \frac{1}{(x_k - x_i)^2(x_k - x_j)^2} + \frac{1}{(x_{k+1} - x_i)^2(x_{k+1} - x_j)^2} \right\}.$$

Since all summands in (2.26) are positive,

$$B_k(t) > 0 \qquad (t > \Lambda). \tag{2.27}$$

Consequently, from (2.23),

$$g_k'(t) > A_k(t) \qquad (t > \Lambda). \tag{2.28}$$

In order to obtain the first desired inequality of (2.19) of Lemma 2.5, note that

$$\sum_{j\neq k,k+1}' \frac{1}{(x_k(t) - x_j(t))^4} < \left(\sum_{j\neq k,k+1}' \frac{1}{(x_k(t) - x_j(t))^2} \right)^2 \qquad (t > \Lambda), \tag{2.29}$$

and thus (see (2.24), (2.27), and (2.29))

$$\begin{aligned} -\frac{A_k(t)}{8} &< \left(\sum_{j\neq k,k+1}' \frac{1}{(x_k(t) - x_j(t))^2} \right)^2 + \left(\sum_{j\neq k,k+1}' \frac{1}{(x_{k+1}(t) - x_j(t))^2} \right)^2 \\ &< \left(\sum_{j\neq k,k+1}' \left[\frac{1}{(x_k(t) - x_j(t))^2} + \frac{1}{(x_{h+1}(t) - x_j(t))^2} \right] \right)^2 \\ &= (g_k(t))^2 \qquad (t > \Lambda), \end{aligned} \tag{2.30}$$

where the last equality follows from the definition in (2.17). Therefore, (2.30) and (2.29) yield the desired lower bound (2.19).

We next turn to the proof of (2.20). Let s be a fixed but arbitrary point in $(\Lambda, 0] \cap (-1/(8g_k(0)), 0)$. Then, from (2.18), $g_k(t) > 0$ for all $t > \Lambda$, and it follows from (2.19) that

$$\int_s^0 \left\{ -\frac{g_k'(t)}{[g_k(t)]^2} \right\} dt < 8 \int_s^0 dt.$$

However, on integrating, this gives

$$\frac{1}{g_k(0)} - \frac{1}{g_k(s)} < -8s,$$

which is equivalent to the desired result of (2.20). ■

3. Proofs of Theorem 1 and Corollary 1 of Section 1

Proof of Theorem 1. Let $\{x_k(0); x_{k+1}(0)\}$ be a Lehmer pair of zeros of H_0. First, if the zeros $x_j(0)$ of H_0 are all real, then $g_k(0)$ of (1.11), as the sum of positive terms, is clearly positive. Thus, if $g_k(0) \le 0$, then H_0 necessarily has some nonreal zeros; whence from (1.4)(ii), $\Lambda > 0$.

Next, assume that $g_k(0) > 0$. Then, since $\{x_k(0); x_{k+1}(0)\}$ is a Lehmer pair of zeros of H_0, (1.11) is valid, so that λ_k, as defined in (1.13), satisfies $-1/[8g_k(0)] < \lambda_k < 0$. Suppose, to the contrary of (1.14), that $\Lambda < \lambda_k < 0$. Set

$$y_k(t) := x_{k+1}(t) - x_k(t) \qquad (t \in [\lambda_k, 0]), \tag{3.1}$$

and observe that $y_k(t) > 0$ for all $t \in [\lambda_k, 0]$, since the hypothesis $\Lambda < \lambda_k$ implies from Corollary 2 that all the zeros of H_s are real and simple for any $s \ge \lambda_k$. As a consequence of (2.12) of Lemma 2.4 and (3.1), consider the associated differential equation

$$\begin{cases} \dfrac{dy_k(t)}{dt} = \dfrac{4}{y_k(t)} - f_k(t) \cdot y_k(t) & (t \in [\lambda_k, 0]), \\ y_k(0) = \Delta_k, \end{cases} \tag{3.2}$$

where Δ_k is defined in (1.10) and where $f_k(t)$ is defined in (2.13). On multiplying the first equation of (3.2) by $2y_k(t)$, this differential equation takes the form

$$\begin{cases} \dfrac{dy_k^2(t)}{dt} + 2f_k(t) \cdot y_k^2(t) = 8 & (t \in [\lambda_k, 0]), \\ y_k^2(0) = \Delta_k^2. \end{cases} \tag{3.3}$$

