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# Asymptotics for the zeros and poles of normalized Padé approximants to $e^z$

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Dedicated to Josef Stoer on the occasion of his 60th birthday

**Summary.** With $s_n(z)$ denoting the $n$-th partial sum of $e^z$, the exact rate of convergence of the zeros of the normalized partial sums, $s_n(nz)$, to the Szegő curve $D_{0,\infty}$ was recently studied by Carpenter et al. (1991), where $D_{0,\infty}$ is defined by

$$D_{0,\infty} := \{z \in \mathbb{C} : |ze^{1-z}| = 1 \text{ and } |z| \leq 1\}.$$

Here, the above results are generalized to the convergence of the zeros and poles of certain sequences of normalized Padé approximants $R_{n,\nu}((n+\nu)z)$ to $e^z$, where $R_{n,\nu}(z)$ is the associated Padé rational approximation to $e^z$.

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## 1. Introduction

With $s_n(z) := \sum_{j=0}^{n} z^j/j!$, $n \geq 1$, denoting the familiar $n$-th partial sum of the exponential function $e^z$, it was shown in 1924 in a remarkable paper by Szegő [10] that the zeros $\{z_n(k)\}_{k=1}^{n}$, of the normalized partial sum $s_n(nz)$, tend, as $n \to \infty$, to the closed curve $D_{0,\infty}$ in the closed unit disk, where

$$D_{0,\infty} := \{z \in \mathbb{C} : |ze^{1-z}| = 1 \text{ and } |z| \leq 1\}. \tag{1.1}$$

Now, it is known (see [1] or [4]) that the zeros $\{z_n(k)\}_{k=1}^{n}$ all lie in the closed unit disk for every $n \geq 1$, and Szegő's result, more precisely, is that each accumulation point (in the closed unit disk) of all these zeros *must lie on* $D_{0,\infty}$, and, conversely, *each* point of $D_{0,\infty}$ is an accumulation point of these zeros!

Subsequently, the *rate of convergence*, as a function of $n$, of the zeros $\{z_n(k)\}_{k=1}^{n}$ to the curve $D_{0,\infty}$ was studied by Buckholtz [2] who showed, with the notation

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$$\text{dist}\ \left[\{z_n(k)\}_{k=1}^{n}; D_{0,\infty}\right] := \max_{1\le k\le n}\left(\text{dist}\ \left[z_n(k); D_{0,\infty}\right]\right),$$

that

$$\text{dist}\ [\{z_n(k)\}_{k=1}^{n}; D_{0,\infty}] \le \frac{2\mathrm{e}}{\sqrt{n}} \quad (n \ge 1), \tag{1.2}$$

which of course implies that

$$\overline{\lim_{n\to\infty}}\left\{\sqrt{n}\cdot\ \text{dist}\ [\{z_n(k)\}_{k=1}^{n}; D_{0,\infty}]\right\} \le 2\mathrm{e} \doteq 5.436\ 563. \tag{1.3}$$

To complement the result of (1.3), it was later shown in Carpenter et al. [4] that

$$\underline{\lim}_{n\to\infty}\left\{\sqrt{n}\cdot\ \text{dist}\ \left[\{z_n(k)\}_{k=1}^{n}; D_{0,\infty}\right]\right\} = \text{Im}\ t_1 + \ \text{Re}\ t_1 \doteq 0.636\ 657, \tag{1.4}$$

where, denoting the complementary error function by

$$\text{erfc}(w) := \frac{1}{\sqrt{\pi}}\int_w^\infty \mathrm{e}^{-t^2}dt \qquad (w \in \mathbb{C}),$$

$t_1$ is the (complex) zero of erfc$(w)$, in the upper half-plane, which is closest to the origin. From the numerical results of Fettis et al. [6], it is known that

$$t_1 \doteq -1.354\ 810 + \mathrm{i}1.991\ 467.$$

Thus, if we express the upper bound of (1.2) as $O(1/\sqrt{n})$, as $n \to \infty$, then (1.4) shows that this upper bound is *best possible* in the sense that $1/\sqrt{n}$ cannot be replaced by a function of $n$ which tends more rapidly to zero, as $n \to \infty$, than does $1/\sqrt{n}$. (It is in this sense that we use the term *best possible* in what is to follow.)

It was also shown in [4] that a quantitatively *faster* convergence, of these zeros to $D_{0,\infty}$, takes place if one stays uniformly away from the point $z = 1$. Specifically, if we cover the point $z = 1$ with the open disk

$$C_\delta := \{z \in \mathbb{C} : |z-1| < \delta\} \quad (0 < \delta < 1), \tag{1.5}$$

then it was shown in [4, Theorem 2] that, for each fixed $\delta$ with $0 < \delta < 1$,

$$\text{dist}\ [\{z_n(k)\}_{k=1}^{n}\backslash C_\delta; D_{0,\infty}] = O\left(\frac{\ln n}{n}\right) \qquad (n \to \infty), \tag{1.6}$$

where the constant, implicit in the right-side of (1.6), is dependent only on $\delta$.

For a more precise location of the zeros of $s_n(nz)$, consider the arc $D_{0,n}$, defined in [4] for each $n \ge 1$ by

$$D_{0,n} := \Big\{z \in \mathbb{C} : |z\mathrm{e}^{1-z}|^n = \tau_n\sqrt{2\pi n}\ \left|\tfrac{1-z}{z}\right|, |z| \le 1 \text{ and } |\arg z| \ge \cos^{-1}\left(\tfrac{n-2}{n}\right)\Big\}, \tag{1.7}$$

where $\tau_n$, from Stirling's formula, is given by the asymptotic series

$$\tau_n := \frac{n!}{n^n\mathrm{e}^{-n}\sqrt{2\pi n}} \cong 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots \quad (n \to \infty).$$

It was shown in [4, Proposition 3] that $D_{0,n}$ is a well-defined arc, and it is further shown in [4, Theorem 4] that, for each fixed $\delta$ with $0 < \delta < 1$,

$$\text{dist}\,[\{z_n(k)\}_{k=1}^{n}\backslash C_\delta\,;D_{0,n}] = O\left(\frac{1}{n^2}\right) \quad (n \to \infty), \tag{1.8}$$

so that the arc $D_{0,n}$ more closely approximates the zeros $\{z_n(k)\}_{k=1}^{n}$ of $s_n(nz)$, than does the Szegő curve $D_{0,\infty}$. The results of (1.6) and (1.8) are both known to be *best possible* (cf. [4]).

Having reviewed the above results for the convergence behavior of the zeros of the normalized partial sums of $e^z$, it is of course well known that the partial sums $s_n(z)$ of $e^z$ are just the special cases of the $(n,0)$-th Padé approximations to $e^z$. The early result of Szegő [10] has subsequently been generalized by Saff and Varga [9] to more general Padé approximations, where one obtains, in the spirit of Szegő, the convergence of the normalized zeros and poles of these Padé approximants to the arcs $D_{\sigma,\infty}$ and $E_{\sigma,\infty}$, defined below in (1.16) and (1.17), in the closed unit disk. The goal of this paper is to obtain the *analogs* of (1.6) and (1.8) in this more general Padé setting, thereby generalizing the results of [4] and [9].

In the remainder of this section, we introduce needed background and known results for this study of Padé rational approximation to $e^z$.

Let $\pi_n$ denote the set of all complex polynomials of degree at most $n\,(n = 0,1,\cdots)$. For each pair $(n,\nu)$ of nonnegative integers, the $(n,\nu)$-th Padé approximant to $e^z$ is the rational function $R_{n,\nu}(z)$ such that

$$R_{n,\nu}(z) = \frac{P_{n,\nu}(z)}{Q_{n,\nu}(z)}, \tag{1.9}$$

where

(i) $P_{n,\nu}(z) \in \pi_n$, and $Q_{n,\nu}(z) \in \pi_\nu$ with $Q_{n,\nu}(0) = 1$, and
(ii) $e^z - R_{n,\nu}(z) = O(|z|^{n+\nu+1})$ as $|z| \to 0$.

