An Extension of a Result of Rivilin on Walsh Equiconvergence

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Abstract. Considering certain best polynomial approximation of a function in l^2-sense, Rivlin [7] has proved an extension of Walsh equiconvergence theorem. Special cases of the main result proved here lead to the foregoing result of Rivlin and a result proved in Cavaretta, Sharma and Varga [2].

§1 Introduction

Let f be a function holomorphic in the disk $D_R := \{z \in \mathbb{C} : |z| < R\}$ for some $R > 1$. For a non-negative integer n, we denote by $L_n(\cdot; f)$ the Lagrange interpolant to f in the $(n+1)^{\text{th}}$ roots of unity and by $S_n(\cdot, f)$ the n^{th} partial sum of the power-series expansion of f about the origin. Then the Walsh equiconvergence theorem [8, p. 153] asserts that

$$\lim_{n\to\infty} [L_n(z; f) - S_n(z; f)] = 0, \quad z \in D_{R^2}, \tag{1.1}$$

the convergence being uniform and geometric in every disk $\overline{D}_\rho$ with $\rho < R^2$.

Rivlin [7] extended (1.1) by considering the polynomial $P_{m,n}(\cdot, f)$ of degree n which best approximates f in the ℓ_2-sense over all polynomials of degree n in the $(m+1)^{\text{th}}$ roots of unity, where $m = q(n+1) - 1$, $q \in \mathbb{N}$. Rivlin showed that

$$\lim_{n\to\infty} [P_{m,n}(z; f) - S_n(z; f)] = 0, \quad z \in D_{R^{1+q}} \tag{1.2}$$

with uniform and geometric convergence in every disk $\overline{D}_\rho$, $\rho < R^{1+q}$.

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In [1], the first author showed that Walsh Theorem (1.1) cannot be carried over to the following more general situation. Let E be a compact subset of the complex plane with a complement E^c which is simply connected in the extended complex plane. According to the Riemann mapping theorem, there exists a conformal map ψ of $\{\omega \in \mathbb{C} : |\omega| > 1\}$ onto E^c normalized at infinity by $\psi(\infty) = \infty$ and $C := \psi'(\infty) > 0$, where C is called the capacity of E. For $R > 1$, let $C_R := \{\psi(\omega) : |\omega| = R\}$ be an outer level curve of E and let A_R denote the class of functions f holomorphic in $G_R :=$ Int $\mathrm{C_R}$, having at least one singularity on C_R. We denote by F_k the k-th Faber polynomial, and for $f \in A_R$, we denote by $S_n(\cdot; f)$ the n^{th} partial sum of the Faber expansion of f with respect to E. For the definition and properties of Faber polynomials, we refer to Curtiss [4] or Gaier [5].

For a non-negative integer n let the interpolation nodes $z_{kn} \in E$ $(k = 0, \cdots, n)$ be given and let $L_n(\cdot; f)$ denote the Lagrange interpolant to $f \in A_R$ in these nodes. The interpolation is to be understood in the Hermite sense if some of these nodes coincide. If we set

$$\omega_n(z) := \prod_{k=0}^{n} (z - z_{kn}) \tag{1.3}$$

we require that the nodes z_{kn} are chosen such that

$$\lim_{n\to\infty} \frac{\omega_n(\psi(\omega))}{C^{n+1}\omega^{n+1}} = 1, \quad |\omega| > 1, \tag{1.4}$$

holds uniformly on every closed subset of $\{\omega \in \mathbb{C} : |\omega| > 1\}$. It follows from a more general result in [1] that if (1.1) holds for all $f \in A_R$ and all $z \in G_\rho$ for some $\rho > R$, then E must be a disk.

