NORTH-HOLLAND

On Classes of Inverse Z-Matrices

Reinhard Nabben
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31
33 501 Bielefeld, Germany

and

Richard S. Varga*
Institute for Computational Mathematics
Kent State University
Kent, Ohio 44242
USA

Dedicated to M. Fiedler and V. Pták.

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ABSTRACT

Recently, a classification of matrices of class Z was introduced by Fiedler and Markham. This classification contains the classes of M-matrices and the classes of N_0- and F_0-matrices studied by Fan, G. Johnson, and Smith. The problem of determining which nonsingular matrices have inverses which are Z-matrices is called the *inverse Z-matrix problem*. For special classes of Z-matrices, such as the M- and N_0-matrices, there exist at least partial results, i.e., special classes of matrices have been introduced for which the inverse of such a matrix is an M-matrix or an N_0-matrix. Here, we define a system of classes of matrices for which the inverse of each matrix of each class belongs to one class of the classification of Z-matrices defined by Fiedler and Markham. Moreover, certain properties of the matrices of each class are established, e.g., inequalities for the sum of the entries of the inverse and the structure of certain Schur complements. We also give a necessary and sufficient condition for regularity. The class of inverse N_0-matrices given here generalizes the class of inverse N_0-matrices discussed by Johnson. All results established here can be applied to a class of distance

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matrices which corresponds to a nonarchimedean metric. This metric arises in p-adic number theory and in taxonomy.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we basically deal with $n \times n$ Z-matrices, i.e., real matrices whose off-diagonal elements are nonpositive. One well-known class of Z-matrices is the class of nonsingular M-matrices, i.e., those matrices of the form $A = tI - B$ where B is a nonnegative matrix ($B \geqslant O$) and where $t > \rho(B)$; here, $\rho(B)$ denotes the spectral radius of B. However, other classes of Z-matrices are also of interest, and have been discussed in the literature. Recently, Fiedler and Markham introduced in [5] a classification of matrices in class Z. They defined the class L_s, for $s = 0, \dots, n$, as consisting of real $n \times n$ Z-matrices which have the form

$$A = tI - B, \qquad \text{where} \quad B \geqslant O \text{ and } \rho_s(B) \leqslant t < \rho_{s+1}(B). \quad (1.1)$$

Here, $\rho_s(B)$ denotes the maximum of the spectral radii of all $s \times s$ principal submatrices of B, where $\rho_0(B) := -\infty$ and $\rho_{n+1}(B) := +\infty$. Thus, the class L_n is just the class of (singular and nonsingular) M-matrices. We note that L_{n-1} is the class of N_0-matrices introduced by G. A. Johnson [20], and this class contains the N-matrices introduced by Fan [4]. Moreover, L_{n-2} is the class of F_0-matrices defined by Smith [20].

As proved in [5], for each s with $1 \leqslant s \leqslant n-1$, the class L_s is equal to the class of Z-matrices for which all principal submatrices of order s are M-matrices, but there exists a principal submatrix of order $s+1$ which is not an M-matrix. Additional properties of some of these classes are given in [2], [19] and [21].

On the other hand, there has been interest in inverse M-matrices, i.e., any nonsingular matrix $B \geqslant O$ whose inverse is an M-matrix. A survey of this topic is given by C. R. Johnson [9]. Recently, the new class of generalized ultrametric matrices was simultaneously introduced by McDonald, Neumann, Schneider, and Tsatsomeros [15] and Nabben and Varga [17]. If nonsingular, these matrices are inverse M-matrices. However, the oldest class of symmetric inverse M-matrices is the class of positive type D matrices defined by Markham [13] in 1972. A matrix $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ is of *type D* if there exist real numbers $\{\alpha_i\}_{i=1}^n$, with $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1$, such that

$$a_{i,j} = \begin{cases} \alpha_i & \text{if } i \leqslant j, \\ \alpha_j & \text{if } i > j. \end{cases} \quad (1.2)$$

As shown in [13], a matrix of positive type D (i.e., with $\alpha_1 > 0$) is nonsingular, and its inverse is a symmetric M-matrix (i.e., a Stieltjes matrix). As an example, for $\alpha_j := j$ for $1 \leqslant j \leqslant n$, where $n \geqslant 2$, the associated $n \times n$ positive type D matrix and its inverse are given by

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-1 & n \end{bmatrix}$$

and

$$A^{-1} = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}.$$

As mentioned in [17], the class of positive type D matrices (i.e., with $\alpha_1 > 0$) is contained in the class of generalized ultrametric matrices.

But not only inverse M-matrices are of interest. For example, the linear complementarity problem can be solved by a single linear problem if the related matrix is an inverse Z-matrix [12]. Thus, it is natural to determine classes of matrices which are inverse Z-matrices. A first step was taken by G. A. Johnson [11], who proved that a matrix of negative type D (i.e., with $\alpha_n < 0$) is an inverse N_0-matrix. Later, Chen [3] studied necessary conditions for a matrix to be an inverse F_0-matrix; however, no sufficient conditions were developed.

Here, we continue discussing inverse Z-matrices. Using the classification of Z-matrices mentioned above, for each choice of s, we introduce sufficient conditions for a matrix to be an inverse L_s-matrix. Thus, we define a system of classes of matrices for which the inverse of each matrix of each class belongs to one class of the classification on Z-matrices defined by Fiedler and Markham.

Our system of classes contains the class of generalized ultrametric matrices, and, as we show below, our classes are defined in terms of ultrametric inequalities. The class of positive type D matrices is included in the class of inverse L_n-matrices (i.e., the inverse M-matrices). We show that the concept of type D matrices can be extended to obtain inverse L_s-matrices for each s with $0 \leqslant s \leqslant n$. Moreover, we establish properties of the matrices of our classes, e.g., inequalities for the sum of the entries of the inverse and the structure of certain Schur complements. We also give a necessary and sufficient condition for their nonsingularity. In Section 3, we apply these results to a class of distance matrices which corresponds to a nonarchimedean metric. This metric arises in p-adic number theory and in taxonomy.

We need the following additional notation. The set of positive integers $\{1, \dots, n\}$ is denoted by N. By ξ_n we denote the vector $[1, \dots, 1]^T \in \mathbb{R}^n$. For $A = [a_{i,j}] \in \mathbb{R}^{n,n}$, we set

$$\begin{aligned} \tau(A) &:= \min\{a_{i,j}: i, j \in N\}, \\ \omega(A) &:= \min\{a_{j,i}: a_{i,j} = \tau(A)\}, \\ \delta(A) &:= \omega(A) - \tau(A), \qquad \text{so that} \quad \delta(A) \geqslant 0. \end{aligned} \tag{1.3}$$

For a block-partitioned matrix $A \in \mathbb{R}^{n,n}$ with

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix},$$

where $A_{1,1} \in \mathbb{R}^{r,r}$ is nonsingular and where $A_{2,2} \in \mathbb{R}^{n-r,n-r}$ with $1 \leqslant r < n$, the expression

$$A/A_{1,1} := A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2} \tag{1.4}$$

denotes the *Schur complement* of A with respect to $A_{1,1}$. Similarly, $A/A_{2,2}$ is defined. Also, $|B|$ denotes the cardinality of a set B, i.e., the number of elements in B.

2. A SUFFICIENT CONDITION

The definition of our system of classes, which leads to a necessary and sufficient condition for a matrix to be an inverse L_s-matrix, makes use of ultrametric inequalities for the entries of the matrix. The differences between the classes depend on the signs of all nonzero off-diagonal entries and on the order of the largest principal submatrix which is a nonsingular M-matrix.

We begin with the following common definition of [15] and [17] for a generalized ultrametric matrix.

DEFINITION 2.1. A matrix $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ is a *generalized ultrametric matrix* if

(i) A has nonnegative entries;
(ii) $a_{i,j} \geqslant \min\{a_{i,k}; a_{k,j}\}$ for all $i, k, j \in N$;
(iii) each triple $\{q, s, t\} \subseteq N^3$ can be reordered as a triple $\{i, j, k\}$ such that
 (iii.i) $a_{j,k} = a_{i,k}$ and $a_{k,j} = a_{k,i}$,
 (iii.ii) $\max\{a_{i,j}; a_{j,i}\} \geqslant \max\{a_{i,k}; a_{k,i}\}$,
(iv) $a_{i,i} \geqslant \max\{a_{i,j}; a_{j,i}\}$ for all $i, j \in N$.

A matrix A is said to be a *strictly generalized ultrametric matrix* if strict inequality holds in (iv) for all $i \neq j$ in N, where, if $n = 1$, this is interpreted as $a_{1,1} > 0$.

We see that if A is a generalized ultrametric matrix in $\mathbb{R}^{n,n}$, then the matrix $B := A + \tau \xi_n \xi_n^T$ satisfies (ii), (iii), and (iv) of Definition 2.1 for *any* real τ. This observation gives rise to

DEFINITION 2.2. A matrix $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ is a *shifted generalized ultrametric matrix* if there is a real number τ such that $A + \tau \xi_n \xi_n^T$ is a generalized ultrametric matrix. If strict inequality holds in (iv) of Definition 2.1 for all $i \in N$, then A is a *strictly shifted generalized ultrametric matrix*. In addition, if $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ is a shifted generalized ultrametric matrix which is not a generalized ultrametric matrix, so that $\tau(A) < 0$, then A is said to be *of type* $U^{(-1)}_{p,n-p}$ if

(v) $a_{i,j} \leqslant 0$ for all $i, j \in N$ with $i \neq j$;
(vi) p is the largest positive integer such that there exists a $p \times p$ principal submatrix of A which is a nonsingular M-matrix. If no such positive integer exists, then $p := 0$.

As is easily seen, there are shifted generalized ultrametric matrices in $\mathbb{R}^{n,n}$, $n \geqslant 2$, which are not generalized ultrametric matrices, and for which the off-diagonal entries are *mixed* in sign. This shows that the sets $U^{(-1)}_{p,n-p}$ represent proper subsets, of all shifted generalized ultrametric matrices not generalized ultrametric matrices, in $\mathbb{R}^{n,n}$ for $n \geqslant 2$.