An integrating factor for the differential equation in (3.3) is $\exp(F_k(t))$, where

$$F_k(t) = -2 \int_t^0 f_k(u)\, du \qquad (t \in [\lambda_k, 0]), \tag{3.4}$$

and the solution of (3.3) is explicitly given by

$$\tfrac{1}{8}(\Delta_k^2 - \exp(F_k(t)) \cdot y_k^2(t)) = \int_t^0 \exp(F_k(u))\, du \qquad (t \in [\lambda_k, 0]). \tag{3.5}$$

To obtain the desired contradiction, we obtain a lower bound for the right-hand side of (3.5). Since $\{x_k(0); x_{k+1}(0)\}$ is a Lehmer pair of zeros of H_0 and since $g_k(0) > 0$, we know that $-1/(8g_k(0)) < \lambda_k$, and from (2.18) that $0 < f_k(t) < g_k(t)$ $(t > \Lambda)$. Thus, from (2.20) of Lemma 2.5,

$$F_k(t) := -2 \int_t^0 f_k(u)\, du \ge -2 \int_t^0 g_k(u)\, du \ge -2 \int_t^0 \left(\frac{g_k(0)}{1 + 8g_k(0)u}\right) du$$

for any $t \in [\lambda_k, 0]$. On directly integrating the last term above,

$$F_k(t) \ge \tfrac{1}{4} \log(1 + 8g_k(0)t) \qquad (t \in [\lambda_k, 0]),$$

so that

$$(3.6)\qquad \int_t^0 \exp(F_k(u))\,du \geq \int_t^0 (1 + 8g_k(0)u)^{1/4}\,du = \frac{1}{10g_k(0)}\{1 - (1 + 8g_k(0)t)^{5/4}\}$$

for any $t \in [\lambda_k, 0]$. Choosing $t = \lambda_k$ in (3.5) and using the lower bound of (3.6) then gives

$$(3.7)\qquad \tfrac{1}{8}[\Delta_k^2 - \exp(F_k(\lambda_k))y_k^2(\lambda_k)] \geq \frac{\{1 - (1 + 8g_k(0)\lambda_k)^{5/4}\}}{10g_k(0)} = \frac{\Delta_k^2}{8},$$

the last equality in (3.7) following directly from the definition of λ_k in (1.13). Next, since (2.16) gives that $f_k(u) > 0$ for all $u > \Lambda$, then $\exp(F_k(\lambda_k)) > 0$, and we necessarily conclude from (3.7) that

$$y_k^2(\lambda_k) := (x_{k+1}(\lambda_k) - x_k(\lambda_k))^2 \leq 0,$$

which is a contradiction (see the line after (3.1)). However, this is a contradiction to the hypothesis that $\Lambda < \lambda_k$, for $\Lambda < \lambda_k$ guarantees that all zeros of H_{λ_k} are real and simple, so that $y_k^2(\lambda_k) = (x_{k+1}(\lambda_k) - x_k(\lambda_k))^2 > 0$. Thus, $\lambda_k \leq \Lambda$. ■

Proof of Corollary 1. Let $\{x_{k_i}(0); x_{k_i}(0)\}_{i=0}^{\infty}$ be Lehmer pairs of zeros of H_0 with $g_{k_i}(0) > 0$ and $\Delta_{k_i} > 0$ for all $i \geq 1$, so that, from (1.11),

$$(3.8)\qquad 0 < \Delta_{k_i}^2 g_{k_i}(0) < \tfrac{4}{5} \qquad \text{for all} \quad i \geq 1.$$