Now the question arises whether it is possible to obtain an equiconvergence result, if we replace $L_n(\cdot, f)$ by a polynomial of the type $P_{m,n}(\cdot; f)$ in certain nodes $z_{km} \in E$. Rivlin [7] proved a similar result when $E = [-1, 1]$ and z_{km} are the zeros of the m^{th} Chebyshev polynomial. Furthermore he showed that $P_{m,n}(\cdot; f) = S_n(\cdot; L_m(z; f))$ when E is the unit disk D and z_{km} are the $(m+1)^{\text{th}}$ roots of unity or when $E = [-1, 1]$ and z_{km} are the Chebyshev nodes. We do not know if this relationship prevails in the general situation described above. Therefore we set $Q_{m,n}(\cdot; f) := S_n(\cdot; L_m(z; f))$ and prove an equiconvergence theorem for the difference $Q_{m,n}(\cdot; f) - S_n(\cdot; f)$ provided E is "nice" and the nodes z_{km} are suitably chosen.

In Section 2, we state the main result for Lagrange interpolants and some known special cases. In Section 3, we sketch an outline of the proof. Section 4 deals with statements of two theorems which can be proved by using the properties of Faber polynomials given in Section 3. The detailed proofs will appear elsewhere.

§2 Lagrange interpolation

Using the notation of Section 1, let the boundary ∂E of E be an analytic Jordan curve. Then the conformal map ψ is continuable to a homeomorphism of $\{\omega \in \mathbb{C} : |\omega| \geq 1\}$ onto $\mathbb{C} \setminus (\text{Int } E)$, so that we may define $z_{km} := \psi(\omega_{km})$, where $\omega_{km} := \exp\left(2\pi i k/(m+1)\right)$, $(k = 0, 1, \ldots, m)$. The points z_{km} are called the $(m+1)^{\text{th}}$ Féjer nodes with respect to E. Following Pommerenke [6], we say that ∂E is an γ_0-analytic curve $(0 \leq \gamma_0 < 1)$, if the conformal map ψ admits a univalent continuation to $\{\omega \in \mathbb{C} : |\omega| > \gamma_0\}$. For $f \in A_R$, let

$$f(z) = \sum_{k=0}^{\infty} a_k F_k(z), \quad z \in G_R \ (= \text{ Int } C_R)$$

be the Faber expansion of f with respect to E. Then $S_n(z; f) := \sum_{k=0}^{n} a_k F_k(z)$, is the $n^{\text{th}}-$section of the Faber expansion of f. For non-negative integers m, n and j with $m \geq n$, we set

$$S_{m,n,j}(z; f) := \sum_{k=0}^{n} a_{k+j(m+1)} F_k(z) \tag{2.1}$$

and for $\ell \in \mathbb{N}$, let

$$D_{m,n,\ell}(z; f) := S_n\big(z; L_m(\cdot; f)\big) - \sum_{j=0}^{\ell-1} S_{m,n,j}(z; f). \tag{2.2}$$

Clearly $S_{m,n,0}(z, f) = S_n(z; f)$ and $S_n\big(z; L_m(z; f)\big)$ is the n^{th} Faber section of the expansion of $L_m(z, f)$ in terms of Faber polynomials. We are now in a position to state

Theorem 1. *Let ∂E be an γ_0-analytic curve for some $\gamma_0 \in [0, 1)$, $f \in A_R$, $m = q(n+1) - 1$, with $q \in \mathbb{N}$ and let $D_{m,n,\ell}(z; f)$ be as defined in (2.2). Then*

$$\lim_{n \to \infty} D_{m,n,\ell}(z; f) = 0, \quad z \in G_\lambda, \tag{2.3}$$

the convergence being geometric and uniform on every subset $\overline{G}_\mu$ for $1 < \mu < \lambda$, where

$$\lambda := \min\left\{R^{1+\ell q}, R/\gamma_0^q, R^q/\gamma_0^{q-1}\right\} \tag{2.4}$$

with $0^k := 0$ for any non-negative integer k and $1/0 := \infty$.

