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Zero Distribution, the Szegő Curve, and Weighted Polynomial Approximation in the Complex Plane

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1 Introduction

Given $f(z) := \sum_{k=0}^{\infty} a_k z^k$ with $\overline{\lim_{k\to\infty}} |a_k|^{1/k} = 1$, so that f is an analytic function in $|z| < 1$, it is natural to ask where the zeros of the partial sums of f, namely,

$$s_n(z; f) := \sum_{k=0}^{n} a_k z^k \quad (n \in \mathbb{N}),$$

are located. (Here, $\mathbb{N}$ and $\mathbb{N}_0$ denote, respectively, the set of positive integers and the set of nonnegative integers.) It was shown in 1917 by Jentzsch [6] that *each* z of $|z| = 1$ is an accumulation point of the zeros of $\{s_n(z; f)\}_{n\in\mathbb{N}}$. (A sharper form of this can be found in Szegő [15].) But for an entire function, the behavior of the zeros of its partial sums is quite different. In 1924, Szegő [16] made a substantial first step in this area by considering the special entire function e^z and its familiar partial sums,

$$s_n(z) := \sum_{k=0}^{n} z^k/k! \quad (n \in \mathbb{N}_0).$$

Specifically, an application of the Eneström–Kakeya Theorem (cf. Marden [9, p. 137, Ex. 2]) to the above $s_n(z)$ shows that, for each $n \in \mathbb{N}$, all zeros of $s_n(z)$ lie in the disk $|z| \leq n$. Calling $s_n(nz)$ the *normalized* partial sum, this implies that

$$\text{all zeros of } \{s_n(nz)\}_{n\in\mathbb{N}} \text{ lie in } |z| \leq 1. \tag{1.1}$$

This is clearly indicated in Fig. 1. Consequently, the infinite set of all zeros of $\{s_n(nz)\}_{n\in\mathbb{N}}$ must have at least one accumulation point in $|z| \leq 1$. On defining the simple closed curve

$$S := \{z \in \mathbb{C} : |ze^{1-z}| = 1 \text{ and } |z| \leq 1\}, \tag{1.2}$$

which is called the *Szegő curve*, we next state the remarkable result of Szegő [16], published in 1924.

Theorem 1.1 [16]. *A complex number ζ is an accumulation point of the zeros of the normalized partial sums $\{s_n(nz)\}_{n\in\mathbb{N}}$ if and only if $\zeta \in S$.*

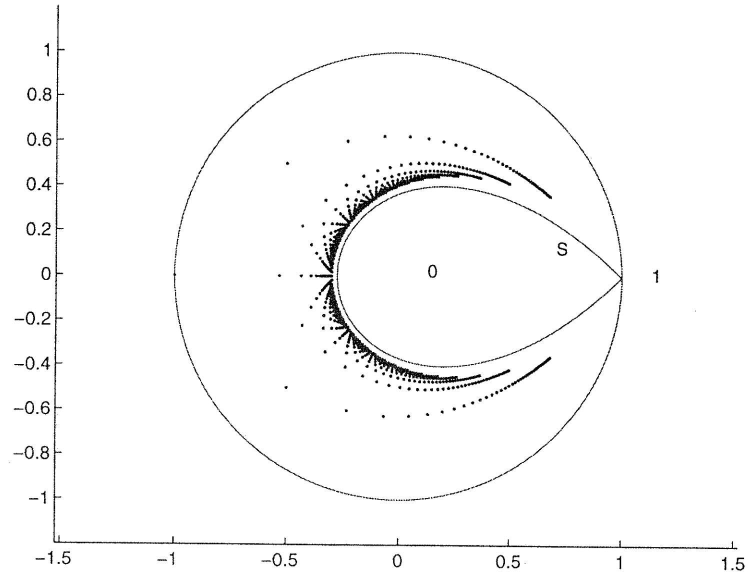

FIG. 1. Zeros of $\{S_n(nz)\}_{n=1}^{40}$

We remark that the Szegő curve S is also shown in Fig. 1. In addition, we note from Fig. 1 that the convergence, of the zeros of $\{s_n(nz)\}_{n\in\mathbb{N}}$ to S, is noticeably *slower* in the neighborhood of the point $z = 1$ of S, and this will have subsequent interesting ramifications for us! We also mention that the curve