EXAMPLE 2.1. Consider the Z-matrix

$$A := \begin{bmatrix} 5 & -4 & -6 \\ -2 & 2 & -6 \\ -2 & -2 & 1 \end{bmatrix}. \tag{2.1}$$

Since

$$A + 6\xi_3\xi_3^T = \begin{bmatrix} 11 & 2 & 0 \\ 4 & 8 & 0 \\ 4 & 4 & 7 \end{bmatrix}$$

can be verified to be a (strictly) generalized ultrametric matrix, then A is a (strictly) *shifted* generalized ultrametric matrix, which is obviously not a generalized ultrametric matrix, since A has some negative entries. Note that the matrix

$$A + 5\xi_3\xi_3^T = \begin{bmatrix} 10 & 1 & -1 \\ 3 & 7 & -1 \\ 3 & 3 & 6 \end{bmatrix}$$

is also a (strictly) shifted generalized ultrametric matrix which is not a generalized ultrametric matrix, but this matrix is not in any $U_{p,3-p}^{(-1)}$ for $0 \leqslant p \leqslant 3$, because it has off-diagonal entries which are of mixed signs [cf. (v) of Definition 2.2]. It is interesting to note that

$$\left(A + 5\xi_3\xi_3^T\right)^{-1} = \frac{1}{441}\begin{bmatrix} 45 & -9 & 6 \\ -21 & 63 & 7 \\ -12 & -27 & 67 \end{bmatrix},$$

so that its inverse is not a Z-matrix.

To determine the value of p for which $A \in U_{p,3-p}^{(-1)}$, we have that

$$A^{-1} = \frac{1}{154}\begin{bmatrix} 10 & -16 & -36 \\ -14 & 7 & -42 \\ -8 & -18 & -2 \end{bmatrix} = \tfrac{1}{154}(10I - B), \tag{2.2}$$

where

$$B := \begin{bmatrix} 0 & 16 & 36 \\ 14 & 3 & 42 \\ 8 & 18 & 12 \end{bmatrix},$$

showing that A is nonsingular and that A^{-1} is a Z-matrix. Since some entries of A^{-1} are negative, A is surely not a nonsingular M-matrix. However, the leading principal 2×2 submatrix of A is a nonsingular M-matrix, so that, in the notation (vi) of Definition 2.2,

$$A \in U_{2,1}^{(-1)}.$$

Also, since A^{-1} is a Z-matrix, it can be verified, using the notation of (1.1), that the matrix B of (2.2) satisfies $B \geqslant O$ with $\rho_0(B) = -\infty < 10 < \rho_1(B) = 12$, so that [cf. (1.1)]

$$A^{-1} \in L_0.$$

The point of this example is this: for the matrix A of (2.1), we have that

$$A \in U_{n-m,m}^{(-1)} \text{ and } A^{-1} \in L_{m-1} \qquad \text{for} \quad n = 3 \text{ and } m = 1,$$

which couples the class $U_{n-m,m}^{(-1)}$ for A to the class L_{m-1} for A^{-1}. As we shall later see, this is a special case of one of our main results, Theorem 2.10.

In Section 3, we discuss the class of (symmetric) nonarchimedean matrices, which are the negatives of symmetric shifted generalized ultrametric matrices of type $U_{0,n}^{(-1)}$ with zero diagonal entries, i.e., $A \in \mathbb{R}^{n,n}$ is a symmetric shifted generalized ultrametric matrix of type $U_{0,n}^{(-1)}$ with zero diagonal entries if and only if $-A$ is a nonarchimedean matrix; see Proposition 3.3.

We next derive a subclass of each class defined above which generalizes the concept of type D matrices due to Markham [13].

DEFINITION 2.3. A matrix $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ is of *type* $D_{1,n-1}^{(1)}$ if there exist real numbers $\{\alpha_i\}_{i=1}^n$, with $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > 0$, such that

$$a_{i,j} = \begin{cases} \alpha_i & \text{if } i \leqslant j, \\ \alpha_j & \text{if } i > j. \end{cases}$$

A matrix $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ is of *type* $D_{p,n-p}^{(-1)}$, where $0 \leqslant p \leqslant n$, if there exist real numbers $\{\alpha_i\}_{i=1}^n$, with $0 > \alpha_n > \alpha_{n-1} > \cdots > \alpha_1$, such that

$$a_{i,j} = \begin{cases} \alpha_i & \text{if } i < j, \\ \alpha_j & \text{if } i > j, \end{cases}$$

where, if $P := \{i \in N: a_{i,i} > 0\}$ and if $M := \{i \in N: a_{i,i} < 0\}$, then $P \cup M = N$, $p := |P|$, $n - p = |M|$, and

$$a_{i,i} > \sum_{j \in P,\, j \neq i} |a_{i,j}| \qquad (i \in P),$$

$$a_{i,i} = \alpha_i \qquad (i \in M).$$

Matrices of type $D_{1,n-1}^{(1)}$ are just the matrices of positive type D of (1.2), defined by Markham, and hence are generalized ultrametric matrices (cf. [17]). Next, for any matrix A of type $D_{p,n-p}^{(-1)}$ of Definition 2.3, it is also easy to verify that A is a shifted generalized ultrametric matrix which is not a generalized ultrametric matrix. Then, as the condition on the set P (with $p = |P|$) in the second part of Definition 2.3 creates a $p \times p$ principal submatrix of A which is a nonsingular M-matrix, and as the remaining diagonal elements of A (associated with the set M) are negative, there can be *no* larger principal submatrix of A which is a nonsingular M-matrix. This shows that $D_{p,n-p}^{(-1)}$ is in fact a subset of $U_{p,n-p}^{(-1)}$, as the notation would suggest.

EXAMPLE 2.2. As an example of a matrix of Definition 2.3, the matrix

$$A := \begin{bmatrix} -4 & -4 & -4 & -4 \\ -4 & -3 & -3 & -3 \\ -4 & -3 & 3 & -2 \\ -4 & -3 & -2 & 3 \end{bmatrix}$$

is in $D_{p,n-p}^{(-1)}$ for $n = 4$, $p = 2$, $\alpha_j := -5 + j$ for $1 \leqslant j \leqslant 4$, $P := \{3, 4\}$, and $M := \{1, 2\}$. Its inverse, given by

$$A^{-1} = \frac{1}{140} \begin{bmatrix} 105 & -140 & 0 & 0 \\ -140 & 180 & -20 & -20 \\ 0 & -20 & 24 & -4 \\ 0 & -20 & -4 & 24 \end{bmatrix} = \tfrac{1}{140}(180I - B)$$

where

$$B := \begin{bmatrix} 75 & 140 & 0 & 0 \\ 140 & 0 & 20 & 20 \\ 0 & 20 & 156 & 4 \\ 0 & 20 & 4 & 156 \end{bmatrix},$$

is such that $\rho_1(B) = 156 < 180 < \rho_2(B) = 182.43533\ldots$, so that $A^{-1} \in L_1$. Thus,

$$A \in D^{(-1)}_{n-m,m} \text{ and } A^{-1} \in L_{m-1} \quad \text{for} \quad m = 2 \text{ and } n = 4,$$

which, as we shall later see, is a special case of Corollary 2.11.

To illustrate the classes of shifted generalized ultrametric matrices, and especially the inequalities for the off-diagonal entries, weighted rooted trees can be used, as indicated in [17], for generalized ultrametric matrices.

Let $G = (V, E)$ be a rooted tree, consisting of the set of vertices V and the set of edges $E \subset \{(x, y): x, y \in V, x \neq y\}$. Let $w \in V$ denote the *root* of the tree, and let $B \subset V$ denote the set of its *leaves*, with $|B| = n$. To each edge (x, y) of E, there are assigned two nonnegative numbers:

$$l(x, y) \geqslant 0 \quad \text{and} \quad r(x, y) \geqslant 0,$$

and l and r are called *weighting functions* for the rooted tree. Then, for any $b \in B$, let $p(b)$ denote the consecutive distinct edges $\{(v_i, v_{i+1})\}_{i=0}^{m-1}$, with $b = v_0$ and $v_m = w$, which forms a path connecting the leaf b with the root w. For any b and $\tilde{b}$ in B, set

$$\begin{aligned} d_l(b,\tilde{b}) &:= \sum_{p(b)\cap p(\tilde{b})} l(v_i, v_{i+1}), \\ d_r(b,\tilde{b}) &:= \sum_{p(b)\cap p(\tilde{b})} r(v_i, v_{i+1}), \\ d(b,b) &:= \max\{d_l(b,b); d_r(b,b)\}, \end{aligned} \tag{2.3}$$

where $p(b) \cap p(\tilde{b})$ denotes the *common edges* of these paths from the leaves $b$and $\tilde{b}$ to the root w. If the leaves are numbered from 1 to n, and if the associated matrix $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ has its entries defined by

$$a_{i,j} := \begin{cases} d_r(i,j) & \text{for} \quad i < j, \\ d(i,j) & \text{for} \quad i = j, \\ d_l(i,j) & \text{for} \quad i > j, \end{cases} \tag{2.4}$$

we obtain, for every rooted tree and for all weighting functions l and r

defined on this rooted tree, a generalized ultrametric matrix. Conversely, for a given generalized ultrametric matrix A, there exists a rooted tree and weighting functions l and r such that the entries of A are given as indicated in (2.3) and (2.4) (see Theorem 3.4 of [17]).

To extend the above connection, between weighted rooted trees and generalized ultrametric matrices, to weighted rooted trees and *shifted* generalized ultrametric matrices, it is only necessary to add, in the discussion above, a new vertex, which becomes the new root, and to add a single edge from the new root to the old root. For the associated weights, we assign to the new edge two *equal* real numbers τ (possibly negative), i.e., $l = r = \tau$ for this edge. The addition of this new root and single edge to a weighted rooted tree analogously defines a shifted generalized ultrametric matrix, and conversely, any shifted generalized ultrametric matrix similarly defines a weighted rooted tree.