It is well known (cf. Perron [8, p. 433], or Saff and Varga [9, p. 242]) that $P_{n,\nu}(z)$ and $Q_{n,\nu}(z)$ of (1.9) are given explicitly, for any pair $(n,\nu)$ of nonnegative integers $n$ and $\nu$, by

$$P_{n,\nu}(z) = \sum_{k=0}^{n} \frac{(n+\nu-k)!n!z^k}{(n+\nu)!k!(n-k)!}, \tag{1.10}$$

and

$$Q_{n,\nu}(z) = \sum_{k=0}^{\nu} \frac{(n+\nu-k)!\nu!(-z)^k}{(n+\nu)!k!(\nu-k)!}. \tag{1.11}$$

The polynomials $P_{n,\nu}(z)$ and $Q_{n,\nu}(z)$ are respectively called the *Padé numerator* and *Padé denominator* of type $(n,\nu)$ for $e^z$. In what follows, we consider, as in [9], any sequence of Padé approximants $\{R_{n_j,\nu_j}(z)\}_{j=1}^{\infty}$ to $e^z$ for which there exists a constant $\sigma$, with $0 \le \sigma < \infty$, such that

$$\lim_{j\to\infty} n_j = \infty \text{ and } \lim_{j\to\infty} \nu_j/n_j = \sigma. \tag{1.12}$$

For any $\sigma$ with $0 < \sigma < \infty$, define the two complex numbers

$$z_\sigma^{\pm} := \left[(1-\sigma) \pm 2\sqrt{\sigma}\mathrm{i}\right]/(1+\sigma), \tag{1.13}$$

which have modulus unity, and consider the complex plane $\mathbb{C}$ slit along the two rays

$$\mathscr{R}_\sigma := \left\{z \in \mathbb{C} : z = z_\sigma^{+} + i\tau \text{ or } z = z_\sigma^{-} - i\tau, \quad \text{for all } \tau \ge 0\right\},$$

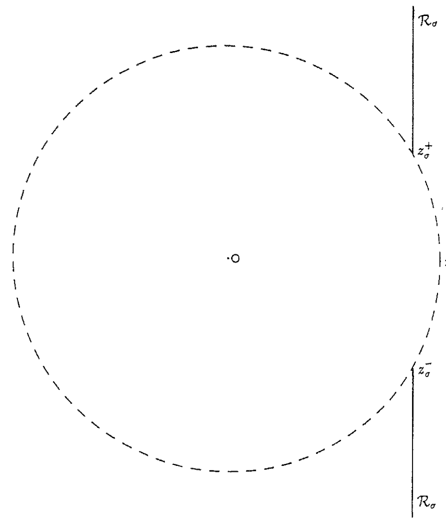


**Fig. 1.** The cut plane $\mathbb{C}\backslash\mathscr{R}_\sigma$

as shown in Fig. 1.

With

$$\lambda_\sigma := \frac{1-\sigma}{1+\sigma},$$

the function $g_\sigma(z)$, defined by

$$g_\sigma(z) := \sqrt{1+z^2-2\lambda_\sigma z}\,, \tag{1.14}$$

has $z_\sigma^\pm$ as branch points, which are the finite extremities of $\mathscr{R}_\sigma$. On setting $g_\sigma(0) := 1$ and on extending $g_\sigma(z)$ analytically on the doubly slit domain $\mathbb{C}\backslash\mathscr{R}_\sigma$, then $g_\sigma(z)$ is analytic and single-valued on $\mathbb{C}\backslash\mathscr{R}_\sigma$. It turns out that $1 \pm z + g_\sigma(z)$ does not vanish on $\mathbb{C}\backslash\mathscr{R}_\sigma$ (cf. Saff and Varga [9, p. 244]).

Next, define $(1+z+g_\sigma(z))^{2/(1+\sigma)}$ and $(1-z+g_\sigma(z))^{2\sigma/(1+\sigma)}$ by requiring that their values at $z=0$ be $2^{2/(1+\sigma)}$ and $2^{2\sigma/(1+\sigma)}$, respectively, and by analytic continuation. These functions are also analytic and single-valued on $\mathbb{C}\backslash\mathscr{R}_\sigma$. For $0<\sigma<\infty$, define the function $w_\sigma(z)$ by

$$w_\sigma(z) := \frac{4\sigma^{\sigma/(1+\sigma)} z e^{g_\sigma(z)}}{(1+\sigma)[1+z+g_\sigma(z)]^{2/(1+\sigma)}[1-z+g_\sigma(z)]^{2\sigma/(1+\sigma)}} \quad (0<\sigma<\infty). \tag{1.15}$$

Then, $w_\sigma(z)$ is analytic and single-valued on $\mathbb{C}\backslash\mathscr{R}_\sigma$, and is also univalent (and starlike) in $|z|<1$ (cf. [9, p. 251]). We remark, on letting $\sigma \to 0$ in (1.15), that it is known (cf. [9, p. 244]) that $w_0(z) := \lim_{\sigma\to 0} w_\sigma(z)$ satisfies

$$w_0(z) = ze^{1-z} \quad (\text{for } |z| \le 1),$$

which connects the above discussion to the Szegő curve of (1.1).

With the function $w_\sigma(z)$ of (1.15) for $0<\sigma<\infty$, the two Jordan arcs, $D_{\sigma,\infty}$ and $E_{\sigma,\infty}$, are defined by

$$(1.16)\quad D_{\sigma,\infty} := \left\{ z \in \mathbb{C} : |w_\sigma(z)| = 1,\ |z| \le 1,\ \text{and } |\arg z| \ge \cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right) \right\},$$

and

$$(1.17)\quad E_{\sigma,\infty} := \left\{ z \in \mathbb{C} : |w_\sigma(z)| = 1,\ |z| \le 1 \text{ and } |\arg z| \le \cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right) \right\},$$

where $-\pi \le \arg z < \pi$, and these arcs are symmetric with respect to the real axis. If

$$J_{\sigma,\infty} := \{ z \in \mathbb{C} : |w_\sigma(z)| = 1 \text{ and } |z| \le 1 \},$$

then we see from (1.16) and (1.17) that

$$J_{\sigma,\infty} = D_{\sigma,\infty} \bigcup E_{\sigma,\infty} \qquad (0 < \sigma < \infty).$$

Thus, for $0 < \sigma < \infty$, $J_{\sigma,\infty}$ is a Jordan curve, consisting of the two Jordan arcs $D_{\sigma,\infty}$ and $E_{\sigma,\infty}$, and $J_{\sigma,\infty}$ lies interior to the unit disk, except for the endpoints of these arcs, namely, $z_\sigma^\pm$, which lie on the boundary of the unit disk. (For $\sigma = 0$, $J_{0,\infty}$ reduces to the Szegő curve $D_{0,\infty}$ of (1.1).)

With the arcs of (1.16) and (1.17), we have the following known result of [9]:

**Theorem A.** *For any $\sigma$ with $0 \le \sigma < \infty$, consider any sequence of Padé approximants $\{R_{n_j,\nu_j}(z)\}_{j=1}^\infty$ to $e^z$ for which (1.12) holds. Then,*

(i) *$z$ is a limit point of the zeros of the normalized Padé approximants $\left\{R_{n_j,\nu_j}\left((n_j+\nu_j)z\right)\right\}_{j=1}^\infty$ if and only if $z \in D_{\sigma,\infty}$.*

(ii) *If $0 < \sigma < \infty$, then $z$ is a limit point of the poles of the normalized Padé approximants $\left\{R_{n_j,\nu_j}\left((n_j+\nu_j)z\right)\right\}_{j=1}^\infty$ if and only if $z \in E_{\sigma,\infty}$.*

The special case of (i) of Theorem A with $\sigma = 0$ and, in addition with $n_j = j$ and $\nu_j = 0$ for all $j \ge 1$, reduces to Szegő's result (cf. [9]). As previously mentioned, the convergence rates of the zeros in this case has been treated in detail in Carpenter et al. [4].