Remarks. (1) If $q = 1$ and $\gamma_0 > 0$, then $\lambda = R$ so that Theorem 1 gives no overconvergence. For arbitrary q, if $\gamma_0 \to 1$, then $\lambda \to R$ and again there is no overconvergence. In the special case $\gamma_0 = 0$, i.e. $E = \overline{D}$, we have $\lambda = R^{1+\ell q}$. For $\ell = 1$, we obtain the result of Rivlin [7, Theorem 1] and for $q = 1$, we obtain a result of Cavaretta, Sharma and Varga [2, Theorem 1]. If $q \geq \ell + 1$ and $\gamma_0 \leq \frac{1}{R^\ell}$, then $\lambda = R^{1+\ell q}$, that is we have the same λ as in the case of the unit-disk, if E is sufficiently close to $\overline{D}$.

(2) We do not know if λ is best possible. However, we are able to improve our result if $E = E_\delta \quad (\delta > 1)$ is an ellipse with half axis $a := \frac{1}{2}(\delta + \frac{1}{\delta})$ and $b := \frac{1}{2}(\delta - \frac{1}{\delta})$, i.e., ∂E_δ is the image of the circle $\{\omega \in \mathbb{C} : |\omega| = \delta\}$ under the map $\omega = \frac{1}{2}(w + \frac{1}{w})$. In this case we have $\psi(\omega) = \frac{1}{2}(\delta\omega + \frac{1}{\delta\omega})$. Then ∂E_δ is a γ_0−analytic curve with $\gamma_0 = \frac{1}{\delta}$ and

$$\omega_m(\psi(\omega)) = (\frac{\delta}{2})^{m+1}(\omega^{m+1} - 1)(1 - \frac{1}{(\delta^2\omega)^{m+1}})$$

which is an improvement on (3.1) (see Sec. 3).

Furthermore, we have

$$F_k(\psi(\omega)) = \omega^k + \frac{1}{\delta^{2k}\omega^k}\,,$$

so that the coefficients $\alpha_{k\nu}$ in (3.4) are explicitly known. Now an examination of the proof of Theorem 1 shows that in this case λ is given by

$$\lambda = \min\,\{R^{1+\ell q}, R^{1+q}/\gamma_0^{2q}, R^{2q-1}/\gamma_0^{2(q-1)}\}.$$

This is best possible, as can be seen by the example $f(z) := \frac{1}{\psi(R)-z}$. If $q = 1$, then $\lambda = R$. Furthermore, we have $\lambda = R^{1+\ell q}$ provided $q \geq \ell + 1$ and $\gamma_0^2 \leq \frac{1}{R^{\ell-1}}$. In particular, $\lambda = R^{1+q}$ for all $\gamma_0 \in (0, 1)$ if $\ell = 1$ and $q \geq 2$.

(3) The previous remark also applies when $\delta = 1$, i.e., $E = [-1, 1]$. (Note that the *Fejér* nodes on $[-1, 1]$ are not mutually different.) Then we obtain $\lambda = R$ for $q = 1$ and $\lambda = R^{1+q}$ for all $q \geq 2$. If we use the zeros of the Chebyshev polynomials as interpolation nodes, we have

$$\omega_m(\psi(\omega)) = (\frac{1}{2})^{m+1}\omega^{m+1}(1 - \frac{1}{\omega^{2(m+1)}}).$$

The proof of Theorem 1 runs through with minor modifications and yields

$$\lambda = \begin{cases} R^{2q-1} & \text{for} \quad \ell = 1 \\ R & \text{for} \quad q = 1 \\ R^{q+1} & \text{for} \quad q, \ell > 1. \end{cases}$$

so that we obtain a generalization and a new proof of Theorem 2 of Rivlin [7].

§3 Proof of Theorem 1

a) Some Properties of Faber Polynomials : Since ∂E is a γ_0−analytic curve, by Lemma 3.1 in [3] for any $\rho > \gamma_0$, we have

$$\omega_m(\psi(\omega)) = C^{m+1}(\omega^{m+1} - 1)(1 + O(\rho^m)) \tag{3.1}$$

uniformly on closed subsets of $\{\omega \in \mathbb{C} : |\omega| > \gamma_0\}$. If

$$f(z) = \sum_{k=0}^{\infty} a_k F_k(z)$$

is the Faber expansion of f, then the Faber coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \int_{|\xi|=\gamma'} \frac{f(\psi(\xi))}{\xi^{k+1}} d\xi, \quad (k = 0, 1, 2, \ldots), \tag{3.2}$$

where $1 < \mu' < R$, so that

$$S_n(z; f) = \frac{1}{2\pi i} \int_{|\xi|=\gamma'} f(\psi(\xi)) \sum_{k=0}^{n} \frac{F_k(z)}{\xi^{k+1}} d\xi, \quad z \in \mathbb{C}.$$