EXAMPLE 2.3. Consider the rooted tree in Figure 1 with leaves (1), ..., (7) and weights $r(i, j)$ and $l(i, j)$, shown respectively on the right and left side of the corresponding edge (i, j). If we add a new root and a

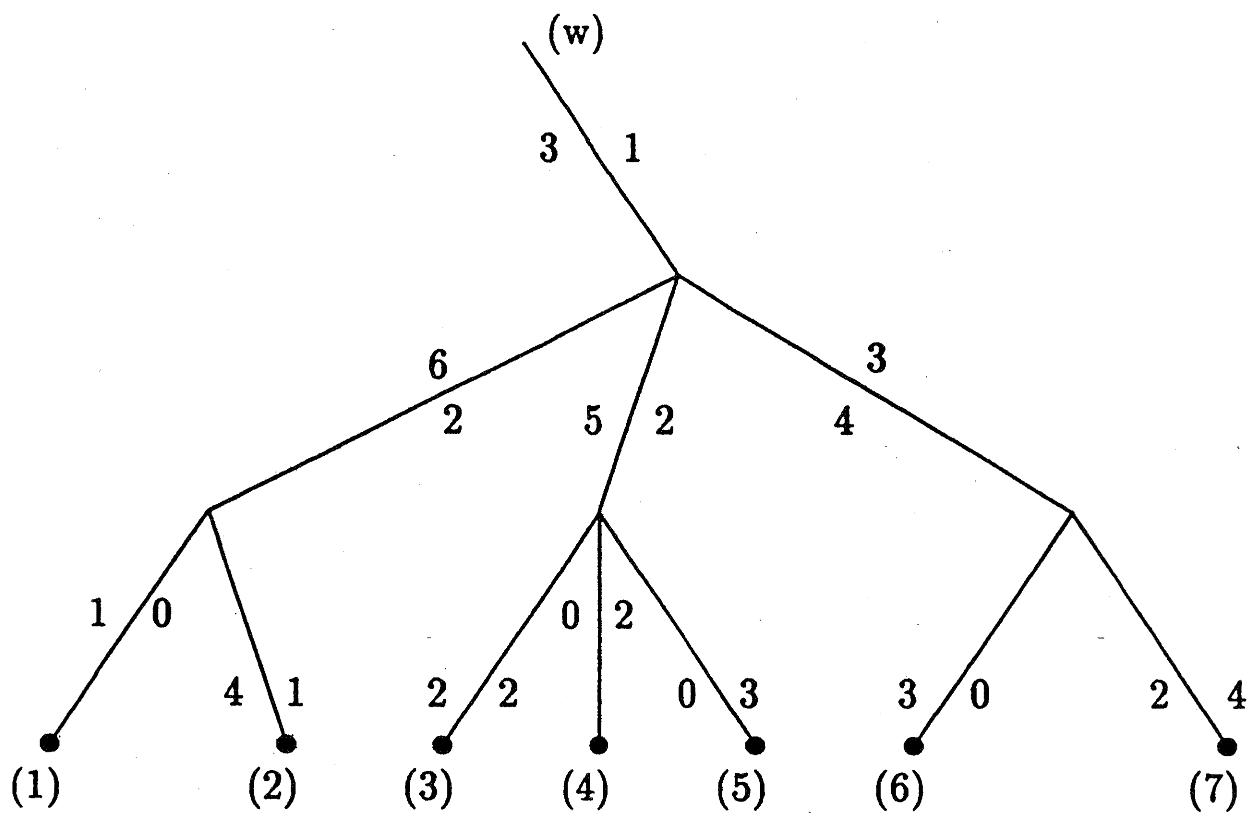


FIG. 1.

single edge to the old root, with weights $l = r = -5$, and use the definitions of (2.3) and (2.4), the associated shifted generalized ultrametric matrix $A = [a_{i,j}] \in \mathbb{R}^{7,7}$ is given by

$$A = \begin{bmatrix} 5 & -2 & -4 & -4 & -4 & -4 & -4 \\ 4 & 8 & -4 & -4 & -4 & -4 & -4 \\ -2 & -2 & 5 & -2 & -2 & -4 & -4 \\ -2 & -2 & 3 & 3 & -2 & -4 & -4 \\ -2 & -2 & 3 & 3 & 3 & -4 & -4 \\ -2 & -2 & -2 & -2 & -2 & 5 & -1 \\ -2 & -2 & -2 & -2 & -2 & 2 & 4 \end{bmatrix}.$$

Throughout this paper, we use the following theorem, which is proved in Theorems 3.7 and 3.8 of [17]:

THEOREM 2.1. *Let $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ be a strictly generalized ultrametric matrix. Then A is nonsingular, and its inverse $A^{-1} \in \mathbb{R}^{n,n}$ is a strictly row and strictly column diagonally dominant M-matrix, with the property that*

$$\omega(A)\,\xi_n^T A^{-1}\xi_n < 1. \tag{2.5}$$

If A is a nonsingular generalized ultrametric matrix, its inverse $A^{-1} \in \mathbb{R}^{n,n}$ is a row and column diagonally dominant M-matrix, with the property that

$$\omega(A)\,\xi_n^T A^{-1}\xi_n \leqslant 1. \tag{2.6}$$

We begin our results with the following lemma which is a small extension of Theorem 3.6 in [15]:

LEMMA 2.2. *Let $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ be nonsingular. Define*

$$\tau_i := \min_{j\in N}\{a_{i,j}\} \quad \text{and} \quad \mu_i := \max_{j \in N}\{a_{i,j}\} \qquad \text{for all} \quad i \in N.$$

If $A \geqslant O$ and if $A^{-1}\xi_n \geqslant 0$, then for each $i \in N$,

$$\begin{aligned} \mu_i(\xi_n^T A^{-1}\xi_n) &\geqslant 1, \\ \tau_i(\xi_n^T A^{-1}\xi_n) &\leqslant 1. \end{aligned} \tag{2.7}$$

If $A \geqslant O$ and if $A^{-1}\xi_n > 0$, then for each $i \in N$,

$$\begin{aligned} \mu_i(\xi_n^T A^{-1}\xi_n) &> 1 \quad \textit{unless} \quad a_{i,j} = \mu_i \quad \textit{for all } j \in N, \\ \tau_i(\xi_n^T A^{-1}\xi_n) &< 1, \quad \textit{unless} \quad a_{i,j} = \tau_i \quad \textit{for all } j \in N. \end{aligned} \tag{2.8}$$

Similarly, if $A \leqslant O$ and if $A^{-1}\xi_n \leqslant 0$, then for each $i \in N$,

$$\begin{aligned} \mu_i(\xi_n^T A^{-1}\xi_n) &\leqslant 1, \\ \tau_i(\xi_n^T A^{-1}\xi_n) &\geqslant 1. \end{aligned} \tag{2.9}$$

If $A \leqslant O$ and if $A^{-1}\xi_n < 0$, then for each $i \in N$,

$$\begin{aligned} \mu_i(\xi_n^T A^{-1}\xi_n) &< 1 \quad \textit{unless} \quad a_{i,j} = \mu_i \quad \textit{for all } j \in N, \\ \tau_i(\xi_n^T A^{-1}\xi_n) &> 1 \quad \textit{unless} \quad a_{i,j} = \tau_i \quad \textit{for all } j \in N. \end{aligned} \tag{2.10}$$

Proof. Assuming $A \geqslant O$ and $A^{-1}\xi_n \geqslant 0$, then

$$\xi_n = A(A^{-1}\xi_n) \leqslant \begin{bmatrix} \mu_1 & \cdots & \mu_1 \\ \vdots & & \vdots \\ \mu_n & \cdots & \mu_n \end{bmatrix} A^{-1}\xi_n = (\xi_n^T A^{-1}\xi_n) \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}.$$

On comparing vector components above, it follows that $1 \leqslant \mu_i(\xi_n^T A^{-1}\xi_n)$ for all $i \in N$, which is the desired first inequality of (2.7). Next, note that if $[y_1, \dots, y_n]^T := A^{-1}\xi_n$, then for each $i \in N$,

$$1 = [A(A^{-1}\xi_n)]_i = \sum_{j=1}^{n} a_{i,j} y_j \leqslant \mu_i \sum_{j=1}^{n} y_j.$$

If $y_i > 0$ for all $i \in N$ (i.e., $A^{-1}\xi_n > 0$, then equality can hold throughout, in the above display, only if $a_{i,j} = \mu_i$ for *all* $j \in N$, and this establishes the first inequality of (2.8). The remaining inequalities are similarly established.

As in the class of generalized ultrametric matrices, shifted generalized ultrametric matrices also satisfy a nested block structure, as given in Theorem 2.3 below.

THEOREM 2.3. *Let $A = [a_{i,j}] \in \mathbb{R}^{n,n}$, $n > 1$, be a shifted generalized ultrametric matrix. Then there exist a permutation matrix $P \in \mathbb{R}^{n,n}$ and a positive integer r, with $1 \leqslant r < n$, such that*

$$PAP^T = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

$$= \begin{bmatrix} C & O \\ \delta(A)\xi_{n-r}\xi_r^T & D \end{bmatrix} + \tau(A)\xi_n\xi_n^T =: M + \tau(A)\xi_n\xi_n^T, \quad (2.11)$$

where $A_{1,1} \in \mathbb{R}^{r,r}$ and $A_{2,2} \in \mathbb{R}^{n-r,n-r}$ are shifted generalized ultrametric matrices. Moreover, the matrices $A_{1,1}$ and $A_{2,2}$ are reduced in the same way as A. The off-diagonal blocks are $A_{1,2} = \tau(A)\xi_r\xi_{n-r}^T$ and $A_{2,1} = \omega(A)\xi_{n-r}\xi_r^T$.