$$\{z \in \mathbb{C} : |ze^{1-z}| = 1\}, \tag{1.3}$$

of which the Szegő curve S of (1.2) is a part, is shown in Fig. 2, and this divides the complex plane into three disjoint domains, i.e., G, Ω_0, and Ω_∞, where G denotes the interior of the Szegő curve S. (The significance of these three domains, namely, G, Ω_0, and Ω_∞ will be clarified later.)

2 Weighted Polynomial Approximation by $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$

The introduction in Section 1 to the zeros of the normalized partial sums $s_n(nz)$ of e^z may seem remote from our goal of considering weighted polynomial approximation of analytic function in the complex plane, but, as was shown in [16], there holds

$$e^{-nz}s_n(nz) = 1 - \frac{\sqrt{n}}{\tau_n\sqrt{2\pi}} \int_0^z (\zeta e^{1-\zeta})^n d\zeta \quad (n \in \mathbb{N}, z \in \mathbb{C}), \tag{2.1}$$

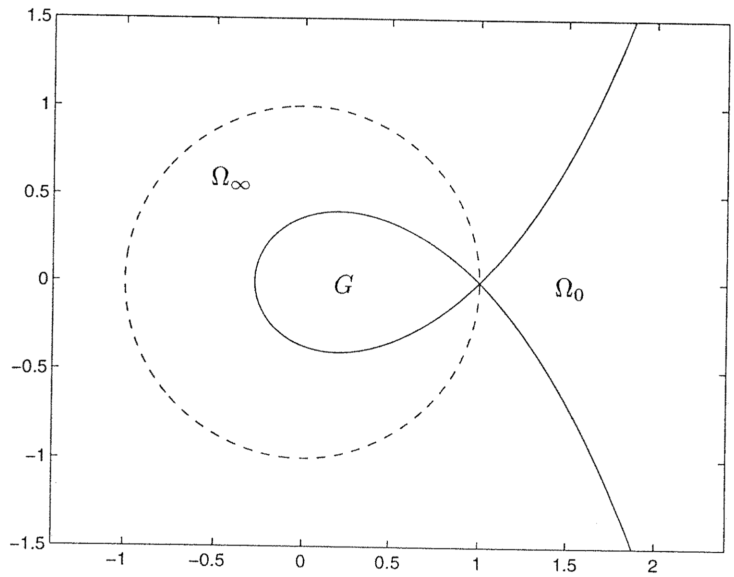

FIG. 2. The Szegő curve and the associated domains

where, from Stirling's asymptotic series formula (cf. Henrici [5, p. 377]),

$$\begin{cases} \tau_n := \dfrac{n!}{n^n e^{-n}\sqrt{2\pi n}} \approx 1 + \dfrac{1}{12n} + \dfrac{1}{288n^2} - \dfrac{139}{51840n^3} + \cdots, \text{ as } n \to \infty, \\ \text{so that } \lim_{n\to\infty} \tau_n = 1. \end{cases} \tag{2.2}$$

We note that the function

$$\varphi(z) := ze^{1-z} \tag{2.3}$$

already appears in the integrand of (2.1), and it also plays a significant role in the asymptotic behavior of the zeros of $\{s_n(nz)\}_{n\in\mathbb{N}}$. It is the case that $\varphi(z)$ of (2.3) is univalent in $|z| < 1$, and $\varphi(z) = w$ conformally maps the open set G onto the open unit disk $\{w \in \mathbb{C} : |w| < 1\}$ in the w-plane, with the Szegő curve S being mapped 1–1 onto the unit circle $T := \{w \in \mathbb{C} : |w| = 1\}$. But note also that $\varphi'(1) = 0$, so that conformality of this mapping is lost at the point $z = 1$.

Based on (2.1), the following can be established.