Then, setting $t_i := 5\Delta_{k_i}^2 g_{k_i}(0)/4$ so that $0 < t_i < 1$ from (3.8), λ_{k_i} can be expressed from (1.13) as

$$(3.9)\qquad \frac{\lambda_{k_i}}{\Delta_{k_i}^2} = \frac{5[(1 - t_i)^{4/5} - 1]}{32t_i} \qquad \text{for all} \quad i \geq 1.$$

However, since $5[(1 - t)^{4/5} - 1]/(32t)$ is monotone decreasing on the interval $(0, 1)$ with

$$-\frac{5}{32} < \frac{5[(1 - t)^{4/5} - 1]}{32t} < -\frac{1}{8} \qquad \text{for all} \quad t \in (0, 1),$$

it follows from (3.9) that $\lim_{i\to\infty} \lambda_{k_i} = 0$ if and only if $\lim_{i\to\infty} \Delta_{k_i}^2 = 0$. Thus, as $\lambda_{k_i} \leq \Lambda$ for all $i \geq 1$ from (1.14) of Theorem 1, then the hypothesis of Corollary 1 that $\lim_{i\to\infty} \Delta_{k_i}^2 = 0$ implies that $0 \leq \Lambda$, the desired result of (1.14′). ■

4. Existence of Lehmer Pairs of Zeros of H_0, and a Numerical Lower Bound for Λ

We now use the theoretical results of the previous sections to show that Lehmer pairs of zeros of H_0 do indeed exist. We begin with

Lemma 4.1. *As in* (1.7), *let*

$$H_0(x) = H_0(0) \prod_{j=1}^{\infty} \left(1 - \frac{x^2}{x_j^2(0)}\right),$$

where (*see* (1.8′)) $\sum_{j=1}^{\infty} |x_j(0)|^{-2} < \infty$. *Then*

$$\sum_{j=1}^{\infty} \frac{1}{x_j^2(0)} = 0.005\,776\,248\,278\ldots. \tag{4.1}$$

Proof. Since $H_0(x)$ is a real even entire function (so that $H_0'(0) = 0$), a calculation using (1.7) gives that

$$\left(\frac{H_0'}{H_0}\right)'(0) = \frac{H_0''(0)}{H_0(0)} = -2 \sum_{j=1}^{\infty} \frac{1}{x_j^2(0)}. \tag{4.2}$$

Next, it is known (see [CNV1]) that the entire function $H_0(x)$ has a Taylor series expansion of the form

$$H_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \hat{b}_m x^{2m}}{(2m)!} \qquad (x \in \mathbf{C}); \tag{4.3}$$

here, the moments $\hat{b}_m$ are defined by

$$\hat{b}_m := \int_0^{\infty} u^{2m} \Phi(u)\, du \qquad (m = 0, 1, 2, \ldots),$$

where Φ is defined in (1.2). As mentioned previously, $\Phi(u) > 0$ for all $u \geq 0$, so that $\hat{b}_m > 0$ $(m = 0, 1, 2, \ldots)$. Thus, (4.2) and (4.3) give

$$\frac{H_0''(0)}{H_0(0)} = -\frac{\hat{b}_1}{\hat{b}_0} \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{1}{x_j^2(0)} = \frac{\hat{b}_1}{2\hat{b}_0}. \tag{4.4}$$

Now in [CNV1], the moments $\{\hat{b}_m\}_{m=0}^{20}$ were determined to an absolute error of 10^{-50}, and, in particular (see Table 4.1 of [CNV1]),

$$\hat{b}_0 = 0.062\,140\,097\,273\ldots \quad \text{and} \quad \hat{b}_1 = 0.000\,717\,873\,259\ldots.$$

Substituting the above numbers in the last expression in (4.4) gives the desired result of (4.1), where the truncated number in (4.1) is accurate to 45 significant digits. ■