From (3.1) we obtain

$$\frac{\omega_m(\psi(\xi)) - \omega_m(\psi(t))}{\omega_m(\psi(\xi))} = \frac{\xi^{m+1} - t^{m+1}}{\xi^{m+1} - 1}(1 + O(\rho^m)) \tag{3.3}$$

uniformly on closed subsets of the set

$$\{\xi \in \mathbb{C} : |\xi| > 1\} \times \{\mathrm{t} \in \mathbb{C} : |\mathrm{t}| > 1\}.$$

Also for the Faber polynomials F_k the following relation holds ([4, Equation (2.7)])

$$F_k(\psi(\omega)) = \omega^k + \sum_{\nu=1}^{\infty} \alpha_{k\nu}\omega^{-\nu}, \quad |\omega| > 1 \tag{3.4}$$

uniformly on closed subsets of $\{\omega \in \mathbb{C} : |\omega| > 1\}$ with certain coefficients $\alpha_{k\nu} \in \mathbb{C}$.

b) Integral Representation of $D_{m,n,\ell}(z; f)$: From the well-known Hermite interpolation formula [5; p. 59] we obtain for any $\gamma' \in (1, R)$

$$L_m(z; f) = \frac{1}{2\pi i} \int_{|\xi|=\gamma'} f(\psi(\xi)) \frac{\psi'(\xi)}{\psi(\xi) - z} \cdot \frac{\omega_m(\psi(\xi)) - \omega_m(z)}{\omega_m(\psi(\xi))} d\xi, \quad z \in \mathbb{C} \tag{3.5}$$

with ω_m defined by (1.3). Then the expansion of $L_m(z;f)$ in terms of Faber polynomials is given by

$$\begin{aligned} S_n(z;L_m(\cdot;f)) &= \frac{1}{2\pi i}\int_{|t|=\gamma} L_m(\psi(t))\cdot\sum_{k=0}^{n}\frac{F_k(z)}{t^{k+1}}dt \\ &= \frac{1}{2\pi i}\int_{|\xi|=\gamma'} f(\psi(\xi))\Big(\frac{1}{2\pi i}\int_{|t|=\gamma}\frac{\psi'(\xi)}{\psi(\xi)-\psi(t)} \\ &\quad\times\frac{\omega_m(\psi(\xi))-\omega_m(\psi(t))}{\omega_m(\psi(\xi))}\sum_{k=0}^{n}\frac{F_k(z)}{t^{k+1}}dt\Big)d\xi, \quad z\in\mathbb{C} \end{aligned} \tag{3.6}$$

where we choose γ and γ' such that $1<\gamma<\gamma'<R$.

Furthermore, (3.2) implies that

$$S_{m,n,j}(z;f) = \frac{1}{2\pi i}\int_{|\xi|=\gamma'} f(\psi(\xi))\frac{1}{\xi^{j(m+1)}}\sum_{k=0}^{n}\frac{F_k(z)}{\xi^{k+1}}d\xi, \quad z\in\mathbb{C}. \tag{3.7}$$

Thus we have

$$\sum_{j=0}^{\ell-1} S_{m,n,j}(z;f) = \frac{1}{2\pi i}\int_{|\xi|=\gamma'} f(\psi(\xi))\frac{\xi^{\ell(m+1)}-1}{\xi^{(\ell-1)(m+1)}(\xi^{m+1}-1)}\sum_{k=0}^{n}\frac{F_k(z)}{\xi^{k+1}}d\xi, \quad z\in\mathbb{C}. \tag{3.8}$$