Proposition 2.1 [11] *For the weighted normalized partial sums $e^{-nz}s_n(nz)$, there holds*

$$|e^{nz}s_n(nz) - 1| \leq \frac{4}{\sqrt{2\pi n}|z-1|} \quad (z \in \overline{G}\backslash\{1\}; n \in \mathbb{N}). \tag{2.4}$$

We remark that the inequality of (2.4) of Proposition 2.1, which is in a form useful for our subsequent developments, is closely related to similar results of Szegő [16, eq. (5′)], eq. and [2, eq. (2.13)], but is not implied by these results.

For the behavior of the weighted normalized partial sums $\{e^{-nz}s_n(nz)\}_{n\in\mathbb{N}}$ outside of $\overline{G}$, we also have the following result:

Theorem 2.2 [11] *For the normalized partial sums of* e^z, *we have*

$$\lim_{n\to\infty} |e^{-nz}s_n(nz)|^{1/n} = |\varphi(z)| \quad (z \in \mathbb{C}\backslash\overline{G}). \tag{2.5}$$

The result (2.5) shows that $e^{-nz}s_n(nz)$ diverges unboundedly in Ω_∞ (where $|\varphi(z)| > 1$), and converges to the identically zero function in Ω_0 (where $|\varphi(z)| < 1$), as the subscripts of these regions indicate. What (2.4) asserts is that the special analytic function, $f(z) \equiv 1$, can be uniformly approximated, on compact subsets of $\overline{G}\backslash\{1\}$, by the weighted normalized partial sums $\{e^{-nz}s_n(nz)\}_{n\in\mathbb{N}}$. This then connects with the theme given in the title of this paper! Moreover, Proposition 2.1 is the precursor of the following more general approximation result for the weighted polynomials $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$. In what follows, all norms used are the uniform (Chebyshev) norms on the indicated sets.

Theorem 2.3 [11] *Let f be analytic in G and continuous on compact subsets of $\overline{G}\backslash\{1\}$. Then, given any compact set $E \subset \overline{G}\backslash\{1\}$, there exists a sequence of complex polynomials $\{P_n(z)\}_{n\in\mathbb{N}_0}$, with* deg $P_n \le n$ *for all $n \in \mathbb{N}_0$, such that*

$$\lim_{n\to\infty} \|e^{-nz}P_n(z) - f(z)\|_E = 0. \tag{2.6}$$

Furthermore, if f is analytic in G, and continuous in $\overline{G}$ with $f(1) = 0$, then there is a sequence of polynomials $\{\tilde{P}(z)\}_{n\in\mathbb{N}_0}$, such that

$$\lim_{n\to\infty} \|e^{-nz}\tilde{P}_n(z) - f(z)\|_{\overline{G}} = 0. \tag{2.7}$$

The difference between (2.6) and (2.7) is in the behavior of f at the sole point $z = 1$ of $\overline{G}$, the point which is exceptional in Proposition 2.1, and for which we know that $\varphi'(1) = 0$. We believe that the second part of Theorem 2.3 *cannot* be further strengthened, in the sense that we make the following:

Conjecture [11] There exists an f, analytic in G and continuous in $\overline{G}$ with $f(1) \neq 0$, such that for *no* sequence of polynomials $\{P_n(z)\}_{n\in\mathbb{N}_0}$, with deg $P_n \le n$ for each $n \in \mathbb{N}_0$, it is true that

$$\lim_{n\to\infty} \|e^{-nz}P_n(z) - f(z)\|_{\overline{G}} = 0. \tag{2.8}$$

Let us examine Theorem 2.3 more closely. Obviously, any polynomial $Q_n(z)$ can itself be locally uniformly approximated (i.e., on compact subsets of $\overline{G}\backslash\{1\}$) by weighted polynomials $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$. Then, by Mergelyan's Theorem (cf.