Recalling our discussion in Section 1 of $g_k(0)$ in (1.12), we continue with

Lemma 4.2. *For positive integers k and N with* $N > k + 1$, *set*

$$g_{k,N}(0) := \sum_{\substack{j=-N \\ j \neq k, k+1}}^{N}{}' \left[\frac{1}{(x_k(0) - x_j(0))^2} + \frac{1}{(x_{k+1}(0) - x_j(0))^2}\right], \tag{4.5}$$

and set

$$R_{k,N+1}(0) := g_k(0) - g_{k,N}(0), \tag{4.6}$$

where $g_k(0)$ *is defined in* (2.17). *Then*

$$|R_{k,N+1}(0)| < 2\left\{\frac{|x_{N+1}^2(0)|}{(|x_{N+1}(0)| - |x_k(0)|)^2} + \frac{|x_{N+1}^2(0)|}{(|x_{N+1}(0)| - |x_{k+1}(0)|)^2} + 2\right\} \cdot \mathrm{Re}\left(\sum_{j=N+1}^{\infty} \frac{1}{x_j^2(0)}\right), \tag{4.7}$$

so that

$$\lim_{N\to\infty} R_{k,N+1}(0) = 0. \tag{4.8}$$

Proof. Because H_0 is an even function, $x_{-j}(0) := -x_j(0)$ and, thus,

$$R_{k,N+1}(0) = \sum_{|j|\geq N+1}\left[\frac{1}{(x_k(0) - x_j(0))^2} + \frac{1}{(x_{k+1}(0) - x_j(0))^2}\right],$$

which can be expressed as

$$R_{k,N+1}(0) = \sum_{j=N+1}^{\infty}\left[\frac{1}{(x_k(0) - x_j(0))^2} + \frac{1}{(x_{k+1}(0) - x_j(0))^2} + \frac{1}{(x_k(0) + x_j(0))^2} + \frac{1}{(x_{k+1}(0) + x_j(0))^2}\right]. \tag{4.9}$$

Now consider the first sum in (4.9), which can be expressed as

$$\sum_{j=N+1}^{\infty} \frac{1}{(x_k(0) - x_j(0))^2} = \sum_{j=N+1}^{\infty} \frac{1}{(1 - x_k(0)/x_j(0))^2} \cdot \frac{1}{x_j^2(0)}. \tag{4.10}$$

Clearly,

$$\frac{1}{|1 - x_k(0)/x_j(0)|^2} \leq \frac{1}{(1 - |x_k(0)/x_j(0)|)^2} \leq \frac{1}{(1 - |x_k(0)/x_{N+1}(0)|)^2} \qquad (j \geq N+1),$$

since, by the ordering used in (1.11), $|x_j(0)| \geq |x_{N+1}(0)|$ for all $j \geq N+1$. On taking moduli in (4.10),

$$\left|\sum_{j=N+1}^{\infty} \frac{1}{(x_k(0) - x_j(0))^2}\right| \leq \frac{|x_{N+1}(0)|^2}{(|x_{N+1}(0)| - |x_k(0)|)^2} \sum_{j=N+1}^{\infty} \frac{1}{|x_j(0)|^2}. \tag{4.11}$$

We next claim that

$$\sum_{j=N+1}^{\infty} \frac{1}{|x_j(0)|^2} \leq 2\,\mathrm{Re}\left(\sum_{j=N+1}^{\infty} \frac{1}{x_j^2(0)}\right). \tag{4.11'}$$

For, on writing $x_j(0) = \alpha + i\beta$, a calculation shows that

$$\frac{1}{|x_j(0)|^2} \leq 2 \operatorname{Re}\left\{\frac{1}{x_j^2(0)}\right\} \tag{4.11''}$$

holds, provided that $\alpha^2 \geq 3\beta^2$. However, since $|\beta| \leq 1$ (see [B]) and since $\alpha \geq 28$ for all $j \geq 1$ (see p. 389 of [T]), (4.11″) is valid for all $j \geq 1$. Hence, (4.11″) gives the result of (4.11′), which in turn, with (4.11), gives

$$\left|\sum_{j=N+1}^{\infty} \frac{1}{(x_k(0) - x_j(0))^2}\right| \leq 2 \frac{|x_{N+1}(0)|^2}{(|x_{N+1}(0)| - |x_k(0)|)^2} \operatorname{Re}\left(\sum_{j=N+1}^{\infty} \frac{1}{x_j^2(0)}\right),$$

the first product term in the desired result of (4.7). The remaining terms in the inequality of (4.7) follow similarly. ■