Using the residue theorem, we obtain

$$\frac{1}{2\pi i}\int_{|t|=\gamma}\frac{\psi'(\xi)}{\psi(\xi)-\psi(t)}\cdot\frac{dt}{t} = \frac{1}{\xi}, \quad |\xi|>\gamma,$$

so that (3.8) can be written as a double integral. Thus we get

$$\begin{aligned} \sum_{j=0}^{\ell-1} S_{m,n,j}(z;f) &= \frac{1}{2\pi i}\int_{|\xi|=\gamma'} f(\psi(\xi))\Big(\frac{1}{2\pi i}\int_{|t|=\gamma}\frac{\psi'(\xi)}{\psi(\xi)-\psi(t)}\cdot\frac{\xi}{t} \\ &\quad\times\frac{\xi^{\ell(m+1)}-1}{\xi^{(\ell-1)(m+1)}(\xi^{m+1}-1)}\sum_{k=0}^{n}\frac{F_k(z)}{\xi^{k+1}}dt\Big)d\xi, \quad z\in\mathbb{C}. \end{aligned} \tag{3.9}$$

Combining (3.6) and (3.9) we are able to obtain an integral representation for $D_{m,n,\ell}(z;f)$. It can be verified that using (3.5), (3.9), (3.3) and (3.4), we can write

$$D_{m,n,\ell}(z;f) = \frac{1}{2\pi i}\int_{|\xi|=\gamma'} f(\psi(\xi))\Big(\frac{1}{2\pi i}\int_{|t|=\gamma}\frac{\psi'(\xi)}{\psi(\xi)-\psi(t)}K_{m,n,\ell}(\omega,\xi,t)dt\Big)d\xi. \tag{3.10}$$

The kernel $K_{m,n,\ell}(\omega,\xi,t)$ can be broken into four parts. We set $K_{m,n,\ell}(\omega,\xi,t) := \sum_{j=1}^{4} K^{(j)}_{m,n,\ell}(\omega,\xi,t)$ where

$$\left\{\begin{aligned} K^{(1)}_{m,n,\ell}(\omega,\xi,t) &:= \frac{\xi^{m+1}-t^{m+1}}{\xi^{m+1}-1}(1+O(\rho^m))\sum_{k=1}^{n}\sum_{\nu=1}^{\infty}\alpha_{k\nu}\omega^{-\nu}t^{-k-1} \\ &\quad -\frac{\xi}{t}\,\frac{\xi^{k(m+1)}-1}{\xi^{(\ell-1)(m+1)}(\xi^{m+1}-1)}\sum_{k=1}^{n}\sum_{\nu=1}^{\infty}\alpha_{k\nu}\omega^{-\nu}\xi^{-k-1} \\ K^{(2)}_{m,n,\ell}(\omega,\xi,t) &:= \frac{\xi^{m+1}(t^{n+1}-\omega^{n+1})}{(t-\omega)(\xi^{m+1}-1)t^{n+1}} \\ &\quad -\frac{\xi}{t}\,\frac{(\xi^{n+1}-\omega^{n+1})}{(\xi-\omega)\xi^{n+1}}\times\frac{\xi^{\ell(m+1)}-1}{\xi^{(\ell-1)(m+1)}(\xi^{m+1}-1)} \end{aligned}\right. \tag{3.11}$$

$$\left\{\begin{aligned} K^{(3)}_{m,n,\ell}(\omega,\xi,t) &= O(\rho^m)\frac{(\xi^{m+1}-t^{m+1})(t^{n+1}-\omega^{n+1})}{(t-\omega)t^{n+1}(\xi^{m+1}-1)} \\ &\text{and} \\ K^{(4)}_{m,n,\ell}(\omega,\xi,t) &= -\frac{t^{m+1}(t^{n+1}-\omega^{n+1})}{(t-\omega)t^{n+1}(\xi^{m+1}-1)}. \end{aligned}\right. \tag{3.12}$$