Since the polynomials of (1.10) and (1.11) satisfy the obvious identity

$$(1.18)\qquad Q_{n_j,\nu_j}(z) = P_{\nu_j,n_j}(-z),$$

it suffices then to investigate only the convergence behavior of the zeros of the normalized Padé approximants $R_{n_j,\nu_j}\left((n_j+\nu_j)z\right)$, or equivalently, only the convergence behavior of the zeros of the normalized Padé numerators $P_{n_j,\nu_j}\left((n_j+\nu_j)z\right)$. Clearly, all subsequent results for the zeros easily translate into results for the poles via (1.18).

## 2. Statements of new results

For any sequence of Padé approximants $\{R_{n_j,\nu_j}(z)\}_{j=1}^\infty$ to $e^z$ which satisfies (1.12) with $\sigma > 0$, Theorem A above gives the precise location of the limit points of the zeros and poles of the normalized Padé approximants $\{R_{n_j,\nu_j}((n_j+\nu_j)z)\}_{j=1}^\infty$. Our interest here is in determining the *convergence behavior* of zeros and poles of these normalized Padé approximants, as this would extend the results of [4] which are explicitly given for the case $\sigma = 0$. But, we note that the results of [4] were specifically determined for the special sequence $\{(n_j, \nu_j)\}_{j=1}^{\infty}$ with $\nu_j := 0$ and $n_j := j$ for all $j \geq 1$, so that

$$\nu_j/n_j = \sigma(= 0) \text{ for each } \quad j \geq 1.$$

On the other hand, when considering rational Padé approximants to $e^z$ which are not polynomials, the second condition of (1.12) may hold, for the case $\sigma > 0$, for sequences $\{(n_j, \nu_j)\}_{j=1}^{\infty}$ having exceedingly *slow* convergence of $\nu_j/n_j$ to $\sigma$, as $j \to \infty$. To indicate this, consider the sequence of pairs of positive integers $\{(n_j, \nu_j)\}_{j=3}^{\infty}$ defined by

$$\nu_j := \left[\left[ j\left(1 + \frac{10}{\ln \ln j}\right)\right]\right] \text{ and } n_j := j, \text{ for all } j \geq 3,$$

(where $[[x]]$ denotes the integer part of $x$), so that (1.12) is satisfied for $\sigma = 1$. But, for $m := 10^6$, we have

$$n_m = 10^6 \text{ and } \nu_m = 4,808,374; \text{ whence, } \frac{\nu_m}{n_m} = 4.808\ 374,$$

which is far removed from the limiting value $\sigma = 1$. In this example, measuring the distance of the zeros or poles of $R_{n_j,\nu_j}((n_j + \nu_j)z)$, with respect to the limiting arc $D_{1,\infty}$ or $E_{1,\infty}$ of Theorem A, is of little value for $j = 10^6$.

Instead, we measure the distance of the zeros $\{z_{n_j,\nu_j}(k)\}_{k=1}^{n_j}$ of $R_{n_j,\nu_j}((n_j + \nu_j)z)$ from the Jordan arc $D_{\sigma_j,\infty}$, where

(2.1) $$\sigma_j := \nu_j/n_j \text{ for all } j \geq 1,$$

and where $D_{\sigma_j,\infty}$ is the arc of (1.16) with $\sigma$ replaced by $\sigma_j$ of (2.1). Similarly, $z^{\pm}_{\sigma_j}$ are defined from (1.13) with $\sigma$ replaced by $\sigma_j$, and, for a fixed $\delta$ with $0 < \delta < 1$, we set

(2.2) $$\tilde{C}_{\delta,\sigma_j} := \{z \in \mathbb{C} : |z - z^{+}_{\sigma_j}| < \delta\} \bigcup \{z \in \mathbb{C} : |z - z^{-}_{\sigma_j}| < \delta\},$$

for all $j \geq 1$. We note that $\tilde{C}_{\delta,\sigma_j}$, consisting of two disks, is the analog of $C_\delta$ of (1.5) for the case $\sigma = 0$.

With the above definition, our first result (to be proved in Sect. 3), which is patterned after the result of (1.6), can be stated as

**Theorem 1.** *Consider any sequence of Padé approximants* $\{R_{n_j,\nu_j}(z)\}_{j=1}^{\infty}$ *to* $e^z$ *for which*

(2.3) $$\lim_{j\to\infty} n_j = \infty \quad \textit{and} \quad \lim_{j\to\infty} \nu_j/n_j = \sigma, \textit{ where } 0 < \sigma < \infty.$$

*If* $\{z_{n_j,\nu_j}(k)\}_{k=1}^{n_j}$ *denotes the zeros of* $R_{n_j,\nu_j}((n_j + \nu_j)z)$, *then for each fixed* $\delta$ *with* $0 < \delta < 1$,

(2.4) $$dist\ [\{z_{n_j,\nu_j}(k)\}_{k=1}^{n_j} \backslash \tilde{C}_{\delta,\sigma_j}; D_{\sigma_j,\infty}] = O\left(\frac{1}{n_j + \nu_j}\right) \quad (j \to \infty).$$

*Moreover, the result of* (2.4) *is best possible.*

We remark that the zeros of $P_{n_j,\nu_j}((n_j + \nu_j)z)$ in (2.4), which are outside of the disks of $\tilde{C}_{\delta,\sigma_j}$, are measured relative to the arc $D_{\sigma_j,\infty}$, where both $\tilde{C}_{\delta,\sigma_j}$ and $D_{\sigma_j,\infty}$ in general *vary* with $j$. Of course, there is a case where $D_{\sigma_j,\infty}$ is a *fixed* arc for all $j \geq 1$, and this is covered in the following immediate corollary of Theorem 1.

**Corollary 2.** *If, under the hypothesis of* (2.3) *of Theorem 1, $\sigma$ is a positive rational number and if the associated sequence of nonnegative pairs of integers $\{(n_j, \nu_j)\}_{j=1}^{\infty}$ satisfies*

$$\nu_j/n_j = \sigma \text{ for all } j \geq 1, \tag{2.5}$$

*then for each fixed $\delta$ with $0 < \delta < 1$,*

$$dist\ [\{z_{n_j,\nu_j}(k)\}_{k=1}^{n_j} \backslash \tilde{C}_{\delta,\sigma}; D_{\sigma,\infty}] = O\left(\frac{1}{n_j + \nu_j}\right) \quad (j \to \infty). \tag{2.6}$$

*Moreover, the result of* (2.6) *is best possible.*

We remark that the case $\sigma = 0$ of the zeros of the normalized partial sums of $e^z$, as discussed in Sect. 1, are also measured, as in Corollary 2, against a *fixed* curve, $D_{0,\infty}$, but we note with interest that the result of (2.6) for $0 < \sigma < \infty$, which is the analog of (1.6), now *eliminates* the (ln $n$) term appearing in (1.6). We also remark that essentially the special case $\sigma = 1$ of Corollary 2 is obtained (via a different technique) in [3].