The matrices $M \in \mathbb{R}^{n,n}$, $C \in \mathbb{R}^{r,r}$, and $D \in \mathbb{R}^{n-r,n-r}$ are generalized ultrametric matrices and $\delta(A) = \omega(A) - \tau(A) \geqslant 0$. Moreover,

$$\omega(A_{1,1}) \geqslant \omega(A) \quad \text{and} \quad \omega(A_{2,2}) \geqslant \omega(A). \quad (2.12)$$

If A is a strictly shifted generalized ultrametric matrix, then M is a strictly generalized ultrametric matrix.

Proof. For a generalized ultrametric matrix, this is proved in Theorem 3.1 of [17]. The general result then follows directly from the definition of a shifted generalized ultrametric matrix in Definition 2.2. ■

In what follows, we assume without loss of generality that the permutation matrix in (2.11) is the identity matrix.

Theorem 2.3, given above for any shifted generalized ultrametric matrix, applies of course to its subsets $U_{p,n-p}^{(-1)}$. We remark, however, that the decomposition of (2.11), when applied to a matrix A in $U_{p,n-p}^{(-1)}$, is such that its associated block submatrices, $A_{1,1}$ and $A_{2,2}$ of (2.11), are shifted generalized ultrametric matrices which may or may not be in U-type sets. For example, the matrix A of (2.1) is in $U_{2,1}^{(-1)}$, but its decomposition, via (2.11), gives

$$A := \left[\begin{array}{cc|c} 5 & -4 & -6 \\ -2 & 2 & -6 \\ \hline -2 & -2 & 1 \end{array}\right] =: \left[\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array}\right],$$

with $A_{1,1} \in U_{2,0}^{(-1)}$, while $A_{2,2}$ is a strictly generalized ultrametric matrix.

The next lemma gives some properties of matrices of type $U_{p,n-p}^{(-1)} \in \mathbb{R}^{n,n}$.

LEMMA 2.4. *Let $A \in \mathbb{R}^{n,n}$, $n > 1$, be a shifted generalized ultrametric matrix given in the block form (2.11).*

(i) *If A is a nonsingular strictly shifted generalized ultrametric matrix of type $U_{p,n-p}^{(-1)}$ for some p with $0 \leqslant p < n$, then*

$$A^{-1}\xi_n < 0 \quad \textit{and} \quad \omega(A)\xi_n^T A^{-1}\xi_n > 1. \tag{2.13}$$

Similarly, if A is a nonsingular shifted generalized ultrametric matrix of type $U_{p,n-p}^{(-1)}$ for some p with $0 \leqslant p < n$, then

$$A^{-1}\xi_n \leqslant 0 \quad \textit{and} \quad \omega(A)\xi_n^T A^{-1}\xi_n \geqslant 1. \tag{2.14}$$

(ii) *If $A_{1,1}$ in (2.11) is nonsingular and of type $U_{p,r-p}^{(-1)}$ for some p with $0 \leqslant p < r$, then the Schur complement $A/A_{1,1}$ is a generalized ultrametric matrix. If $A_{1,1}$ is a nonsingular M-matrix, i.e., of type $U_{r,0}^{(-1)}$, and if $A_{2,2} \in U_{q,n-r-q}^{(-1)}$ with $0 \leqslant q \leqslant n-r$, then $A/A_{1,1}$ is of type $U_{\tilde{p},n-r-\tilde{p}}^{(-1)}$ for a $\tilde{p}$ with $0 \leqslant \tilde{p} \leqslant q$. The same holds for $A/A_{2,2}$.*

Proof. (i) We consider the nested block structure of (2.11) for A. Since A is by hypothesis a nonsingular strictly shifted generalized ultrametric matrix, then M of (2.11) is a strictly generalized ultrametric matrix from Theorem 2.3, which is also nonsingular from Theorem 2.1. Using the Sherman-Morrison formula (cf. [8, p. 52]) gives that

$$A^{-1} = \left[M + \tau(A)\xi_n^T\xi_n\right]^{-1} = M^{-1} - \frac{\tau(A)}{1 + \tau(A)\xi_n^T M^{-1}\xi_n} M^{-1}\xi_n\xi_n^T M^{-1},$$

where $\tau(A)\xi_n^T M^{-1}\xi_n \neq -1$. Thus,

$$A^{-1}\xi_n = \frac{1}{1 + \tau(A)\xi_n^T M^{-1}\xi_n} M^{-1}\xi_n. \tag{2.15}$$

If $\tau(A)\xi_n^T M^{-1}\xi_n > -1$, we would obtain from (2.15) that $A^{-1}\xi_n =: u$ is a positive vector, since $M^{-1}\xi_n > 0$ (because M^{-1} is a strictly row diagonally dominant matrix from Theorem 2.1). Now, the hypotheses of (i) give that A is a Z-matrix. But as $Au = \xi_n$, this would imply (cf. [2, p. 136, I_{28}]) that A is a nonsingular M-matrix, which contradicts the hypothesis that $A \in U_{p,n-p}^{(-1)}$

for some p with $p < n$. Thus, $\tau(A)\xi_n^T M^{-1}\xi_n < -1$, and (2.15) gives $A^{-1}\xi_n < 0$, the desired first inequality of (2.13). It also follows from (2.15) that

$$\omega(A)\,\xi_n^T A^{-1}\xi_n > 1 \quad \Leftrightarrow \quad [\omega(A) - \tau(A)]\,\xi_n^T M^{-1}\xi_n < 1. \quad (2.16)$$

From (2.11), it is evident that $\omega(M) = \delta(A) = \omega(A) - \tau(A)$. Because A is by hypothesis a strictly shifted generalized ultrametric matrix, M is a strictly generalized ultrametric matrix from Theorem 2.3, and hence [cf. (2.5) of Theorem 2.1]

$$\omega(M)\,\xi_n^T M^{-1}\xi_n = [\omega(A) - \tau(A)]\,\xi_n^T M^{-1}\xi_n < 1.$$

Thus from (2.16), $\omega(A)\xi_n^T A^{-1}\xi_n > 1$, the desired final inequality of (2.13).

To establish (2.14), assume that A is a nonsingular shifted generalized ultrametric matrix of type $U_{p,n-p}^{(-1)}$ for some p with $0 \leqslant p < n$. Because A is a Z-matrix, we can write

$$A = tI - B, \qquad \text{where} \quad B \in \mathbb{R}^{n,n} \quad \text{with} \quad B \geqslant O.$$

Since A is nonsingular and not a nonsingular M-matrix, then $\rho(B) > t$, and for all $\epsilon > 0$ sufficiently small,

$$A + \epsilon I = (t + \epsilon)I - B, \qquad \text{where} \quad \rho(B) > t + \epsilon.$$

Hence, $A + \epsilon I$ is a nonsingular *strictly* shifted generalized ultrametric matrix for all $\epsilon > 0$ sufficiently small. On applying the two inequalities of (2.13) to $A + \epsilon I$, and on letting $\epsilon \downarrow 0$, the desired results of (2.14) follow.

(ii) If $A_{1,1}$ is nonsingular, its Schur complement $A/A_{1,1}$ from (1.4) and (2.11), is given by

$$A/A_{1,1} = A_{2,2} - \tau(A)\,\omega(A)\left(\xi_r^T A_{1,1}^{-1}\xi_r\right)\xi_{n-r}\xi_{n-r}^T.$$

As $A_{2,2}$ is a shifted generalized ultrametric matrix from Theorem 2.3, it follows from (2.17) and Definition 2.2 that $A/A_{1,1}$ also is.

Next, by hypothesis we have that $A_{1,1}$ is of type $U_{p,r-p}^{(-1)}$ for some p with $0 \leqslant p < r$. With (2.14), we have

$$\xi_r^T A_{1,1}^{-1}\xi_r \leqslant 0 \quad \text{and} \quad \omega(A_{1,1})\left(\xi_r^T A_{1,1}^{-1}\xi_r\right) \geqslant 1.$$

As these inequalities imply that $\xi_r^T A_{1,1}^{-1} \xi_r \neq 0$, we have

$$\xi_r^T A_{1,1}^{-1} \xi_r < 0 \quad \text{and} \quad \omega(A_{1,1}) \xi_r^T A_{1,1}^{-1} \xi_r \geqslant 1,$$

from which it follows that $\omega(A_{1,1}) < 0$. In addition, since $\omega(A_{1,1}) \geqslant \omega(A)$ from (2.12), we further deduce that

$$\omega(A) \xi_r^T A_{1,1}^{-1} \xi_r \geqslant 1.$$

Also, since $\omega(A) \geqslant \tau(A)$ from (1.3), we have $\tau(A) < 0$. Then recalling, from (2.11) of Theorem 2.3, that $A_{2,2} = D + \tau(A)\xi_{n-r}\xi_{n-r}^T$ where D in $\mathbb{R}^{n-r,n-r}$ is a generalized ultrametric matrix, the expression in (2.17) can be written as

$$A/A_{1,1} = D + [-\tau(A)][\omega(A)\xi_r^T A_{1,1}^{-1} \xi_r - 1]\, \xi_{n-r}\xi_{n-r}^T. \tag{2.18}$$

But, as the scalar multiplier of $\xi_{n-r}\xi_{n-r}^T$ in (2.18) is nonnegative, we see from Definition 2.1 that $A/A_{1,1}$ is also a generalized ultrametric matrix.