Walsh [19, p. 367]), any function which is analytic interior to a given compact set $E \subset \overline{G}\backslash\{1\}$ and is continuous on E, is itself uniformly approximable by polynomials, if E has a connected complement. This gives us the result:

Corollary 2.4 [11] *If E is a compact set, such that $E \subset \overline{G}\backslash\{1\}$ and such that its complement $\overline{\mathbb{C}}\backslash E$ is connected, then there exists a sequence of polynomials $\{P_n(z)\}_{n\in\mathbb{N}_0}$, with* deg $P_n \leq n$, *such that*

$$\lim_{n\to\infty} \|e^{-nz}P_n(z) - f(z)\|_E = 0. \tag{2.9}$$

Many questions naturally arise from Theorem 2.3. For example, one can ask if it is possible to give *rates of convergence* in (2.6), of these weighted polynomial approximations to a given f. This is done below, and is reminiscent of the classical Bernstein–Walsh overconvergence results (cf. [19, pp. 75-78]). To state this result, we use the notation

$$S_r := \{z \in \mathbb{C} : |\varphi(z) = ze^{1-z}| = r, |z| \leq 1 \text{ and } 0 < r \leq 1\}, \tag{2.10}$$

so that S_r is a *level curve* of the mapping φ. We also set

$$G_r := \text{int } S_r \quad (0 < r \leq 1), \tag{2.11}$$

so that $G = G_1$.

Theorem 2.5 [11] *Let (r, R) be a pair of numbers satisfying $0 < r < R \leq 1$. Then, a function f is analytic in G_R if and only if there exists a sequence of polynomials $\{P_n(z)\}_{n\in\mathbb{N}_0}$, with* deg $P_n \leq n$, *such that*

$$\overline{\lim_{n\to\infty}} \|e^{-nz}P_n(z) - f(z)\|_{S_r}^{1/n} \leq \frac{r}{R} < 1. \tag{2.12}$$

We remark that the last inequality in (2.12) implies the *geometric convergence* of the sequence $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}}$ to $f(z)$ in S_r, for $0 < r < R \leq 1$.

It also turns out that the case of equality in (2.12) can be studied more carefully. For notation, if f is analytic in G_R, where $0 < r < R < 1$, set

$$E_n^{\exp(-z)}(f, \overline{G}_r) := \inf_{P_n\in\pi_n} \|e^{-nz}P_n(z) - f(z)\|_{S_r}, \tag{2.13}$$

where π_n denotes the collection of all complex polynomials of degree at most n. Thus, $E_n^{\exp(-z)}(f, \overline{G}_r)$ is the best uniform weighted polynomial approximation, from π_n, of f in $\overline{G}_r$, with the weight function $W(z) := e^{-z}$, for each $n \in \mathbb{N}_0$. We then have:

Corollary 2.6 [11] *The function $f(z)$, which is analytic in G_R, has a singularity on S_R if and only if*

$$\overline{\lim_{n\to\infty}} \{E_n^{\exp(-z)}(f, \overline{G}_r)\}^{1/n} = \frac{r}{R}. \tag{2.14}$$

We next consider the possibility of uniform approximability of functions by $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$, in sets *other* than those such as compact subsets of $\overline{G}\backslash\{1\}$, or $\overline{G}$, already treated in Theorem 2.3. This includes the following *shift-invariant property* of weighted polynomial approximation, when the weight function is $W(z) = e^{-z}$.

Theorem 2.7 [11] *Let E be a compact set in $\mathbb{C}$ which has the property that any function, analytic in E and continuous on E, can be uniformly approximated on E by the weighted polynomials $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$. Then, the same is true for the translated set $\zeta + E := \{\zeta + z : z \in E\}$, for* **any** $\zeta \in \mathbb{C}$.

Using the above idea, we give the following negative result which shows, up to translations (as considered in the statement of Theorem 2.7), that the open set G, the interior of the Szegő curve S, is the largest *universal* domain of this shape in the complex plane for which locally uniform weighted polynomial approximation, of the form $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$, to arbitrary analytic functions is possible.

Theorem 2.8 [11]. *If a domain H, or any of its translations, contains $\overline{G}$, then no analytic function in H (except the identically zero function) can be approximated, locally uniformly in H, by weighted polynomials $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$.*

We next show that the weighted polynomials $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$ *can* give rise to locally uniform approximation of analytic functions in domains having shapes *different* from G or its translations. As an example, consider an open disk in $\mathbb{C}$ having a radius r, where $r > 0$. Our next result incorporates the shift invariant property of the weighted polynomials $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$.