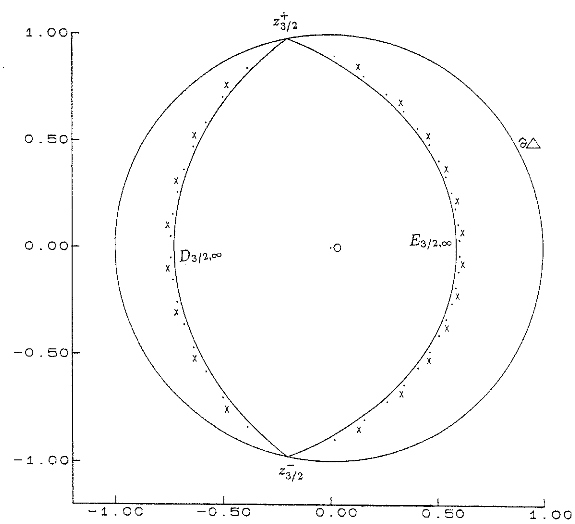


**Fig. 2.** Zeros and poles of $R_{8,12}(20z)$ and $R_{16,24}(40z)$, and the arcs $D_{3/2,\infty}$ and $E_{3/2,\infty}$

To illustrate the results of Theorem 1 and Corollary 2, we have graphed in Fig. 2 the 8 zeros and 12 poles of $R_{8,12}(20z)$, marked by $\times$'s, as well as the 16 zeros and 24 poles of $R_{16,24}(40z)$, marked by dots, in relation to the arcs $D_{3/2,\infty}$ and $E_{3/2,\infty}$, for the case $\sigma = 3/2$. Note that the zeros and poles of $R_{8,12}(20z)$ are, respectively, about *twice* as far from the curves $D_{3/2,\infty}$ and $E_{3/2,\infty}$, as are the zeros and poles of $R_{16,24}(40z)$, which is in agreement with (2.4) of Theorem 1. Similar results are shown in Fig. 3, for $R_{10,10}(20z)$ and $R_{20,20}(40z)$ for the case $\sigma = 1$, and in Fig. 4, for $R_{12,8}(20z)$ and $R_{24,16}(40z)$ for the case $\sigma = 2/3$.

For our next result, we need some additional notation. As in [5, p. 22], for $0 < \sigma < \infty$ we set

$$\hat{N}_\sigma(z) := \frac{g_\sigma(z) + 1 - \lambda_\sigma \cdot z}{z\sqrt{1 - \lambda_\sigma^2}} \qquad \left(z \in \mathbb{C} \backslash (\mathscr{R}_\sigma \cup \{0\})\right), \tag{2.7}$$

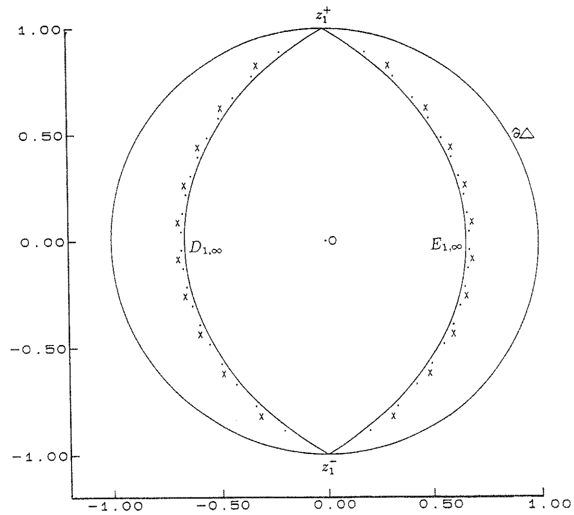


**Fig. 3.** Zeros and poles of $R_{10,10}(20z)$ and $R_{20,20}(40z)$, and the arcs $D_{1,\infty}$ and $E_{1,\infty}$

where it is known that $\hat{N}_\sigma(z)$ is analytic and single-valued on $\mathbb{C}\backslash(\mathscr{R}_\sigma \cup \{0\})$. Then, in analogy with the arc $D_{0,n}$ of (1.7), we define, for each pair $(n_j, \nu_j)$ of positive integers, the arcs

$$D_{\sigma_j, n_j+\nu_j} := \Big\{z \in \mathbb{C} : |w_{\sigma_j}(z)|^{n_j+\nu_j} = |\hat{N}_{\sigma_j}(z)|, |z| \leq 1, \text{ and } |\arg z| \geq \cos^{-1}\left(\frac{n_j-\nu_j-2}{n_j+\nu_j}\right)\Big\} \tag{2.8}$$

and

$$E_{\sigma_j, n_j+\nu_j} := \Big\{z \in \mathbb{C} : |w_{\sigma_j}(z)|^{n_j+\nu_j} = |\hat{N}_{\sigma_j}(z)|, |z| \leq 1, \text{ and } |\arg z| \leq \cos^{-1}\left(\frac{n_j-\nu_j+2}{n_j+\nu_j}\right)\Big\}, \tag{2.9}$$

where $\sigma_j := \nu_j/n_j$ and where $-\pi \leq \arg z < +\pi$. It is shown in Sect. 4 that these arcs of (2.8) and (2.9) are *well-defined*.

With the above definitions, our next result (to be proved in Sect. 5), which is patterned after the result of (1.8), can be stated as

**Theorem 3.** *Under the hypothesis of* (2.3) *of Theorem 1,*

$$\text{dist}\ \left[\{z_{n_j,\nu_j}(k)\}_{k=1}^{n_j} \backslash \tilde{C}_{\delta,\sigma_j}; D_{\sigma_j,n_j+\nu_j}\right] = O\left(\frac{1}{(n_j+\nu_j)^2}\right) \quad (j \to \infty). \tag{2.10}$$

We remark that a special case of (2.10) of Theorem 3 was previously established in de Bruin et al. [5]. Specifically, for the case $\sigma = 1$ and $n_j$ odd for all $j \geq 1$ of (2.3), it was shown in [5, eq. (9.31)] that the negative real zero $z_{n_j,\nu_j}$ of $R_{n_j,\nu_j}((n_j+\nu_j)z)$ satisfies

$$z_{n_j,\nu_j} = \hat{z}_{n_j,\nu_j} + O\left(\frac{1}{(n_j+\nu_j)^2}\right) \quad (j \to \infty), \tag{2.11}$$

where $\hat{z}_{n_j,\nu_j}$ denotes the real point of the arc $D_{\sigma_j,n_j+\nu_j}$. We also remark that the result of Theorem 3, for essentially the case $\sigma_j = 1$, is obtained in [3].