Next, let $A_{1,1}$ be a shifted generalized ultrametric matrix of type $U_{r,0}^{(-1)}$, i.e., $A_{1,1}$ is a nonsingular M-matrix; hence $\xi_r^T A_{1,1}^{-1} \xi_r > 0$. Since the off-diagonal entries of $A_{1,1}$ are nonpositive, then $\tau(A)$ and $\omega(A)$ are both nonpositive, whence

$$\tau(A)\omega(A)(\xi_r^T A_{1,1}^{-1} \xi_r)\xi_{n-r}\xi_{n-r}^T \geqslant O. \tag{2.19}$$

If $A_{2,2}$ is of type $U_{q,n-r-q}^{(-1)}$, its off-diagonal entries are nonpositive, and it follows from (2.17) and (2.19) that the same is true for $A/A_{1,1}$. From this, it is easy to see that $A/A_{1,1} \in U_{\tilde{p},n-r-\tilde{p}}^{(-1)}$ for some $\tilde{p}$ with $0 \leqslant \tilde{p} \leqslant q$. ∎

Since we are interested in nonsingular matrices, we establish a necessary and sufficient condition for a shifted generalized ultrametric matrix, with at least one negative diagonal entry, to be nonsingular.

THEOREM 2.5. *Let $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ be a shifted generalized ultrametric matrix with a negative diagonal entry. Then A is nonsingular if and only if A contains no zero row or zero column and no two rows or two columns which are the same.*

Proof. As it is evident that if A is nonsingular, then A contains no zero row or zero column and no two rows or two columns which are the same, we need only to show the reverse implication. Thus, our hypothesis is that A in

$\mathbb{R}^{n,n}$ is a shifted generalized ultrametric matrix, with a negative diagonal entry, which has no zero row or zero column and no two rows or two columns which are the same, and our goal is to establish that A is nonsingular. If $n = 1$, this is certainly true, and we may assume that $n \geqslant 2$. Applying Theorem 2.3 to $A = [a_{i,j}]$ gives, up to a permutation of indices, that there is a positive integer r, with $1 \leqslant r < n$, such that [cf. (2.11)]

$$\begin{aligned} A &= \begin{bmatrix} A_{1,1} & \tau(A)\xi_r \xi_{n-r}^T \\ \omega(A)\xi_{n-r}\xi_r^T & A_{2,2} \end{bmatrix} \\ &= \begin{bmatrix} C & O \\ \delta(A)\xi_{n-r}\xi_r^T & D \end{bmatrix} + \tau(A)\xi_n\xi_n^T =: M + \tau(A)\xi_n\xi_n^T, \end{aligned} \tag{2.20}$$

where $A_{1,1} \in \mathbb{R}^{r,r}$ and $A_{2,2} \in \mathbb{R}^{n-r,n-r}$ are shifted generalized ultrametric matrices, and where $M \in \mathbb{R}^{n,n}$, $C \in \mathbb{R}^{r,r}$, and $D \in \mathbb{R}^{n-r,n-r}$ are generalized ultrametric matrices. Because A has a negative diagonal entry, it follows from (iv) of Definition 2.1 and from the representation in (2.20) that

$$\tau(A) \leqslant \omega(A) < 0. \tag{2.21}$$

First, suppose that there is a zero row or zero column in the generalized ultrametric matrix M in (2.20). Then, there is a permutation of indices such that A and M can be expressed, from (2.20), as

$$\begin{aligned} A &= \left[\begin{array}{c|ccc} \tau & \tau & \cdots & \tau \\ \hline \tau & & & \\ \vdots & & \tilde{A}_{2,2} & \\ \tau & & & \end{array}\right] = M + \tau\xi_n\xi_n^T \\ &= \left[\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \tilde{M}_{2,2} & \\ 0 & & & \end{array}\right] + \tau\xi_n\xi_n^T \qquad [\tau := \tau(A)]. \end{aligned} \tag{2.22}$$

Now, the principal submatrix $\tilde{M}_{2,2}$ of M (which is also a generalized ultrametric matrix) cannot have a zero row or zero column, since, from (2.22), A would then have two rows or two columns which are the same, contradicting our initial hypothesis. By the same reasoning, $\tilde{M}_{2,2}$ does not have two rows or two columns which are the same. Using Theorem 4.4 of [15], it follows that $\tilde{M}_{2,2}$ is nonsingular. If we set $\tilde{A}_{1,1} := [\tau(A)] \in \mathbb{R}^{1,1}$, then $\tilde{A}_{1,1}$ is nonsingular from (2.21), with $\det \tilde{A}_{1,1} = \tau(A)$. Using the Schur complement [cf. (1.4)] of A in (2.22), with respect to $\tilde{A}_{1,1}$ we have (cf. [7, p. 22]) that

$$\det A = \det \tilde{A}_{1,1} \det\left(A/\tilde{A}_{1,1}\right) = \tau(A) \det\left[\tilde{A}_{2,2} - \tau(A)\xi_{n-1}\xi_{n-1}^T\right].$$

But as $\tau(A) < 0$ and as [cf. (2.22)] $\tilde{A}_{2,2} - \tau(A)\xi_{n-1}\xi_{n-1}^T = \tilde{M}_{2,2}$ is nonsingular, the previous display gives that A is nonsingular.

Next, we consider the case where there is no zero row or zero column in M. If M contains two rows or two columns which are the same, it would follow from (2.20) that A has two rows or two columns which are the same, which is again a contradiction to our initial hypothesis. Hence, it again follows Theorem 4.4 of [15] that M is nonsingular. Let $a_{k,k}$ be a negative diagonal entry of A. If $a_{k,k} = \tau(A)$, it would follow from (iv) of Definition 2.1 that the entries in the kth row and kth column of A are all $\tau(A)$; whence, from (2.20), the kth row and the kth column of M would be zero, a contradiction. Thus, $a_{k,k} > \tau(A)$. On writing $M := [m_{i,j}] \in \mathbb{R}^{n,n}$, then from (2.20), $m_{k,k} = a_{k,k} - \tau(A)$, and, as $a_{k,k} < 0$, this implies that

$$m_{k,k} < -\tau(A). \tag{2.23}$$

Now, assume that A is singular, i.e., there exists an $x \neq 0$ with $Ax = 0$. On setting $\alpha := \xi_n^T x$, we have

$$0 = Ax = Mx + \tau(A)\alpha\xi_n. \tag{2.24}$$

Note that α is not zero, since M is nonsingular. Then (2.24) implies that

$$x = -\tau(A)\alpha M^{-1}\xi_n \qquad \text{whence} \quad \alpha = \xi_n^T x = -\tau(A)\alpha\xi_n^T M^{-1}\xi_n.$$

As $\alpha \neq 0$, this gives

$$-\tau(A)\xi_n^T M^{-1}\xi_n = 1, \tag{2.25}$$

where M is a nonsingular generalized ultrametric matrix. Thus, we have from Theorem 2.1 that M^{-1} is a row and column diagonally dominant M-matrix, so that $M \geqslant O$ and $M^{-1}\xi_n \geqslant 0$. Applying (2.7) of Lemma 2.2 to the matrix

M gives that

$$\mu_i(M)(\xi_n^T M^{-1}\xi_n) \geqslant 1 \qquad \text{for all} \quad i \in N.$$

In particular, with $M = [m_{i,j}]$, we see from (iv) of Definition 2.1 that $m_{k,k} = \mu_k(M)$, so that the above inequality becomes $m_{k,k}(\xi_n^T M^{-1}\xi_n) \geqslant 1$, which, with (2.23), gives $-\tau(A)(\xi_n^T M^{-1}\xi_n) > 1$. As this contradicts (2.25), A is then nonsingular. ∎

EXAMPLE 2.4. The hypothesis in Theorem 2.5, namely, that A has a negative diagonal entry, cannot in general be weakened to A having a nonpositive diagonal entry. To show this, consider the matrix

$$A := \left[\begin{array}{cc|cc} 1 & -1 & -3 & -3 \\ 0 & 0 & -3 & -3 \\ \hline 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

so that

$$A + 3\xi_4\xi_4^T = \left[\begin{array}{cc|cc} 4 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ \hline 3 & 3 & 4 & 2 \\ 3 & 3 & 3 & 4 \end{array}\right].$$

As $A + 3\xi_4\xi_4^T$ can be verified to be a generalized ultrametric matrix, then A is a shifted generalized ultrametric matrix. Note that A is evidently singular, but A has no zero row or no zero column, and no two rows or two columns of A are the same.

As an application of Theorem 2.5, we have

COROLLARY 2.6. *Let* $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ *be a matrix of type* $D_{p,n-p}^{(-1)}$ *for some* p *with* $0 \leqslant p \leqslant n$. *Then* A *is nonsingular.*

Proof. As previously noted, any matrix in $D_{p,n-p}^{(-1)}$ is a shifted generalized ultrametric matrix. If A is of type $D_{p,n-p}^{(-1)}$ for some p with $0 \leqslant p < n$, then from Definition 2.3, A has a negative diagonal entry, and A has no zero row or zero column, and no two rows or two columns which are the same. Thus, A is nonsingular from Theorem 2.5. If A is of type $D_{p,n-p}^{(-1)}$ with $p = n$, then A is strictly diagonally dominant from Definition 2.3, and hence is nonsingular. ∎

Having characterized the nonsingular shifted generalized ultrametric matrices of type $U^{(-1)}_{p,n-p}$, we consider the determinant of these matrices.

THEOREM 2.7. *Let $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ be a nonsingular shifted generalized ultrametric matrix of type $U^{(-1)}_{p,n-p}$. If $0 \leqslant p < n$, then*

$$\det A < 0.$$

Moreover, each principal minor of order m with $p < m < n$ is nonpositive, and there exists a positive principal minor of order p. If $p = n$ (i.e., A is of type $U^{(-1)}_{n,0}$) or if A is a nonsingular generalized ultrametric matrix, then $\det A > 0$.

Proof. If A is a nonsingular generalized ultrametric matrix, then (cf. Theorem 2.1) A^{-1} is a row and column diagonally dominant M-matrix, so that (cf. [2, p. 134, A_1]) $\det A^{-1} > 0$. As $1 = \det(AA^{-1}) = \det A \det A^{-1}$, then $\det A > 0$. Similarly, if A is of type $U^{(-1)}_{n,0}$ (i.e., A is a nonsingular M-matrix), then $\det A > 0$. This establishes the last part of Theorem 2.7.