In searching for Lehmer pairs of zeros of H_0, close pairs of consecutive simple real zeros, with the property that the neighboring zeros are located "relatively far away," are sought. In Tables A and B of [CRV] some "super close" pairs of zeros of H_0 are listed. Here we consider the particular pair of close consecutive simple real zeros of H_0, which was first observed by Lehmer [L], namely,

$$\begin{cases} x_{6709}(0) = 14{,}010.125\,732\,349\,841\ldots, \\ x_{6710}(0) = 14{,}010.201\,129\,345\,293\ldots, \end{cases} \tag{4.12}$$

so that

$$\Delta_{6709} := x_{6710}(0) - x_{6709}(0) = 0.075\,396\,995\,452\ldots. \tag{4.13}$$

It may be of interest to note here that the neighboring zeros are indeed "relatively far away" from $x_{6709}(0)$ and $x_{6710}(0)$:

$$x_{6708}(0) = 14{,}008.087\,446\,998\,657\ldots,$$
$$x_{6711}(0) = 14{,}013.479\,324\,767\,898\ldots.$$

Lemma 4.3. *The pair of zeros* $\{x_{6709}(0); x_{6710}(0)\}$ *of* H_0 *is a Lehmer pair of zeros of* H_0.

Proof. With $k := 6709$ and $N := 14999$, a computation shows (see (4.5)) that

$$\begin{cases} g_{6709;14999}(0) = \displaystyle\sum_{\substack{j=-14999 \\ j\neq 6709,6710}}^{14999}{}' \left[\frac{1}{(x_{6709}(0) - x_j(0))^2} + \frac{1}{(x_{6710}(0) - x_j(0))^2}\right] \\ \qquad = 1.219\,499\,547\,968\ldots. \end{cases} \tag{4.14}$$

The calculation of the sum in (4.14) makes explicit use of the numerical results of te Riele [R1, Table 1], where the numbers $\{\gamma_n\}_{n=1}^{15000}$ have been determined each to an accuracy of 28 significant digits. Each such γ_n is a zero of the function ξ of (1.1), so that, from (1.3′), $H_0(2\gamma_n) = 0$, i.e., $x_n(0) = 2\gamma_n$.

We next estimate the remainder (see (4.6))

$$R_{6709;15000} = g_{6709}(0) - g_{6709;14999}(0).$$

To this end, a computation similar to the one used in (4.14) gives

$$\sum_{j=1}^{14999} \frac{1}{x_j^2(0)} = \tfrac{1}{4} \sum_{j=1}^{14999} \frac{1}{\gamma_j^2} = 5.751\,559\,959\,199\ldots \cdot 10^{-3}, \tag{4.15}$$

so that, from (4.1) of Lemma 4.1,

$$\sum_{j=15000}^{\infty} \frac{1}{x_j^2(0)} = \sum_{j=1}^{\infty} \frac{1}{x_j^2(0)} - \sum_{j=1}^{14999} \frac{1}{x_j^2(0)} = 2.468\,831\,965\,543\ldots \cdot 10^{-5}. \tag{4.16}$$

Since (see Table 1 of [R1])

$$x_{15000}^2(0) = 7.885\,380\,542\,387\ldots \cdot 10^{8}, \tag{4.17}$$

and since

$$\frac{x_{15000}^2(0)}{(x_{6709}(0) - x_{15000}(0))^2} + \frac{x_{15000}^2(0)}{(x_{6710}(0) - x_{15000}(0))^2} + 2 = 9.965\,586\,691\,733\ldots, \tag{4.18}$$

we obtain from (4.7) of Lemma 4.2 that

$$|R_{6709;15000}| < 4.920\,671\,795\,989\ldots \cdot 10^{-4}. \tag{4.19}$$

Thus, with (4.19), an upper estimate for $g_{6709}(0)$ is

$$g_{6709}(0) = g_{6709;14999}(0) + R_{6709;15000}(0) < 1.219\,991\,615\,148\ldots, \tag{4.20}$$

so that (4.13) and (4.20) give

$$\Delta_{6709}^2 \cdot g_{6709}(0) < 6.935\,294\,780\,918\ldots \cdot 10^{-3} < \tfrac{4}{5}. \tag{4.21}$$