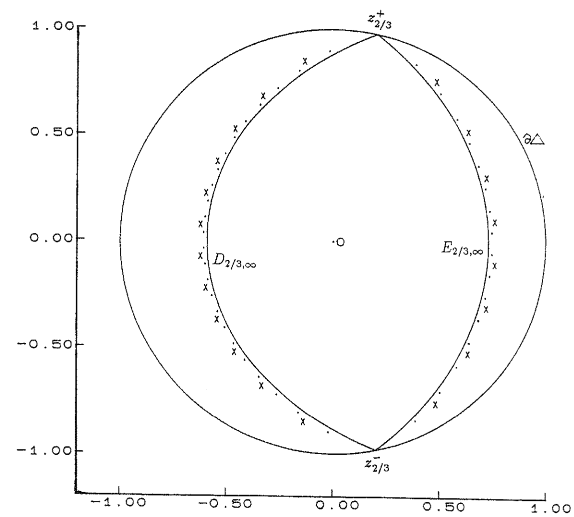


**Fig. 4.** Zeros and poles of $R_{12,8}(20z)$ and $R_{24,16}(40z)$, and the arcs $D_{2/3,\infty}$ and $E_{2/3,\infty}$

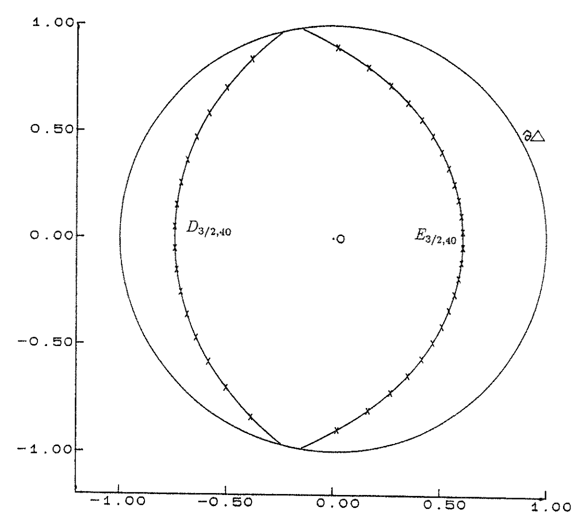


**Fig. 5.** Zeros and poles of $R_{16,24}(40z)$, and the arcs $D_{3/2,40}$ and $E_{3/2,40}$

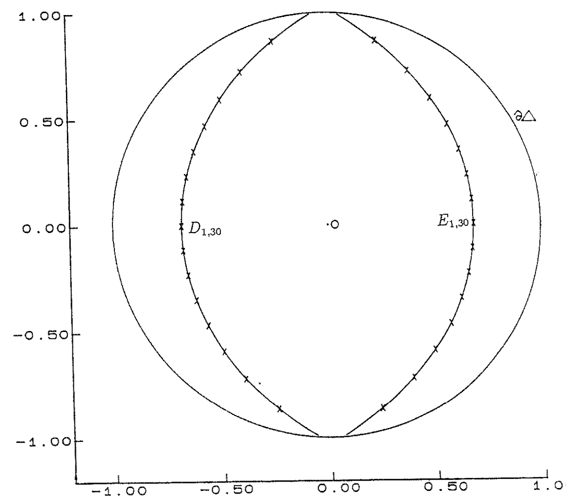


**Fig. 6.** Zeros and poles of $R_{15,15}(30z)$, and the arcs $D_{1,30}$ and $E_{1,30}$

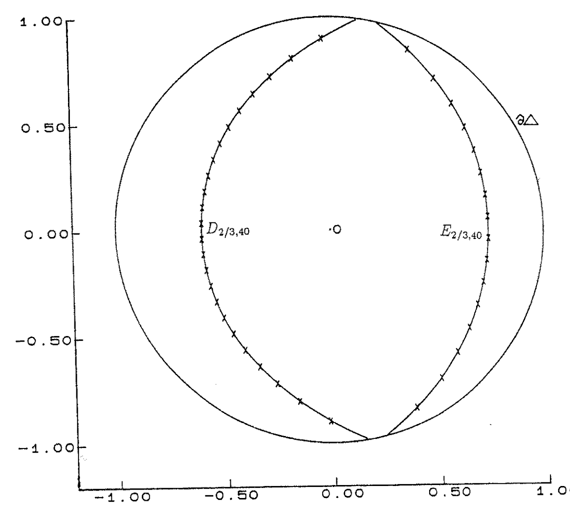


**Fig. 7.** Zeros and poles of $R_{24,16}(40z)$, and the arcs $D_{2/3,40}$ and $E_{2/3,40}$

To illustrate the result of Theorem 3, we have graphed in Figs. 5–7 the following cases. For the normalized Padé approximant $R_{16,24}(40z)$ to $e^z$, for which $n_j = 16$ and $\nu_j = 24$ and $\sigma_j = 3/2$, we show in Fig. 5 the curves $D_{3/2,40}$ and $E_{3/2,40}$, along with the 16 zeros and 24 poles (marked by the ×'s) of $R_{16,24}(40z)$. Figs. 6 and 7 show the corresponding results for $R_{15,15}(30z)$ and $R_{24,16}(40z)$. Up to plotting accuracy, it appears that the zeros and poles of these normalized Padé approximants *lie on* the respective arcs $D_{\sigma_j,n_j+\nu_j}$ and $E_{\sigma_j,n_j+\nu_j}$!

## 3. Proof of Theorem 1

We begin with the following

**Lemma 1.** *For any $\tau$ with $0 < \tau < \infty$, consider the sectorial set*

$$S_\tau := \left\{ z = re^{i\psi} : 0 < r \leq 1 \text{ and } \cos^{-1}\left(\frac{1-\tau}{1+\tau}\right) \leq \psi \leq 2\pi - \cos^{-1}\left(\frac{1-\tau}{1+\tau}\right) \right\}, \tag{3.1}$$

*as shown in Fig. 8. Then,*

$$\min\{|\hat{N}_\tau(z)| : z \in S_\tau\} = 1, \tag{3.2}$$

*with equality holding only at the points $z_\tau^\pm$. Thus* (cf. (2.2)), *for any fixed $\delta$ with $0 < \delta < 1$,*

$$\min\{|\hat{N}_\tau(z)| : z \in S_\tau \backslash \tilde{C}_{\delta,\tau}\} > 1. \tag{3.3}$$

*Proof.* From the definitions of (2.7) and (1.13) and the fact that $g_\tau(z_\tau^\pm) = 0$, it is readily verified that

$$|\hat{N}_\tau(z_\tau^\pm)| = 1. \tag{3.4}$$

Next, as mentioned in de Bruin et al. [5, p. 22], $|\hat{N}_\tau(e^{i\psi})|$ is strictly increasing in $\psi$ on the interval $[\cos^{-1}(\frac{1-\tau}{1+\tau}), \pi]$ and $|\hat{N}_\tau(e^{i\psi})|$ is strictly decreasing in $\psi$ on the interval $[\pi, 2\pi - \cos^{-1}(\frac{1-\tau}{1+\tau})]$. In addition (cf. [5, p.22]), for any fixed $\psi$ with $\cos^{-1}(\frac{1-\tau}{1+\tau}) \leq \psi \leq 2\pi - \cos^{-1}(\frac{1-\tau}{1+\tau})$, $|\hat{N}_\tau(re^{i\psi})|$ is strictly decreasing in $r$ on the interval $0 < r \leq 1$, where we note from (2.7) that $|\hat{N}_\tau(0)| = +\infty$. With (3.4) and the definition of (3.1), we see geometrically that (3.2) is valid, where from (3.4), equality holds in (3.2) only at the points $z_\tau^\pm$. Finally, since $\tilde{C}_{\delta,\tau}$, from its definition in (2.2), contains the points $z_\tau^\pm$, then (3.3) follows directly from (3.2). □

This brings us to the

*Proof of Theorem 1.* With the hypothesis of (2.3) of Theorem 1, it is known (cf. [5, eq. (9.24)] and [9, eq. (4.30)]) that if $z$ is any zero of $R_{n_j,\nu_j}\left((n_j+\nu_j)z\right)$, then

$$|w_{\sigma_j}(z)|^{n_j+\nu_j} = |\hat{N}_{\sigma_j}(z)| \cdot \left\{1 + O\left(\frac{1}{n_j+\nu_j}\right)\right\} \quad (j \to \infty), \tag{3.5}$$

uniformly on any compact subset of $\mathbb{C}\backslash(\mathscr{R}_\sigma \cup \{0\})$. Furthermore, it is known (cf. [9, Theorem 1.1]) that, for $\nu_j \geq 0$ and $n_j \geq 2$, all zeros of $R_{n_j,\nu_j}((n_j+\nu_j)z)$ must lie in the infinite sector

$$\left\{ z \in \mathbb{C} : |\arg z| > \cos^{-1}\left(\frac{n_j - \nu_j - 2}{n_j+\nu_j}\right)\right\},$$

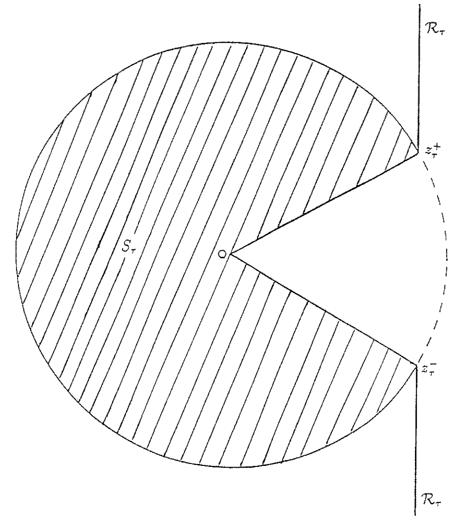


**Fig. 8.** The sectorial set $S_\tau$

and that (cf. [9, Theorem 2.3]) these zeros must have a positive density in the sectorial set $S_\sigma$.