Next, assume that $A \in \mathbb{R}^{n,n}$ is a nonsingular shifted generalized ultrametric matrix of type $U^{(-1)}_{p,n-p}$ with $0 \leqslant p < n$. If $n = 1$ so that $p = 0$, it follows that $A = [\alpha] \in \mathbb{R}^{1,1}$ with $\alpha < 0$; whence, $\det A < 0$. For $1 \leqslant m < n$, assume the inductive hypothesis that any $B \in \mathbb{R}^{m,m}$ which is a nonsingular shifted generalized ultrametric matrix of type $U^{(-1)}_{q,m-q}$ for some q with $0 \leqslant q < m$ satisfies $\det B < 0$ (where, from the above discussion, if $B \in U^{(-1)}_{m,0}$, then $\det B > 0$).

For $n > 1$, A can be represented from Theorem 2.3, up to a permutation of indices, as

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}, \tag{2.26}$$

where $A_{1,1} \in \mathbb{R}^{r,r}$ and $A_{2,2} \in \mathbb{R}^{n-r,n-r}$, $1 \leqslant r < n$, are shifted generalized ultrametric matrices. Moreover, $A_{1,2} = \tau(A)\xi_r\xi^T_{n-r}$ and $A_{2,1} = \omega(A)\xi_{n-r}\xi^T_r$.

Since A is assumed to be nonsingular, then so is $A + \epsilon I$ for all $\epsilon > 0$ sufficiently small. Hence, this shift allows us to assume that both $A_{1,1}$ and $A_{2,2}$ in (2.26) are nonsingular. As in the proof of Theorem 2.5, we have that

$$\begin{aligned} \det A &= \det A_{1,1} \det(A/A_{1,1}) \\ &= \det A_{1,1} \det\left[A_{2,2} - \tau(A)\,\omega(A)\left(\xi^T_r A^{-1}_{1,1}\xi_r\right)\xi_{n-r}\xi^T_{n-r}\right]. \end{aligned}$$

We now consider various cases.

If $A_{1,1}$ is of type $U_{r,0}^{(-1)}$ (i.e., $A_{1,1}$ is a nonsingular M-matrix), then det $A_{1,1} > 0$. Now, assume that the Schur complement $A/A_{1,1} = A_{2,2} - \tau(A)\omega(A)(\xi_r^T A_{1,1}^{-1}\xi_r)\xi_{n-r}\xi_{n-r}^T$ is a nonsingular M-matrix. If $\tilde{A}$ is any leading principal submatrix of A in (2.26) whose order is greater than the order of $A_{1,1}$, the Schur complement $\tilde{A}/A_{1,1}$ is a principal submatrix of $A/A_{1,1}$. Hence, det$(\tilde{A}/A_{1,1}) > 0$, and as det $\tilde{A}$ = det $A_{1,1}$ det$(\tilde{A}/A_{1,1})$, then det $\tilde{A} > 0$. As all leading principal minors of A are then positive, A is a nonsingular M-matrix (cf. [2, p. 135, E_{17}]), which contradicts our assumption on A. Therefore, $A/A_{1,1}$ is not a nonsingular M-matrix, but again, as det A = det $A_{1,1}$ det$(A/A_{1,1})$ where det $A \neq 0$ and where det $A_{1,1} > 0$, then $A/A_{1,1}$ is nonetheless nonsingular. Hence,, by the inductive hypothesis, it follows that det$(A/A_{1,1}) < 0$, which implies det $A < 0$.

If $A_{1,1}$ is not of type $U_{r,0}^{(-1)}$, i.e., $A_{1,1}$ is not a nonsingular M-matrix, then by the induction hypothesis, det $A_{1,1} < 0$. Moreover, $A/A_{1,1}$ is a generalized ultrametric matrix from (ii) of Lemma 2.4, so that from the first part of the proof, det$(A/A_{1,1}) > 0$. Thus, det $A < 0$. [In all cases, we have actually shown that det$(A + \epsilon I) < 0$ for any $\epsilon > 0$ sufficiently small, so that the conclusion remains unchanged on letting $\epsilon \to 0$, since A is by hypothesis nonsingular.]

Since p is the order of the largest principal submatrix which is a nonsingular M-matrix, the remaining statements follow immediately. ■

For arbitrary Z-matrices A, det $A > 0$ is a necessary condition for A to be a nonsingular M-matrix. We next show that, with our additional structure for the off-diagonal entries, det $A > 0$ is also sufficient.

COROLLARY 2.8. *If a Z-matrix $A \in \mathbb{R}^{n,n}$ is a shifted generalized ultrametric matrix, then A is a nonsingular M-matrix if and only if* det $A > 0$.

Proof. If A is a nonsingular M-matrix, then it is well known that det $A > 0$. On the other hand, if A is a nonsingular shifted generalized ultrametric matrix, then A is a matrix of type $U_{p,n-p}^{(-1)}$ for exactly one integer p with $0 \leqslant p \leqslant n$. If det $A > 0$, we obtain with Theorem 2.7 that $p = n$. Thus, A is a nonsingular M-matrix.

THEOREM 2.9. *Let $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ be a nonsingular shifted generalized ultrametric matrix. If A is a generalized ultrametric matrix, then A^{-1} is in L_n. If A is of type $U_{n-m,m}^{(-1)}$, for an m satisfying $1 \leqslant m \leqslant n$, then A^{-1} is in L_{m-1}, i.e., A^{-1} is of the form (cf. (1.1))*

$$A^{-1} = tI - B,$$

with $B \geqslant O$ and $\rho_{m-1}(B) \leqslant t < \rho_m(B)$. Moreover, $\rho_{m-1}(B) = t$ if and only *if there exists a principal submatrix, of order $n - m + 1$, in A which is singular.*

Proof. For nonsingular generalized ultrametric matrices, this is proved in [15] and [17]. For the other cases, let us first assume that A is a nonsingular *strictly* shifted generalized ultrametric matrix of type $U_{n-m,m}^{(-1)}$ where m satisfies $1 \leqslant m \leqslant n$. Then from Theorem 2.3,

$$A = M + \tau(A)\xi_n \xi_n^T,$$

where M is a strictly generalized ultrametric matrix. Hence from Theorem 2.1, M is nonsingular. Using the Sherman-Morrison formula, A^{-1} can be expressed as

$$\begin{aligned} A^{-1} &= \left[M + \tau(A)\xi_n \xi_n^T\right]^{-1} \\ &= M^{-1} - \frac{\tau(A)}{1 + \tau(A)\xi_n^T M^{-1}\xi_n} M^{-1}\xi_n \xi_n^T M^{-1}, \end{aligned} \tag{2.27}$$

where $\tau(A)\xi_n^T M^{-1}\xi_n \neq 1$. From Theorem 2.1, M^{-1} is an M-matrix which satisfies $M^{-1}\xi_n > 0$. Since $m \neq 0$, i.e., A is not a nonsingular M-matrix, we have from the proof of (i) of Lemma 2.4 that $\tau(A)\xi_n^T M^{-1}\xi_n < -1$. With this, it follows from (2.27) that A^{-1} is also a Z-matrix, since all the entries in the final matrix in (2.27) are negative. If A is a nonsingular shifted generalized ultrametric matrix of $U_{n-m,m}^{(-1)}$, then A is a limit of strictly shifted generalized ultrametric matrices. Therefore, the inverse of A is also a Z-matrix.

It is well known (cf. [7, p. 21]) for conformally partitioned nonsingular matrices, with square diagonal submatrices, that with

$$T = \begin{bmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{bmatrix} \quad \text{and} \quad T^{-1} = S = \begin{bmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{bmatrix},$$

we have

$$\det T_{2,2} = (\det S_{1,1})(\det T^{-1})^{-1}. \tag{2.28}$$

Then, choose $T = A$ and $S = A^{-1}$ in (2.28), where A is assumed to be a nonsingular shifted generalized ultrametric matrix of type $U_{n-m,m}^{(-1)}$, where m satisfies $1 \leqslant m \leqslant n$. From Theorem 2.7, $\det A < 0$ [so that $(\det T^{-1})^{-1} < 0$

in (2.28)], and each principal minor of A, of order s with $n - m < s < n$, is nonpositive, while there exists a principal minor of A, of order $n - m$, which is positive. But, as (2.28) gives a coupling of a principal minor of A with its associated *complementary* minor of A^{-1}, it follows that each principal minor of A^{-1}, of order j with $1 \leqslant j \leqslant m - 1$, is nonnegative and there exists a principal minor of A^{-1}, of order m, which is negative. Recalling that A^{-1} is a Z-matrix, we can write that $A^{-1} = tI - B$ where $B \geqslant O$, and, with the notation of (1.1), the above discussion gives that

$$\rho_{m-1}(B) \leqslant t < \rho_m(B).$$

Hence, $A^{-1} \in L_{m-1}$ for any $1 < m \leqslant n$. If $m = 1$, we again have $\rho_1(B) > t$, and as $\rho_0(B) := -\infty$, then $\rho_1(B) > t > \rho_0(B)$, whence $A^{-1} \in L_0$.

Finally, from (2.28) we see that $\rho_{m-1}(B) = t$, for some m with $1 < m \leqslant n$, if and only if there exists a principal submatrix in A, of order $n - m + 1$, which is singular. ■

With Theorem 2.9, we have one of our main results.