Theorem 2.9 [11]. *Any function, analytic in an open disk of radius $1/2$, can be approximated by weighted polynomials $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$, uniformly on compact subsets of this disk. Furthermore, if H is a domain containing a closed disk of radius $1/2$, then no analytic function (except the identically zero function) in H can be approximated locally uniformly in H by such weighted polynomials $\{e^{-nz}P_n\}_{n\in\mathbb{N}_0}$.*

We note that Theorem 2.9 determines the *largest* disk in which such weighted polynomial approximation, to arbitrary analytic functions, is locally uniformly convergent. In Fig. 3, we similarly show, as in the case of Fig. 2, the case $r = 1/2$ of the curve, defined by

$$\{z \in \mathbb{C} : \log\frac{|z|}{r} + \left(\frac{r^2}{|z|^2} - 1\right)\cdot \mathrm{Re} z = 0, \text{ and } |z| \geq r\}, \tag{2.15}$$

which divides the complex plane into three disjoint domains, $|z| < 1/2, \tilde{\Omega}_0$, and $\tilde{\Omega}_\infty$. These three domains play exactly the analogous roles, in the convergence of $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$, as did G, Ω_0, and Ω_∞ for $\{e^{-nz}s_n(nz)\}_{n\in\mathbb{N}_0}$, (cf. Proposition 2.1 and Theorem 2.2.)

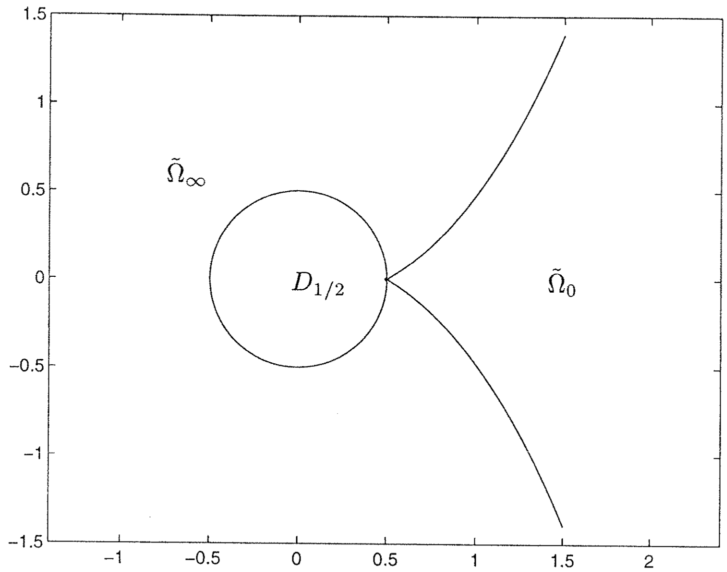

FIG. 3. The extremal disk $D_{1/2}$ and the associated domains

3 General Weighted Polynomial Approximation

In the previous section, we have seen how weighted polynomials, of the form $\{e^{-nz}P_n(z)\}_{n\in\mathbb{N}_0}$, stemming from the work of Szegő [16], can locally uniformly approximate analytic functions in the domain G, defined as the interior of the Szegő curve S of (1.2), or in any of its translations, as well as in the open disk $\{z \in \mathbb{C} : |z| < 1/2\}$, or in any of its translations. But, the broader theoretical question that can be investigated is this. Given the pair

$$(G, W), \tag{3.1}$$

where

$$\begin{cases} \text{i)} & G \text{ is an open bounded set, in the complex plane } \mathbb{C}, \text{ which can be represented as a finite or countable union of disjoint simply connected domains, i.e., } G = \bigcup_{\ell=1}^{\sigma} G_\ell \text{ (where } 1 \le \sigma \le \infty\text{);} \\ \text{ii)} & W(z), \text{ the weight function, is analytic in } G \text{ with } W(z) \neq 0 \text{ for any } z \in G, \end{cases} \tag{3.2}$$