Consequently, $\{x_{6709}(0), x_{6710}(0)\}$ is indeed, by definition, a Lehmer pair of zeros of H_0. ∎

Our next result, on applying Theorem 1, is

Theorem 2. *If Λ is the de Bruijn–Newman constant, then*

$$-7.113 \cdot 10^{-4} < \Lambda. \tag{4.22}$$

Proof. Since

$$G(s, \Delta_{6709}) := \frac{1}{8s} [(1 - \tfrac{5}{4} s \Delta_{6709}^2)^{4/5} - 1]$$

is a strictly decreasing function of s when $0 < s < 4/(5\Delta_{6709}^2)$, a calculation, using that $s_0 := 1.219\,991\,615\,148\ldots > g_{6709}$ by (4.20), shows that

$$-7.112\,065\,292\,499\ldots\cdot 10^{-4} = G(s_0, \Delta_{6709}) < G(g_{6709}(0), \Delta_{6709}).$$

However, as $G(g_{6709}(0), \Delta_{6709}) =: \lambda_{6709}$ from (1.13), the above expression reduces to

$$-7.112\,065\,292\,499\ldots\cdot 10^{-4} < \lambda_{6709}.$$

Thus, since $\{x_{6709}(0); x_{6710}(0)\}$ is a Lehmer pair of zeros of H_0 from Lemma 4.3, then (1.14) of Theorem 1 gives

$$\text{(4.22')} \qquad -7.112\,065\,292\,499\ldots\cdot 10^{-4} < \lambda_{6709} \leq \Lambda.$$

Although this lower bound in (4.22′) is actually accurate to 15 significant digits, this has been rounded down to the desired result of (4.22), since the extra digits do not have any special significance. ■

It is natural to ask how accurate the lower bound, $\lambda_k \leq \Lambda$, of (1.14) of Theorem 1 is, in light of the seemingly generous inequalities used in (2.29) and (2.30), to obtain this lower bound. It turns out that the lower bound, numerically determined in (4.22) of Theorem 2, is quite *insensitive* to the upper bound used for $g_{6709}(0)$ in (4.20). To indicate this, suppose that the upper bound in (4.20) is *increased* by more than a factor of 10, i.e., we estimate $g_{6709}(0)$ from above by

$$\text{(4.23)} \qquad g_{6709}(0) < 12.2,$$

rather than using the upper bound of (4.20). With Δ_{6709} from (4.13) and with the upper bound from (4.23), the following lower bound for Λ, from (1.13) and (1.14), was obtained:

$$\text{(4.24)} \qquad -7.169\,729\,856\,312\ldots\cdot 10^{-4} < \Lambda.$$

Similarly, if we *decrease* the upper bound in (4.20) by roughly a factor of 10, i.e., we estimate $g_{6709}(0)$ from above by

$$\text{(4.23')} \qquad g_{6709}(0) < 0.122,$$

then the following lower bound for Λ was obtained:

$$\text{(4.24')} \qquad -7.106\,500\ldots\cdot 10^{-4} < \Lambda.$$

Thus, in either case, the lower bound for Λ still agrees with the bound of (4.22) to two significant digits! Of course, this insensitivity can be also seen directly from the following Taylor series of λ_k of (1.13), in terms of $\Delta_k^2 g_k(0)$ and its powers:

$$\text{(4.25)} \qquad \lambda_k = -\frac{\Delta_k^2}{8} - \frac{\Delta_k^4 g_k(0)}{64} - \frac{\Delta_k^6 g_k^2(0)}{128} - \frac{11\Delta_k^8 g_k^3(0)}{2048} - \cdots,$$

and we note, from (4.13), that just the first term of (4.25), when $k = 6709$, is