THEOREM 2.10. *If a nonsingular Z-matrix $A \in \mathbb{R}^{n,n}$ is a shifted generalized ultrametric matrix, then $A^{-1} \in L_{m-1}$ if and only if A is of type $U^{(-1)}_{n-m,m}$ for an m with $1 \leqslant m \leqslant n$.*

Proof. If $A \in U^{(-1)}_{n-m,m}$ for an m with $1 \leqslant m \leqslant n$, then $A^{-1} \in L_{m-1}$ follows from Theorem 2.9. If $A^{-1} \in L_{m-1}$ for an m with $1 < m \leqslant n$, then $A^{-1} = tI - B$, with $B \geqslant O$, satisfies $\rho_{m-1}(B) \leqslant t < \rho_m(B)$. Hence, there exists a principal minor of A^{-1}, of order m, which is negative, and that each principal minor of A^{-1}, of order $m - 1$, is nonnegative. From the proof of Theorem 2.9, this implies that A has a principal minor, of order $n - m$, which is positive, and that each principal minor of A, of order s with $n - m < s < n$, is nonpositive. From Definition 2.2, it follows that $A \in U^{(-1)}_{n-m,m}$. The proof for the case $m - 1$ is similar. ■

Since the classes of type $D^{(-1)}_{p,n-p}$-matrices are subclasses of the classes of type $U^{(-1)}_{p,n-p}$-matrices, we immediately obtain

COROLLARY 2.11. *If $A \in \mathbb{R}^{n,n}$ is of type $D^{(1)}_{1,n-1}$, then A^{-1} is in L_n. If $A \in \mathbb{R}^{n,n}$ is of type $D^{(-1)}_{n-m,m}$, with $1 \leqslant m \leqslant n$, then A^{-1} is in L_{m-1}, i.e., A^{-1} is of the form*

$$A^{-1} = tI - B,$$

with $B \geqslant O$ and $\rho_{m-1}(B) < t < \rho_m(B)$.

Proof. The proof follows directly from Theorem 2.9 and Corollary 2.6, which states that matrices of type $D_{p,n-p}^{(-1)}$ are nonsingular. If $m = 1$, one diagonal entry of A^{-1} is negative; hence, $A^{-1} \in L_0$. ■

EXAMPLE 2.5. Consider the matrix

$$A := \begin{bmatrix} 5 & -4 & -4 & -4 \\ -4 & 5 & -3 & -3 \\ -4 & -3 & -2 & -2 \\ -4 & -3 & -2 & -1 \end{bmatrix},$$

which is in $D_{p,n-p}^{(-1)}$ for $n = 4$, $p = 2$, $d_j := -5 + j$ for $1 \leqslant j \leqslant 4$, $P := \{1, 2\}$, and $M := \{3, 4\}$. The inverse of A is given by

$$A^{-1} = \frac{1}{239}\begin{bmatrix} 19 & -4 & -32 & 0 \\ -4 & 26 & -31 & 0 \\ -32 & -31 & 230 & -239 \\ 0 & 0 & -239 & 239 \end{bmatrix}$$

$$= I - \frac{1}{239}\begin{bmatrix} 220 & 4 & 32 & 0 \\ 4 & 213 & 31 & 0 \\ 32 & 31 & 9 & 239 \\ 0 & 0 & 239 & 0 \end{bmatrix} =: I - B.$$

Using the definitions of (1.1), we have that $\rho_1(B) = 0.9205\ldots$, and $\rho_2(B) = 1.0190\ldots$, so that, since $\rho_1(B) < 1 < \rho_2(B)$, $A^{-1} \in L_1$, which is in agreement with Corollary 2.11.

3. NONARCHIMEDEAN MATRICES

The results of the previous section can be applied to a special class of distance matrices, the so-called *nonarchimedean* matrices, which we consider in this section. Nonarchimedean matrices arise in p-adic number theory [22] and in taxonomy [1]. We will show that these matrices are negative symmetric shifted ultrametric matrices of type $U_{0,n}^{(-1)}$. Moreover, we will see that nonarchimedean matrices are closely related to symmetric ultrametric matrices.

Recall that a valuation $|\cdot|_v \colon K \to \mathbb{R}_0^+$, where K denotes a field and $\mathbb{R}_0^+$ denotes the nonnegative real numbers, is called *nonarchimed-*

ean if

$$|a+b|_v \leqslant \max\{|a|_v; |b|_v\} \qquad \text{for all} \quad a, b \in K.$$

Using a nonarchimedean valuation, we obtain a metric d on the field K which satisfies

$$d(a,b) \leqslant \max\{d(a,c), d(c,b)\} \tag{3.1}$$

for all $a, b, c \in K$. A metric, which satisfies the strong triangular inequality (3.1), is called a *nonarchimedean metric* or an *ultrametric metric.*

EXAMPLE 3.1. Let p be a fixed prime number. If $\mathbb{Q}$ denotes the set of all nonzero real rational numbers, then any a in $\mathbb{Q}$ can be uniquely written as

$$a = \frac{r}{s}p^n \qquad \text{with} \quad s > 0,$$

where r and s are integers which do not have a common divisor, p does not divide rs, and n is an integer (positive, negative, or zero). Then, we obtain for any p a nonarchimedean valuation (called the p-adic valuation), defined by

$$|a|_p := \begin{cases} 0 & \text{for} \quad a = 0 \\ p^{-n} & \text{for} \quad a \neq 0. \end{cases}$$

As Ostrowski [18] proved, the valuations $|\cdot|_p$ and the absolute value are the only *nonequivalent* valuations of $\mathbb{Q}$. Here, for two valuations $|\cdot|$ and $\|\cdot\|$ to be *equivalent* means that $|q| < |p|$ if and only if $\|q\| < \|p\|$ for all $p, q \in \mathbb{Q}$.

If the set K is a finite set, we obtain a distance matrix, which we call *nonarchimedean.*

DEFINITION 3.1. A matrix $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ is called *nonarchimedean* if

$$A \text{ is symmetric} \quad \text{and} \quad A \geqslant O, \tag{3.2a}$$

$$a_{i,i} = 0 \qquad \text{for all} \quad i \in N, \tag{3.2b}$$

$$a_{i,j} \leqslant \max\{a_{i,k}; a_{k,j}\} \qquad \text{for all} \quad i, j, k \in N. \tag{3.2c}$$

It is evident that (3.2c) implies

$$-a_{i,j} \geqslant \min\{-a_{i,k}; -a_{k,j}\} \qquad \text{for all} \quad i, j, k \in N.$$

Hence, on comparing Definition 3.1, for a nonarchimedean matrix, and Definition 2.2, for a matrix in $U_{0,n}^{(-1)}$, and noting that the diagonal entries of a matrix in $U_{0,n}^{(-1)}$ are all nonpositive, we directly obtain

PROPOSITION 3.1. *Let A be a nonarchimedean matrix in $\mathbb{R}^{n,n}$. If A is not the null matrix, then $-A$ is a symmetric shifted generalized ultrametric matrix of type $U_{0,n}^{(-1)}$ with zero diagonal entries. Conversely, if B is a symmetric shifted generalized ultrametric matrix of type $U_{0,n}^{(-1)}$ with zero diagonal entries, then $-B$ is a nonarchimedean matrix.*

Moreover, the nonarchimedean matrices and the symmetric generalized ultrametric matrices are closely related in another easily verified way.

PROPOSITION 3.2. *Let A be a nonarchimedean matrix in $\mathbb{R}^{n,n}$. Then, for all $c \in \mathbb{R}$ with $c \geqslant \mu(A)$, where $\mu(A)$ denotes the maximal entry of A, the matrix $B \in \mathbb{R}^{n,n}$ with*

$$B := c\xi_n \xi_n^T - A$$

is a symmetric generalized ultrametric matrix. Conversely, if $B = [b_{i,j}] \in \mathbb{R}^{n,n}$ is a symmetric generalized ultrametric matrix, then

$$A := c\xi_n \xi_n^T - B - \operatorname{diag}(c - b_{1,1}; \ldots; c - b_{n,n}), \tag{3.3}$$

where $c \geqslant \mu(B)$, is a nonarchimedean matrix. Here, $\operatorname{diag}(c - b_{1,1}; \cdots; c - b_{n,n})$ *denotes a diagonal matrix in $\mathbb{R}^{n,n}$ having $c - b_{i,i}$ as diagonal entries.*

In the following, we use the previous propositions to establish properties of nonarchimedean matrices. We begin with a representation of nonarchimedean matrices.

PROPOSITION 3.3. *Let $A \in \mathbb{R}^{n,n}$, $n > 1$, be a nonarchimedean matrix. Then, up to a suitable permutation, A is given by*

$$A\begin{bmatrix} C & O \\ O & D \end{bmatrix} + \mu(A)(uv^T + vu^T), \tag{3.4}$$

where $C \in \mathbb{R}^{r,r}$ *and* $D \in \mathbb{R}^{n-r,n-r}$, $1 \leqslant r < n$, *are nonarchimedean matrices, and*

$$u := \left[\xi_r^T, 0, \ldots, 0\right]^T \in \mathbb{R}^n \quad \textit{and} \quad v := \left[0, \ldots, 0, \xi_{n-r}^T\right]^T \in \mathbb{R}^n.$$

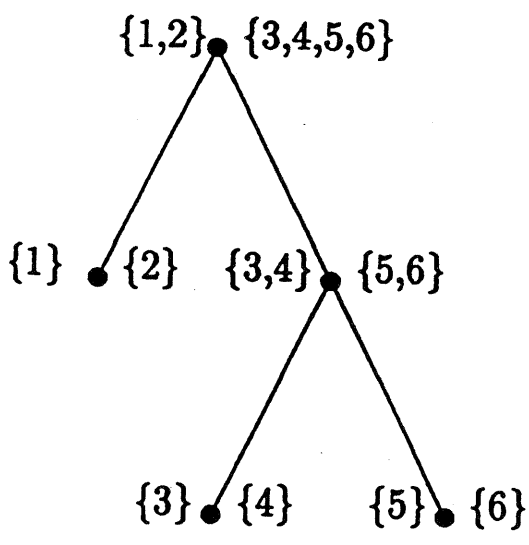
Conversely, if $C \in \mathbb{R}^{r,r}$ *and* $D \in \mathbb{R}^{n-r,n-r}$, $1 \leqslant r < n$, *are nonarchimedean matrices, and if* $\tau \in \mathbb{R}$ *satisfies* $\tau \geqslant \mu(C)$ *and* $\tau \geqslant \mu(D)$, *then*

$$A = \begin{bmatrix} C & O \\ O & D \end{bmatrix} + \tau(uv^T + vu^T) \tag{3.5}$$

is a nonarchimedean matrix.