when is it the case that this pair (G, W) has the **approximation property**, i.e.,

$$\begin{cases} \text{for any } f(z) \text{ which is analytic in } G \text{ and for any compact subset} \\ E \text{ of } G, \text{ there exists a sequence of polynomials } \{P_n(z)\}_{n=0}^{\infty}, \\ \text{with } \deg P_n \le n \text{ for all } n \in \mathbb{N}_0, \text{ such that} \\ \qquad \lim_{n\to\infty} \|f - W^n P_n\|_E = 0. \end{cases} \tag{3.3}$$

The material of Section 2 corresponded to the specific choice of the weight function $W(z) := e^{-z}$.

Given a pair (G, W), as in (3.1) , we state below our main result, Theorem 3.1, which gives a characterization, in terms of potential theory, for the pair (G, W) to have the approximation property. For notation, let $\mathcal{M}(E)$ be the space of all positive unit Borel measures on $\mathbb{C}$ which are supported on a compact set E, i.e., for any $\mu \in \mathcal{M}(E)$, we have $\mu(\mathbb{C}) = 1$ and $\operatorname{supp}\mu \subset E$. The logarithmic potential of a compactly supported measure μ is then defined (cf. Tsuji [18, p. 53]) by

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|}\, d\mu(t). \tag{3.4}$$

Theorem 3.1 [12] *A pair* (G, W), *as in* (3.1), *has the approximation property* (3.3) *if and only if there exist a measure* $\mu(G, W) \in \mathcal{M}(\partial G)$ *and a constant* $F(G, W)$ *such that*

$$U^{\mu(G,W)}(z) - \log|W(z)| = F(G, W), \qquad \textit{for any } z \in G. \tag{3.5}$$

Remark It is well known that any open set in the complex plane is a finite or countable union of disjoint domains, and this is more general than the assumption on the open set G in (3.2(i)). However, we note that the approximation property (3.3) *cannot hold*, even in the classical case where $W(z) \equiv 1$ for all $z \in G$, if $G = \bigcup_{\ell=1}^{\sigma} G_\ell$, when some G_ℓ is multiply connected (cf. Walsh [19, p. 25]). In this sense, our initial assumptions on G in (3.2(i)) are quite general.

Remark The condition that $W(z) \neq 0$ for all $z \in G$ cannot be dropped, for if $W(z_0) = 0$ for some $z_0 \in G_\ell$, where $G = \bigcup_{\ell=1}^{\sigma} G_\ell$, then the necessarily null sequence $\{W^n(z_0)P_n(z_0)\}_{n=0}^{\infty}$ trivially fails to converge to any $f(z)$, analytic in G, with $f(z_0) \neq 0$; whence, the approximation property fails. Even more decisive is the result, to be proved in Section 5, that if $W(z_0) = 0$ for some $z_0 \in G_\ell$, then the sequence $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$ can converge, locally uniformly in G, to $f(z)$, only if $f(z) \equiv 0$ in G_ℓ. In this sense, the assumptions on $W(z)$ in (3.2(ii)) are also quite general.

Remark In the case $W(z) \equiv 1$ of Theorem 3.1, the result that the approximation property (3.3) holds is a known classical result in complex approximation theory (cf. [19, p. 26]). This also follows from Theorem 3.1 because the measure $\mu(G, 1)$ exists by Theorems III.12 and III.14 of Tsuji [18], and is the classical equilibrium distribution measure (in the sense of logarithmic potential theory) for $\overline{G}$.

The topic of weighted approximation by $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, on the real line, has been extensively and thoroughly treated in the recent books of Saff and Totik [13] and Totik [17]. Here, we emphasize weighted approximation *in the complex plane*, which has received far less attention in the current approximation theory literature, with the exception of the recent papers by Borwein and Chen [1] and Pritsker and Varga [11, 12].

We shall present in Section 4 a number of applications of Theorem 3.1 to special pairs (G, W). The proofs of some results and remarks on weighted approximation, stated in Sections 3 and 4, are given in Section 5. Finally, we conclude this chapter with Section 6, where further remarks, open problems, and a discussion of possible generalizations are given.