$$-\frac{\Delta_{6709}^2}{8} = -7.105\,883\,654\,038\ldots\cdot 10^{-4}.$$

5. Other Lehmer Pairs of Zeros of H_0

In Tables A and B of [CRV] there is a list of 36 *super differences* of consecutive simple zeros $\{\frac{1}{2} + i\gamma_n; \frac{1}{2} + i\gamma_{n+1}\}$ of the Riemann ζ-function, where $\gamma_{n+1} - \gamma_n$ is defined to be a super difference if it is smaller than *all* previous differences $\gamma_{j+1} - \gamma_j$ for $1 \leq j < n$. (We remark that this list is *complete* for all $n \leq 2 \cdot 10^6$.) Since $x_n(0) = 2\gamma_n$ from (1.3′), these tables easily convert to a corresponding list of pairs of consecutive simple positive zeros $\{x_n(0); x_{n+1}(0)\}$ of H_0. Now some of the early entries in the corresponding 36 pairs $\{x_n(0); x_{n+1}(0)\}$ of zeros of H_0 are definitely *not* "Lehmer pairs," in the sense of Definition 1 of this paper, but most likely *all* 11 pairs, with $n \geq 6709$, *are Lehmer pairs of zeros of* H_0.

However, because the first term, i.e., $-\Delta_k^2/8$, of the expansion in (4.25) is dominant and because our interest is in finding the *best* lower bound for the de Bruijn–Newman constant Λ, it seemed prudent to consider only the *final* super pair from this list of pairs of zeros of H_0, namely,

$$x_{1,115,578} = 1{,}326{,}637.016\,620\ldots, \qquad (5.1)$$
$$x_{1,115,579} = 1{,}326{,}637.022\,538\ldots,$$

which gives the smallest value (see (1.10)) of Δ_k for all such pairs. Here, we make use of the zeros $\{\gamma_n\}_{n=1}^{2\cdot 10^6}$ of the Riemann ξ-function, which had been compiled by Odlyzko [Od]. These zeros were calculated to an accuracy of six or seven decimal digits.

We first establish

Lemma 5.1. *The pair of zeros* $\{x_{1,115,578}(0); x_{1,115,579}(0)\}$ *of* H_0 *is a Lehmer pair of zeros of* H_0.

Proof. The proof is similar to that of Lemma 4.3. With $K := 1{,}115{,}578$, we first obtain an upper bound for (see (1.12))

$$g_K(0) := \sum_{j \neq K, K+1}{}' \left\{ \frac{1}{(x_K(0) - x_j(0))^2} + \frac{1}{(x_{K+1}(0) - x_j(0))^2} \right\}. \qquad (5.2)$$

To this end, it was sufficient to use only the 5000 zeros of H_0 on either side of $x_K(0)$ and $x_{K+1}(0)$, i.e., the quantity

$$M_{K,5000}(0) := \sum_{\substack{j=K-5000 \\ j \neq K, K+1}}^{K+5001} \left\{ \frac{1}{(x_K(0) - x_j(0))^2} + \frac{1}{(x_{K+1}(0) - x_j(0))^2} \right\} = 2.476\,269\ldots \qquad (5.3)$$

was numerically summed using only a portion of Odlyzko's $2 \cdot 10^6$ numbers. Now it is evident that $g_K(0)$ can be expressed as the sum

$$g_K(0) = M_{K,5000}(0) + I_{K,5001}(0) + R_{K,K+5002}(0), \qquad (5.4)$$

where (on writing, for convenience, x_j for $x_j(0)$)

$$(5.5) \qquad I_{K,5001}(0) := \sum_{j=-K-5001}^{K-5001}{}' \left\{ \frac{1}{(x_K - x_j)^2} + \frac{1}{(x_{K+1} - x_j)^2} \right\},$$

and where

$$(5.6) \qquad R_{K,K+5002}(0) := \sum_{|j| \geq K+5002} \left\{ \frac{1}{(x_K - x_j)^2} + \frac{1}{(x_{K+1} - x_j)^2} \right\}.$$

We separately bound above $|I_{K,5001}(0)|$ and $|R_{K,K+5002}(0)|$.