Proof. The proof follows immediately from Theorem 2.3, since, from Proposition 3.2, there exists a symmetric generalized ultrametric matrix B such that $A = \mu(A)\xi_n \xi_n^T - B$. ■

The reduction in Proposition 3.3 can be applied again to the matrices C and D. Thus, Proposition 3.3 describes the nested block structure of a nonarchimedean matrix. As introduced in [16] for strictly ultrametric matrices and in [17] for generalized ultrametric matrices, an associated binary rooted tree seems to be the most convenient way to illustrate this procedure. This tree determines, at each vertex, the two disjoint sets of indices which



$u_1 := (1,1,0,0,0,0)^T$; $v_1 := (0,0,1,1,1,1)^T$

$u_2 := (1,0,0,0,0,0)^T$; $v_2 := (0,1,0,0,0,0)^T$
$u_3 := (0,0,1,1,0,0)^T$; $v_3 := (0,0,0,0,1,1)^T$

$u_4 := (0,0,1,0,0,0)^T$; $v_4 := (0,0,0,1,0,0)^T$
$u_5 := (0,0,0,0,1,0)^T$; $v_5 := (0,0,0,0,0,1)^T$,

FIG. 2.

correspond to the matrices C and D, and to the vectors u and v, as well as the scalar μ. In contrast to generalized ultrametric matrices, the rooted tree consists just of $n-1$ vertices, since the diagonal entries of a nonarchimedean matrix are zero.

EXAMPLE 3.2. The binary rooted tree shown in Figure 2, together with $(\mu_1, \dots, \mu_5) = (5, 1, 3, 2, 1)$, yields the nonarchimedean matrix

$$A = \sum_{i=1}^{5} \mu_i (u_i v_i^T + v_i u_i^T)$$

with

$$A = \begin{bmatrix} 0 & 1 & 5 & 5 & 5 & 5 \\ 1 & 0 & 5 & 5 & 5 & 5 \\ 5 & 5 & 0 & 2 & 3 & 3 \\ 5 & 5 & 2 & 0 & 3 & 3 \\ 5 & 5 & 3 & 3 & 0 & 1 \\ 5 & 5 & 3 & 3 & 1 & 0 \end{bmatrix}.$$

Hence, we obtain

PROPOSITION 3.4. *Let $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ be a nonarchimedean matrix. Then there exists a associated binary rooted tree for $N = \{1, 2, \dots, n\}$, consisting of $n-1$ vertices, such that*

$$A = \sum_{i=1}^{n-1} \mu_i (u_i v_i^T + v_i u_i^T), \tag{3.6}$$

where the vectors u_i and v_i in (3.6) are nonzero vectors, having only 0 and 1 entries, determined from the vertices of the tree, and the μ_i's are nonnegative. Moreover, these μ_i's, which correspond to a path from the root of the tree to a leaf, build a nonincreasing sequence. Conversely, given any binary rooted tree for $N = \{1, \dots, n\}$ which determines the vectors u_i and $v_i \in \mathbb{R}^n$, *and given any nonnegative constants* $\{\mu_i\}_{i=1}^{n-1}$ *such that these* μ_i*'s, which corresponds to a path from the root to a leaf, do not increase, then*

$$\sum_{i=1}^{n-1} \mu_i\left(u_i v_i^T + v_i u_i^T\right)$$

is a nonarchimedean matrix.

Since we are interested in nonsingular nonarchimedean matrices, we formulate the following result.

THEOREM 3.5. *Let* $A = [a_{i,j}] \in \mathbb{R}^{n,n}$, $n > 1$, *be a nonarchimedean matrix. Then A is singular if and only if* $a_{i,j} = 0$ *for at least one pair* (i, j) *with* $i \neq j$, $i, j \in N$.

Proof. The diagonal entries of a nonarchimedian matrix are, by definition, zero. Thus, it follows from the nested block structure of a nonarchimedean matrix that two rows or two columns in the matrix are the same if there exists a zero off-diagonal entry. If no zero off-diagonal entry exists, then no two rows or two columns are the same. Moreover, as we then have that $a_{i,i} < \max\{a_{i,k}\colon k \in N \setminus \{i\}\}$, we can apply, to $-A$, the same proof as in the proof of Theorem 2.5, and the desired result follows. ■

Using Theorem 3.5, we note that we would obtain a nonsingular matrix if we make use of the property of a metric that

$$d(a, b) = 0 \quad \text{if and only if} \quad a = b$$

in the definition of a nonarchimedean matrix.

The relation between nonarchimedean matrices and symmetric shifted generalized ultrametric matrices of type $U_{0,n}^{(-1)}$, as described in Proposition 3.1, gives rise to the following theorem which summarizes properties of nonsingular nonarchimedean matrices.

THEOREM 3.6. *Let* $A \in \mathbb{R}^{n,n}$, $n > 1$, *be a nonsingular nonarchimedean matrix, given in the block form of* (3.4). *Then* $A_{1,1}$ *and* $A_{2,2}$ *are nonsingular, and the Schur complements* $S_1 := A/A_{1,1}$ *and* $S_2 := A/A_{2,2}$ *are negative, nonsingular symmetric generalized ultrametric matrices, i.e.,* $-S_1$ *and* $-S_2$ *are nonsingular symmetric generalized ultrametric matrices. The sign of the determinant of A is given by*

$$\begin{aligned} &\det A < 0 \quad \text{if } n \text{ is even},\\ &\det A > 0 \quad \text{if } n \text{ is odd}. \end{aligned} \tag{3.7}$$

The inverse of A, $A^{-1} := [\alpha_{i,j}]$, *is a* $-N_0$*-matrix, i.e.,* $-A^{-1}$ *is a* N_0*-matrix, or equivalently* $-A^{-1} \in L_{n-1}$, *and*

$$\begin{aligned} \alpha_{i,i} &= 0 \quad \textit{for all } i \in N \qquad \textit{if} \quad n = 2, \\ \alpha_{i,i} &< 0 \quad \textit{for all } i \in N \qquad \textit{if} \quad n > 2. \end{aligned} \tag{3.8}$$

In addition, as the diagonal entries of A *are all zero, all* n *principal submatrices of* A^{-1}, *of order* $n-1$, *are singular. Moreover,* A^{-1} *satisfies*

$$A^{-1}\xi_n > 0 \quad \textit{and} \quad \omega(-A)\,\xi_n^T A^{-1}\xi_n < -1. \tag{3.9}$$

Proof. All statements of this result follow immediately with Proposition 3.1 and the corresponding results of the previous section. Equations (3.8) and (3.9) follow from the fact that the off-diagonal entries of A and A^{-1} are nonnegative. ∎

EXAMPLE 3.3. To illustrate the results of Theorem 3.6, consider the matrix $A \in \mathbb{R}^{6,6}$, of Example 3.2. From Theorem 3.5, it is seen by inspection that A is nonsingular. Next, computation shows [cf. (3.7)] that $\det A = -1732$, and its inverse $A^{-1} := [\alpha_{i,j}] \in \mathbb{R}^{6,6}$ is given [cf. (3.8)] by

$$A^{-1} = \frac{1}{866}\begin{bmatrix} -450 & 416 & 25 & 25 & 20 & 20 \\ 416 & -450 & 25 & 25 & 20 & 20 \\ 25 & 25 & -266 & 167 & 47 & 47 \\ 25 & 25 & 167 & -266 & 47 & 47 \\ 20 & 20 & 47 & 47 & -482 & 384 \\ 20 & 20 & 47 & 47 & 384 & -482 \end{bmatrix}.$$

Thus, $-A^{-1}$ is a Z-matrix, which can be expressed as $-A^{-1} = \frac{1}{866}(482I - B)$, where

$$B := \begin{bmatrix} 32 & 416 & 25 & 25 & 20 & 20 \\ 416 & 32 & 25 & 25 & 20 & 20 \\ 25 & 25 & 216 & 167 & 47 & 47 \\ 25 & 25 & 167 & 216 & 47 & 47 \\ 20 & 20 & 47 & 47 & 0 & 384 \\ 20 & 20 & 47 & 47 & 384 & 0 \end{bmatrix}.$$

It can be verified that all six principal submatrices, of order 5, of $-A^{-1}$ are singular, so that $\rho_5(B) = 482$, and that $\rho_6(B) = 528.1440\ldots$. Consequently, from (1.1) we have that $-A^{-1} \in L_5$. In addition,

$$A^{-1}\xi_6 = [56, 56, 45, 45, 36, 36]^T/866 > 0,$$

and

$$\omega(-A)\,\xi_6^T A^{-1}\xi_6 = -1.5819\ldots,$$

in accordance with (3.9).

We close this section with the following observations concerning nonarchimedean metrices again. If we have a nonarchimedean metric d on a finite set $K = \{1, \ldots, n\}$, then we obtain a distance matrix $A \in \mathbb{R}^{n,n}$, with $a_{i,j} := d(i,j)$. This matrix is nonsingular and there exists a strictly ultrametric matrix B such that

$$A = \mu(A)\,\xi_n \xi_n^T - B.$$

With Theorem 3.4 of [17], there exists a weighted rooted tree such that the entries of B are given, as indicated in (2.3) and (2.4). Since the matrices are symmetric, we need just one weighting function l. If $p(i)$ denotes the path with distinct edges from leaf i to the root of the tree, we have, for $i, j = 1, \ldots, n$,

$$d(i,j) = \mu(A) - \sum_{p(i)\cap p(j)} l(v_i, v_{i+1}) \qquad \text{for} \quad i \neq j,$$

$$d(i,i) = 0.$$

Thus, we have established a proof, which is based on matrix theory, of the following known result (cf. [1]):

PROPOSITION 3.7. *Each nonarchimedean metric on a finite set is representable by the metric on a rooted tree, as indicated above.*

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