Thus we have

$$D_{m,n,\ell}(z;f) = \sum_{j=1}^{4} D^{(j)}_{m,n,\ell}(z;f)$$

where

$$D^{(j)}_{m,n,\ell} = \frac{1}{2\pi i}\int_{|\xi|=\gamma'} f(\psi(\xi))\Big(\frac{1}{2\pi i}\int_{|t|=\gamma}\frac{\psi'(\xi)}{\psi(\xi)-\psi(t)}K^{(j)}_{m,n,\ell}(\omega,\xi,t)dt\Big)d\xi.$$

c) Estimates of $D^{(j)}_{m,n,\ell}$ $(j=1,2,3,4)$: We begin with $D^{(1)}_{m,n,\ell}$. Letting $n\to\infty$ and observing that $|t|<|\xi|$, we obtain

$$\begin{aligned} K_1(\omega,\xi,t) &:= \lim_{n\to\infty} K^{(1)}_{m,n,\ell}(\omega,\xi,t) \\ &= \sum_{k=1}^{\infty}\sum_{\nu=1}^{\infty}\alpha_{k\nu}\omega^{-\nu}t^{-k-1} - \frac{\xi}{t}\sum_{k=0}^{\infty}\sum_{\nu=1}^{\infty}\alpha_{k\nu}\omega^{-\nu}\xi^{-k-1} \end{aligned}$$

where the double sums on the right hand side are convergent uniformly on closed subsets of $\{\omega\in\mathbb{C} : |\omega|>1\}\times\{t\in\mathbb{C} : |t|>1\}$. Now the residue theorem implies that

$$\frac{1}{2\pi i}\int_{|t|=\gamma}\frac{\psi'(\xi)}{\psi(\xi)-\psi(t)}K_1(\omega,\xi,t)dt = 0, \tag{3.13}$$

for all $\xi \in \mathbb{C}$ with $|\xi| > \gamma > 1$. Thus $\lim_{n\to\infty} D^{(1)}_{m,n,\ell} = 0$ uniformly on $\mathbb{C}$.

Similarly, again using the residue theorem, we have

$$\frac{1}{2\pi i}\int_{|t|=\gamma}\frac{\psi'(\xi)}{\psi(\xi)-\psi(t)}K^{(2)}_{m,n,\ell}(\omega,\xi,t)dt = K^{(2)}_{m,n,\ell}(\omega,\xi,\xi)$$

$$= \frac{\xi^{n+1}-\omega^{n+1}}{(\xi-\omega)\xi^{((\ell-1)q+1)(n+1)}(\xi^{q(n+1)}-1)}$$

$$= O(1)\Big(\frac{|\omega|}{(\gamma')^{1+\ell q}}\Big)^n, \tag{3.14}$$

if $|\omega| > \gamma'$. This yields $\lim_{n\to\infty} D^{(2)}_{m,n,\ell}(z;f) = 0$ uniformly on $\overline{G}_\mu$ for every $\mu < R^{1+\ell q}$.

In order to estimate $D^{(3)}_{m,n,\ell}(z;f)$ we observe that for $|\omega| > \gamma'$, we have

$$K^{(3)}_{m,n,\ell}(\omega,\xi,t) = O(1)\Big(\frac{|\omega|\rho^q}{\gamma}\Big)^n \tag{3.15}$$

and that $\lim_{n\to\infty} D^{(3)}_{m,n,\ell}(z;f) = 0$ uniformly on $\overline{G}_\mu$ for every $\mu < R/\gamma_0^q$.

Similarly, in order to estimate $D^{(4)}_{m,n,\ell}(z;f)$, we recall ([4], Eq. (2.9)) that

$$\frac{\psi'(\xi)}{\psi(\xi)-\psi(t)} = \frac{1}{\xi-t} + \sum_{\mu=1}^{\infty}\sum_{k=1}^{\infty}\alpha_{\mu k}t^{-k}\xi^{-\mu-1} \tag{3.16}$$

uniformly on closed subsets of $\{t \in \mathbb{C} : |t| > 1\} \times \{\xi \in \mathbb{C} : |\xi| > 1\}$, where the coefficients $\alpha_{\mu k}$ are defined by (3.4). Again applying the residue theorem, we get

$$F_{m,n}(\omega,\xi) := \frac{1}{2\pi i}\int_{|t|=\gamma}\frac{\psi'(\xi)}{\psi(\xi)-\psi(t)}K^{(4)}_{m,n,\ell}(\omega,\xi,t)dt$$

$$:= \frac{1}{\xi^{m+1}-1}\sum_{\mu=1}^{\infty}\sum_{\nu=0}^{n}\alpha_{\mu,m-\nu+1}\omega^{\nu}\xi^{-\mu-1}.$$