For the zero $z$ of $R_{n_j,\nu_j}((n_j+\nu_j)z)$, we may assume from the above discussion that $z$ is in $S_\sigma$, and that $|z| < 1$. Writing $z = re^{i\theta}$, let $\tilde{z} := \tilde{r}e^{i\theta}$ be such that $|w_{\sigma_j}(\tilde{z})| = 1$, i.e., $\tilde{z} \in D_{\sigma_j,\infty}$. On defining $z - \tilde{z} =: se^{i\theta}$, $|s|$ then measures the distance between $z$ and $\tilde{z}$. Because $w_{\sigma_j}(z)$ is analytic and single-valued on $\mathbb{C}\backslash\mathscr{R}_{\sigma_j}$, the Taylor expansion of $w_{\sigma_j}(z)$ about the point $\tilde{z}$ is

$$w_{\sigma_j}(z) = w_{\sigma_j}(\tilde{z}) + se^{i\theta}w'_{\sigma_j}(\tilde{z}) + O(s^2), \tag{3.6}$$

and, since the nearest singularities of $w_{\sigma_j}(z)$ to $\tilde{z}$ in $\mathbb{C}\backslash\mathscr{R}_{\sigma_j}$ occur at the points $z^{\pm}_{\sigma_j}$, this Taylor expansion is convergent in the open disk with center $\tilde{z}$ and radius

$$\min\{|\tilde{z} - z^+_{\sigma_j}|; |\tilde{z} - z^-_{\sigma_j}|\}.$$

With [9, eq. (4.2)], it is known in general that

$$w'_\tau(u) = w_\tau(u)g_\tau(u)/u \qquad (\text{any } \ 0 \le \tau < \infty, \text{ any } u \in \mathbb{C}\backslash\mathscr{R}_\tau). \tag{3.7}$$

Thus, with $\tau = \sigma_j$ and $u = \tilde{z}$ in (3.7), we can express (3.6), on factoring out $w_{\sigma_j}(\tilde{z})$, as

$$w_{\sigma_j}(z) = w_{\sigma_j}(\tilde{z})\left\{1 + \frac{sg_{\sigma_j}(\tilde{z})}{\tilde{r}} + O(s^2)\right\}, \tag{3.8}$$

where, since $w_{\sigma_j}(0) = 0$ from (1.15), this modified Taylor expansion in (3.8) is now convergent in the open disk with center $\tilde{z}$ and radius

$$\min\{|\tilde{z} - z^+_{\sigma_j}|; |\tilde{z} - z^-_{\sigma_j}|; |\tilde{z}|\}. \tag{3.9}$$

Similarly, we expand $\hat{N}_{\sigma_j}(z)$ in a Taylor series about $\tilde{z}$. Because in general (cf. [5, eq.(9.23)]), (3.10) $\hat{N}'_\tau(u) = -\hat{N}_\tau(u)/(g_\tau(u)\cdot u)$ (any $0<\tau<\infty$, any $u\in\mathbb{C}\backslash(\mathscr{R}_\tau\cup\{0\})$,

the Taylor expansion for $\hat{N}_{\sigma_j}(z)$ can be analogously expressed as

$$\hat{N}_{\sigma_j}(z) = \hat{N}_{\sigma_j}(\tilde{z})\left\{1-\frac{s}{\tilde{r}g_{\sigma_j}(\tilde{z})}+O(s^2)\right\}, \tag{3.11}$$

where, as $\hat{N}_{\sigma_j}(z)$ is analytic and single-valued in $\mathbb{C}\backslash(\mathscr{R}_{\sigma_j}\cup\{0\})$, the modified Taylor expansion of (3.11) is convergent in the same open disk with center $\tilde{z}$ and radius given by (3.9). With the expressions of (3.8) and (3.11), we derive from (3.5), since $|w_{\sigma_j}(\tilde{z})|=1$, that

$$\begin{aligned}&\left\{1+\frac{s}{\tilde{r}}\mathrm{Re}\left(g_{\sigma_j}(\tilde{z})\right)+O(s^2)\right\}^{n_j+\nu_j}\\&\quad=|\hat{N}_{\sigma_j}(\tilde{z}))|\left\{1-\frac{s}{\tilde{r}}\mathrm{Re}\left(\frac{1}{g_{\sigma_j}(\tilde{z})}\right)+O(s^2)\right\}\left\{1+O\left(\frac{1}{n_j+\nu_j}\right)\right\}.\end{aligned}$$

On taking logarithms in the above display and on dividing by $(n_j+\nu_j)$, we obtain

$$\begin{aligned}&\ln\left\{1+\frac{s}{\tilde{r}}\mathrm{Re}\left(g_{\sigma_j}(\tilde{z})\right)+O(s^2)\right\}\\&=\frac{\ln|\hat{N}_{\sigma_j}(\tilde{z})|}{(n_j+\nu_j)}+\frac{1}{(n_j+\nu_j)}\ln\left\{1-\frac{s}{\tilde{r}}\mathrm{Re}\left(\frac{1}{g_{\sigma_j(\tilde{z})}}\right)+O(s^2)\right\}\\&\quad+O\left(\frac{1}{(n_j+\nu_j)^2}\right),\end{aligned}$$

and for $s$ small, this reduces to

$$\frac{s}{\tilde{r}}\mathrm{Re}\left(g_{\sigma_j}(\tilde{z})\right)+O(s^2)=\frac{\ln|\hat{N}_{\sigma_j}(\tilde{z})|}{(n_j+\nu_j)}-\frac{s\mathrm{Re}(1/g_{\sigma_j}(\tilde{z}))}{(n_j+\nu_j)}+O\left(\frac{1}{(n_j+\nu_j)^2}\right).$$

Thus, we see that

$$s=\frac{(\ln|\hat{N}_{\sigma_j}(\tilde{z})|)\tilde{r}}{(n_j+\nu_j)\cdot\mathrm{Re}\left(g_{\sigma_j}(\tilde{z})\right)}+O\left(\frac{1}{(n_j+\nu_j)^2}\right)\qquad(j\to\infty). \tag{3.12}$$

Now from [9, eq. (4.1)], it is known that $\mathrm{Re}(g_\tau(z))>0$ on $\mathbb{C}\backslash\mathscr{R}_\tau$, and as $g_\tau(z)$ vanishes only at its branch points $z_\tau^\pm$, then $1/\mathrm{Re}(g_\tau(z))$ is uniformly bounded at all points of the unit disk not in $\tilde{C}_{\delta,\tau}$. Next, since $|N_\tau(z)|>1$ on $S_\tau\backslash\tilde{C}_{\delta,\tau}$ from Lemma 1 and since (cf. (1.16)) $D_{\tau,\infty}$ never passes through $z=0$ because $w_\tau(0)=0$, it follows from (3.12) that

(3.13) $\quad s=O\left(\dfrac{1}{n_j+\nu_j}\right)$ for any zero $z$ of $R_{n_j+\nu_j}((n_j+\nu_j)z)$ not in $\tilde{C}_{\delta,\sigma_j}$

$(j\to\infty)$. But as $|s|$ measures the distance from $z$ to a *particular* point, $\tilde{z}$, of $D_{\sigma_j,\infty}$, then dist $[z;D_{\sigma_j,\infty}]\le|s|$ for *any* zero $z$ of $R_{n_j,\nu_j}((n_j+\nu_j)z)$ not in $\tilde{C}_{\delta,\sigma_j}$ and it follows from (3.13) that

$$\mathrm{dist}\left[\{z_{n_j,\nu_j}(k)\}_{k=1}^{n_j}\backslash\tilde{C}_{\delta,\sigma_j};D_{\sigma_j,\infty}\right]=O\left(\frac{1}{n_j+\nu_j}\right)\qquad(j\to\infty),$$

which is the desired result of (2.4) of Theorem 1.