First, from (5.5), we can generously bound $|I_{K,5001}(0)|$ above by

$$\begin{aligned} |I_{K,5001}(0)| &\leq \sum_{j=-K-5001}^{K-5001}{}' \left\{ \frac{1}{|x_K - x_j|^2} + \frac{1}{|x_{K+1} - x_j|^2} \right\} \\ &< 2 \sum_{j=-K-5001}^{K-5001}{}' \frac{1}{|x_K - x_{K-5001}|^2} \\ &< \frac{2 \cdot (2K+1)}{|x_K - x_{K-5001}|^2}, \end{aligned}$$

so that, with the known zeros x_K and x_{K-5001}, this gives

$$(5.7) \qquad |I_{K,5001}(0)| < 0.151\,205\ldots.$$

To similarly generously bound $|R_{K,K+5002}(0)|$ above, we deduce from (4.7) of Lemma 4.2 that

$$\begin{aligned} |R_{K,K+5002}(0)| &< 2\left\{ \frac{x_{K+5002}^2}{(x_{K+5002} - x_K)^2} + \frac{x_{K+5002}^2}{x_{K+5002} - x_{K+1})^2} + 2 \right\} \cdot \operatorname{Re} \sum_{j=K+5002}^{\infty} \frac{1}{x_j^2} \\ &< 2\left\{ \frac{x_{K+5002}^2}{(x_{K+5002} - x_K)^2} + \frac{x_{K+5002}^2}{(x_{K+5002} - x_{K+1})^2} + 2 \right\} \cdot \sum_{j=15{,}000}^{\infty} \frac{1}{x_j^2}, \end{aligned}$$

so that, with the explicit constant from (4.16) and the known values of $x_j(0)$, we have

$$(5.8) \qquad |R_{K,K+5002}(0)| < 5.941\,160\ldots.$$

Thus, with (5.3), (5.7), and (5.8), we have the following upper bound for $g_K(0)$:

$$(5.9) \qquad g_K(0) < 8.568\,635\ldots.$$

However, as (see (5.1)) $\Delta_K = 0.005\,918\ldots$, then, with (5.9),

$$(5.10) \qquad \Delta_K^2 g_K(0) < 3.000\,969\ldots \cdot 10^{-4} < \tfrac{4}{5},$$

proving that the pair of zeros of (5.1) is a Lehmer pair of zeros of H_0. ■

As our final result, we apply Theorem 1 to (5.10), which directly gives, as claimed in our Abstract, the result of

Theorem 3. *If Λ is the de Bruijn–Newman constant Λ, then*

$$-4.379 \cdot 10^{-6} < \lambda_{1,115,578} \leq \Lambda. \tag{5.11}$$

We remark that Theorem 1, applied to (5.10), actually yields that

$$-4.378\,004\ldots \cdot 10^{-6} < \Lambda,$$

but this has been rounded down to give the result of (5.11). We also remark that

$$-\frac{\Delta^2_{1,115,578}}{8} = -4.377\,840\,500\ldots \cdot 10^{-6}, \tag{5.12}$$

which again shows that the deduced lower bound, λ_k of Λ, is quite insensitive to estimates of $g_k(0)$.

To conclude this paper we mention that Lehmer *predicted* (see p. 179 of [E]) that there are *infinitely* many pairs of consecutive simple positive zeros of H_0 which are incredibly close, and this prediction could be interpreted as saying that there are infinitely many "Lehmer pairs" $\{x_{n_i}(0); x_{n_i+1}(0)\}$ of zeros of H_0 which satisfy our Definition 1. As such, it is probable that further *improved* lower bounds for the de Bruijn–Newman constant Λ, using the vehicle of Theorem 1, will result from new extended calculations of zeros of H_0 (or of the Riemann ξ-function).

Acknowledgments. We heartily thank Dr. A. M. Odlyzko for his kindness in providing computed values of $\{\gamma_n\}_{n=1}^{2\cdot 10^6}$, from which the lower bound of (5.11) was derived. Support by the National Science Foundation for this research is also gratefully acknowledged.

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