Since ∂E is γ_0–analytic, we have [4, Eq. (4.2)] $\alpha_{\mu k} = O(1)\rho^{\mu+k}$ for all μ, $k \in \mathbb{N}$ and any $\rho > \gamma_0$. Thus

$$F_{m,n}(\omega,\xi) = O(1)\Big(\frac{|\omega|}{(\gamma')^q\rho^{1-q}}\Big)^n$$

which implies that $\lim_{n\to\infty} D^{(4)}_{m,n,\ell}(z;f) = 0$ uniformly on $\overline{G}_\mu$ for all $\mu < \mathcal{R}^q/\gamma_0^{q-1}$.

Combining the estimates for $D^{(j)}_{m,n,\ell}$ $(j = 1,2,3,4)$, we obtain the result which completes the proof. ■

§4 Hermite interpolation

We shall state without proof two similar results for Hermite interpolation. For $f \in A_R$ and for $s \in \mathbb{N}$, we denote by $H_{s(m+1)-1}(\cdot; f)$ the Hermite interpolatory polynomial to $f, f', \ldots, f^{(s-1)}$ in the $(m+1)^{\text{th}}$ *Fejér* nodes on E. Then for $p, q \in \mathbb{N}$ with $m = q(n+1) - 1$ and with $sq \geq p$, we set

$$\Delta^{p,s}_{m,n}(z; f) := S_{p(n+1)-1}\big(z; H_{s(m+1)-1}(\cdot; f)\big) - S_{p(n+1)-1}(z; f). \tag{4.1}$$

We can then prove

Theorem 2. *Under the assumptions on ∂E in Theorem 1, the following holds:*

$$\lim_{n\to\infty} \Delta^{p,s}_{m,n}(z, f) = 0 \quad for \quad z \in G_\lambda \tag{4.2}$$

the convergence being uniform and geometric on every subset $\overline{G}_\mu$ for $1 < \mu < \lambda$, where

$$\lambda := \begin{cases} \min\{R^{1+\frac{sq}{p}}, R/\gamma_0^{q/p}, R^{sq/p}/\gamma_0^{\frac{q}{p}-1}\}, & q \geq p \\ \min\{R^{(sq+1)/p}, R/\gamma_0^{q/p}, R^{sq/p}/(1-[1-\gamma_0])\}, & q < p, \end{cases}$$

with $0^x := 0$ for any non-negative real number x and $1/0 := \infty$.

Finally, we consider a case of mixed Hermite and Lagrange interpolation. For positive integers p and s with $s \geq \max\{p, 2\}$, we set

$$D^*_{p,s,n}(z; f) := S_{p(n+1)-1}\big(z; H_{s(n+1)-1}(\cdot; f) - L_{s(n+1)-1}(\cdot; f)\big).$$

Then we can prove

Theorem 3. *Under the assumptions of Theorem 1, the following holds:*

$$\lim_{n\to\infty} D^*_{p,s,n}(z; f) = 0, \quad z \in G_\lambda \tag{4.3}$$

the convergence being uniform and geometric on every subset $\overline{G}_\mu$ for $1 < \mu < \lambda$, where if $\gamma_0 = 0$

$$\lambda := \begin{cases} R^{s+2} & for \quad p = 1 \quad and \quad s \quad odd \\ R^{(s+1)/p} & otherwise \end{cases}$$

and if $\gamma_0 > 0$, we have

$$\lambda := \min\{R/\gamma_0^{1/p}, R^{s/p}\}.$$

Proofs of Theorems 2 and 3 will be given elsewhere.

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