We also remark that since all the factors, appearing in the first term on the right in (3.12), are *positive*, then $s > 0$ (for all $j$ sufficiently large), which means, from our construction, that the associated zeros of $R_{n_j,\nu_j}((n_j+\nu_j)z)$ must lie to the *left* of the arc $D_{\sigma_j,\infty}$. Similarly, because of (1.18), the associated poles of $R_{n_j,\nu_j}\left((n_j+\nu_j)z\right)$ must lie to the *right* of the arc $E_{\sigma_j,\infty}$. This can be explicitly seen in Figs. 2-4.

Finally, to show that the result of (2.4) of Theorem 1 is *sharp*, the multiplier of $(n_j+\nu_j)^{-1}$ in the first term on the right in (3.12) is but a special case of $\sigma_j = \tau$ of

$$\frac{(\ln|\hat{N}_\tau(\tilde{z})|)\cdot|\tilde{z}|}{\mathrm{Re}\,(g_\tau(\tilde{z}))},\ \text{where } \tilde{z}\in D_{\tau,\infty}\backslash\tilde{C}_{\delta,\tau}.$$

But from the discussion above, it also follows that, for any $\tau$ with $0<\tau<\infty$ and any fixed $\delta$ with $0<\delta<1$, there exist constants $M_1(\tau)$ and $M_2(\tau,\delta)$ such that

$$0 < M_1(\tau) \le \frac{(\ln|\hat{N}_\tau(\tilde{z})|)\cdot|\tilde{z}|}{\mathrm{Re}\,(g_\tau(\tilde{z}))} \le M_2(\tau,\delta)\ \text{for } all\ \tilde{z}\in D_{\tau,\infty}\backslash\tilde{C}_{\delta,\tau}. \tag{3.14}$$

Hence, because $\sigma_j \to \sigma$ as $j\to\infty$ (where $0<\sigma<\infty$) and because of the bounds of (3.14), it follows that the first term on the right in (3.12) is *exactly* of order $(n_j+\nu_j)^{-1}$, as $j\to\infty$, which shows that the result (2.4) of Theorem 1 is sharp. □

## 4. The arcs $D_{\sigma_j,n_j+\nu_j}$ and $E_{\sigma_j,n_j+\nu_j}$

Here, we show that the arcs $D_{\sigma_j,n_j+\nu_j}$ and $E_{\sigma_j,n_j+\nu_j}$, defined in (2.8) and (2.9), are well-defined for $j$ sufficiently large, where we assume, as in Theorem 1, that (2.3) is valid. Because the treatment of the arcs $E_{\sigma_j,n_j+\nu_j}$ is similar, we consider below only the arcs $D_{\sigma_j,n_j+\nu_j}$.

For a given $\tau$ with $0<\tau<\infty$, consider the function defined by

$$U_{\tau,m}(z) := (w_\tau(z))^m / \hat{N}_\tau(z) \tag{4.1}$$

for any positive integer $m$. It can be verified that $U_{\tau,m}(z)$ is analytic and single-valued on $\mathbb{C}\backslash\mathscr{R}_\tau$. On fixing any $\theta$ with

$$\cos^{-1}\left(\frac{1-\tau}{1+\tau}\right) < \theta < 2\pi - \cos^{-1}\left(\frac{1-\tau}{1+\tau}\right),$$

we know from the discussion in Sect. 3 that $|U_{\tau,m}(re^{i\theta})|$ is strictly increasing on the interval $0\le r\le 1$, where $U_{\tau,m}(0)=0$. In addition, since the arc $D_{\tau,\infty}$ lies completely in the open unit disk (except for its endpoints), we have that $|w_\tau(e^{i\theta})|>1$, and from Lemma 1, we similarly have $|\hat{N}_\tau(e^{i\theta})|>1$. It follows from (4.1) that, for *all* $m$ sufficiently large, say $m\ge m_o(\tau,\theta)$,

$$|U_{\tau,m}(e^{i\theta})|>1 \qquad (m\ge m_o(\tau,\theta)).$$

But the strict increase of $|U_{\tau,m}(re^{i\theta})|$, as a function of $r$ on [0,1], gives that there is a unique $\hat{r}=\hat{r}(\theta,\tau,m)$ with $0<\hat{r}<1$, such that

$$|U_{\tau,m}(\hat{r}e^{i\theta})| = 1,$$

i.e., from (2.8), $\hat{r}e^{i\theta}$ necessarily lies on the arc $D_{\tau,m}$. It is also evident that $\hat{r}(\theta,\tau,m') < \hat{r}(\theta,\tau,m)$ for $m' > m \geq m_o(\tau,\theta)$, since, by definition, $|w_\tau(\hat{r}e^{i\theta})|^m = |\hat{N}_\tau(\hat{r}e^{i\theta})| > 1$, where the last inequality follows from Lemma 1.

Next, with the assumption of (2.3) in Theorem 1, we see that the arcs $D_{\sigma_j,\infty}$, defined in (1.16), *converge*, uniformly as $j \to \infty$, to the fixed arc $D_{\sigma,\infty}$, where $0 < \sigma < \infty$. But as the arcs $D_{\sigma_j,\infty}$ lie in the open unit disk (with the exception of its endpoints) for any $j$, then for any $\epsilon > 0$ sufficiently small and for any $\theta$ satisfying $\cos^{-1}(\frac{1-\sigma}{1+\sigma}) + \epsilon \leq \theta \leq 2\pi - \cos^{-1}(\frac{1-\sigma}{1+\sigma}) - \epsilon$, it follows that $|w_{\sigma_j}(e^{i\theta})| > 1$ for all $j$ sufficiently large, so that

$$|w_{\sigma_j}(e^{i\theta})|^{n_j+\nu_j}/|\hat{N}_{\sigma_j}(e^{i\theta})| > 1 \qquad \text{for all } j \text{ sufficiently large.}$$

Hence, the argument above shows that, for all $j$ sufficiently large, there is an $r(j)$ with $0 < r(j) < 1$ such that $|w_{\sigma_j}(r(j)e^{i\theta})|^{n_j+\nu_j} = |\hat{N}_{\sigma_j}(r(j)e^{i\theta})|$, i.e., $r(j)e^{i\theta}$ lies on $D_{\sigma_j,n_j+\nu_j}$. This thus establishes

**Lemma 2.** *Under the hypothesis of* (2.3) *of Theorem 1, the arcs* $D_{\sigma_j,n_j+\nu_j}$ *and* $E_{\sigma_j,n_j+\nu_j}$ *of* (2.8) *and* (2.9), *are well-defined, for all* $j$ *sufficiently large.*

## 5. Proof of Theorem 3

Under the hypothesis of (2.3) of Theorem 1, the sets $D_{\sigma_j,n_j+\nu_j}$ and $E_{\sigma_j,n_j+\nu_j}$ are well-defined from Lemma 2 for all $j$ sufficiently large, say $j \geq j_0$. For $j \geq j_0$, let $z$ be any zero of $R_{n_j,\nu_j}\big((n_j+\nu_j)z\big)$ in the unit disk and write $z = re^{i\theta}$. Then, let $\hat{z} = \hat{r}e^{i\theta}$ be on the arc $D_{\sigma_j,n_j+\nu_j}$ and, as before, set $z - \hat{z} = se^{i\theta}$, where $|s|$ measures the distance between $z$ and $\hat{z}$. Since $z$ is a zero of $R_{n_j,\nu_j}((n_j+\nu_j)z)$, we have from (3.5) that

$$|w_{\sigma_j}(z)|^{n_j+\nu_j} = |\hat{N}_{\sigma_j}(z)| \cdot \left\{1 + O\left(\frac{1}{n_j+\nu_j}\right)\right\} \quad (j \to \infty),$$

or, in the notation of (4.1),

$$|U_{\sigma_j,n_j+\nu_j}(z)| = 1 + O\left(\frac{1}{n_j+\nu_j}\right), \quad (j \to \infty). \tag{5.1}$$

Similarly, as in the proof of Theorem 1 in Sect. 3, we expand $U_{\sigma_j,n_j+\nu_j}(z)$ in a Taylor series about $\hat{z}$, i.e.,

$$U_{\sigma_j,n_j+\nu_j}(z) = U_{\sigma_j,n_j+\nu_j}(\hat{z}) + se^{i\theta}U'_{\sigma_j,n_j+\nu_j}(\hat{z}) + O(s^2).$$

Using the definition of $U_{\sigma_j,n_j+\nu_j}$ in (4.1), along with the identities of (3.7) and (3.10) for the derivatives of $w_\tau(z)$ and $\hat{N}(z)$, it can be verified that

$$\begin{aligned} &U_{\sigma_j,n_j+\nu_j}(z) \\ &= U_{\sigma_j,n_j+\nu_j}(\hat{z}) \cdot \left\{1 + se^{i\theta}\left[\frac{(n_j+\nu_j)g_{\sigma_j}(\hat{z})}{\hat{z}} + \frac{1}{g_{\sigma_j}(\hat{z})\cdot\hat{z}}\right] + O(s^2)\right\}. \end{aligned} \tag{5.2}$$

But since $|U_{\sigma_j,n_j+\nu_j}(\hat{z})| = 1$, it follows from (5.1) and (5.2), on taking moduli, that

$$(5.3)\qquad 1+\frac{s}{\hat{r}}\left[(n_j+\nu_j)\mathrm{Re}\left(g_{\sigma_j}(\hat{z})\right)+\mathrm{Re}\left(\frac{1}{g_{\sigma_j}(\hat{z})}\right)\right]+O(s^2)=1+O\left(\frac{1}{n_j+\nu_j}\right).$$

As in Sect. 3, we know in general that $\mathrm{Re}\,(g_\tau(z))$ and $1/\mathrm{Re}\,(g_\tau(z))$ are uniformly bounded at all points of the unit disk not in $\tilde{C}_{\delta,\tau}$. In particular, for any zero $z$ of $R_{n_j,\nu_j}((n_j+\nu_j)z))$ in the unit disk not in $\tilde{C}_{\delta,\sigma_j}$, we see from (5.3) that

$$\frac{s}{\hat{r}}=O\left(\frac{1}{(n_j+\nu_j)^2}\right)\qquad (j\to\infty).$$

But since the arc $D_{\sigma_j,n_j+\nu_j}$ cannot pass through 0 and since $\hat{r}$ is bounded above by unity, then

$$s=O\left(\frac{1}{(n_j+\nu_j)^2}\right),$$

for any zero $z$ of $R_{n_j,\nu_j}\left((n_j+\nu_j)z\right)$ not in $\tilde{C}_{\delta,\sigma_j}$, as $j\to\infty$. Again, as $|s|$ measures the distance from $z$ to a particular point, $\hat{z}$, of $D_{\sigma_j,n_j+\nu_j}$, then dist $\left[z;D_{\sigma_j,n_j+\nu_j}\right]\le|s|$ for *any* zero of $R_{n_j,\nu_j}((n_j+\nu_j)z)$, in the unit disk not in $\tilde{C}_{\delta,\sigma_j}$, and it follows that

$$dist\left[\{z_{n_j,\nu_j}(k)\}_{k=1}^{n_j}\backslash\tilde{C}_{\delta,\sigma_j};D_{\sigma_j,n_j+\nu_j}\right]=O\left(\frac{1}{(n_j+\nu_j)^2}\right)\qquad (j\to\infty),$$

which is the desired result of (2.10) of Theorem 3. □

## 6. Final comments

The results of the previous sections deal with the convergence of the zeros (and poles) of the Padé approximants $\left\{R_{n_j,\nu_j}((n_j+\nu_j)z)\right\}_{j=1}^{\infty}$ in relationship to the arcs $D_{\sigma_j,\infty}$ and $D_{\sigma_j,n_j+\nu_j}$ *outside* of the disks $\tilde{C}_{\delta,\sigma_j}$. It is thus natural to ask *what* the convergence rate of these zeros is in the neighborhood of the points $z_{\sigma_j}^{\pm}$, which are explicitly *excluded* in the results of (2.4) of Theorem 1 and (2.10) of Theorem 3. It turns out that, on applying a result of [9, eq.(1.9)], we also have the following result:

**Theorem 4.** *Under the hypothesis of* (2.3) *of Theorem 1, the Padé approximant* $R_{n_j,\nu_j}\left((n_j+\nu_j)z\right)$ *has zeros and poles of the form*

$$(6.1)\qquad z_{\sigma_j}^{\pm}+O\left(\frac{1}{(n_j+\nu_j)^{2/3}}\right)\qquad (j\to\infty).$$

The importance of Theorem 4 lies in the fact that (6.1) is valid for *any* $\sigma$ with $0<\sigma<\infty$, and Theorem 4 shows that there is a substantially *slower* convergence of the zeros and poles of $R_{n_j,\nu_j}((n_j+\nu_j)z)$ to $D_{\sigma_j,\infty}$ and $E_{\sigma_j,\infty}$ in neighborhoods of the branch points, $z_\sigma^{\pm}$, of $g_\sigma(z)$, which are exactly the points which have been *excluded* with our use of the disks $\tilde{C}_{\delta,\sigma}$ of (2.2). In this sense, (6.1) of Theorem 4 is the analog of result (1.2) of Buckholtz [2]. We conjecture that the results of (6.1) are best possible!

Next, we remark that a careful examination shows that the arcs $D_{1,\infty}$ and $E_{1,\infty}$ in Fig. 3 make an angle of $2\pi/3$ as they meet at the points $\pm i$. This has been theoretically established in Olver [7, p.336, Fig. 3] for the special case $\sigma=1$, and we remark that

**Table 1.**

| $k$ | $\arg(z_{16,24}(k))$ | $\lvert z_{16,24}(k)\rvert$ | $\lvert \hat{z}_{16,24}(k)\rvert$ |
|---|---|---|---|
| 1 | 114.677512621653575 | 0.924518801046913 | 0.923788102974373 |
| 2 | 125.542379165422707 | 0.864754514607011 | 0.864321571680979 |
| 3 | 134.974717991547015 | 0.825049238737003 | 0.824715883665449 |
| 4 | 143.729355307749157 | 0.796647826203171 | 0.796364645259408 |
| 5 | 152.091849890036330 | 0.776141233642082 | 0.775887263220768 |
| 6 | 160.211234669319259 | 0.761805526883314 | 0.761569325056279 |
| 7 | 168.181077469080952 | 0.752680657788123 | 0.752454813408059 |
| 8 | 176.069064193694423 | 0.748237220620139 | 0.748016178753669 |

this *same* angle appears also in Figs. 2 and 4. We similarly conjecture that this same angle will appear in *all* cases where $0 < \sigma < \infty$, because of the fact that (6.1) of Theorem 4 is valid for all $0 < \sigma < \infty$.

Finally, because the results of Figs. 5–7 are almost too good to believe, we consider again, as in Fig. 5, the actual zeros $\{z_{16,24}(k)\}_{k=1}^{8}$ of $R_{16,24}(40z)$ in the upper half-plane, ordered by increasing argument. In columns 2 and 3 of Table 1 above, we give respectively the arguments and moduli of these eight zeros. Then, $\hat{z}_{16,24}(k)$ is defined (as in the proof of Theorem 3 in Sect. 5) as the point on the arc $D_{3/2,40}$ having the *same* argument as $z_{16,24}(k)$, and the last column of Table 1 gives the corresponding moduli of the points $\{\hat{z}_{16,24}(k)\}_{k=1}^{8}$. The differences of these corresponding moduli in columns 3 and 4 vary between $7.3 \cdot 10^{-4}$ and $2.2 \cdot 10^{-4}$, which is why, up to plotting accuracy, the zeros of $R_{16,24}(40z)$ appear to lie on the arc $D_{3/2,40}$!

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