4 Applications

Finding the measure $\mu(G, W)$ of Theorem 3.1 or verifying its existence is a nontrivial problem in general. Since $U^{\mu(G,W)}(z)$ in (3.5) is harmonic in $\mathbb{C}\backslash \operatorname{supp}\mu(G, W)$ and, since it can be shown from (3.5), if $\log|W(z)|$ is continuous on $\overline{G}$ and if G is a *finite* union of $G_\ell, \ell = 1, 2, \ldots, \ell_0$, that $U^{\mu(G,W)}(z)$ is equal to $\log|W(z)| + F(G, W)$ on $\operatorname{supp}\mu(G, W)$, then $U^{\mu(G,W)}(z)$ can be found as the solution of the corresponding Dirichlet problems. The measure $\mu(G, W)$ can be recovered from its potential, using the Fourier method described in Section IV.2 of Saff and Totik [13]. This method has already been used successfully by the authors in [11] to study the approximation of analytic functions by the weighted polynomials $\{e^{-nz}P_n(z)\}_{n=0}^{\infty}$, i.e., when $W(z) := e^{-z}$.

In contrast to the above procedure, we next consider a different method, dealing with specific weight functions, which allows us to deduce "explicit" expressions for the measure $\mu(G, W)$ of Theorem 3.1, and to treat some important cases of pairs (G, W). With G, as defined in (3.2(i)) with σ *finite*, we denote the unbounded component of $\overline{\mathbb{C}}\backslash\overline{G}$ by Ω. Let ν_1 and ν_2 be two unit positive Borel measures on $\mathbb{C}$ with compact supports satisfying

$$\operatorname{supp}\nu_1 \subset \overline{\mathbb{C}}\backslash G \quad \text{and} \quad \operatorname{supp}\nu_2 \subset \overline{\mathbb{C}}\backslash G, \tag{4.1}$$

such that

$$\nu_1(\mathbb{C}) = \nu_2(\mathbb{C}) = 1. \tag{4.2}$$

For real numbers α and β, assume that $W(z)$, satisfying

$$\log|W(z)| = -\left(\alpha U^{\nu_1}(z) + \beta U^{\nu_2}(z)\right), \quad z \in G, \tag{4.3}$$

is analytic in G. Then, we state, as an application of Theorem 3.1, our next result as

Theorem 4.1 [12] *Given any pair of real numbers α and β, given an open bounded set $G = \bigcup_{\ell=1}^{\sigma} G_\ell$ as in (3.2(i)) with σ finite, and given the weight function $W(z)$ of (4.3), then the pair (G, W) has the approximation property (3.3) if and only if the measure*

$$\mu := (1+\alpha+\beta)\omega(\infty,\cdot,\Omega) - \alpha\hat{\nu}_1 - \beta\hat{\nu}_2 \tag{4.4}$$

is positive, where $\omega(\infty,\cdot,\Omega)$ *is the harmonic measure at* ∞ *with respect to* Ω*; here,* $\hat{\nu}_1$ *and* $\hat{\nu}_2$ *are, respectively, the balayages of* ν_1 *and* ν_2 *from* $\overline{\mathbb{C}}\backslash\overline{G}$ *to* $\overline{G}$.

Furthermore, if μ *of* (4.4) *is a positive measure, then* (cf. Theorem 3.1)

$$\mu(G,W) = \mu \quad \textit{and} \quad \operatorname{supp}\mu(G,W) \subset \partial G. \tag{4.5}$$

We point out that the harmonic measure $\omega(\infty,\cdot,\Omega)$ (cf. Nevanlinna [10] and Tsuji [18]) is the same as the equilibrium distribution measure for $\overline{G}$, in the sense of classical logarithmic potential theory [18]. For the notion of balayage of a measure, we refer the reader to Chapter IV of Landkof [7] or Section II.4 of Saff and Totik [13].

In the following series of subsections, we consider various classical weight functions and find their corresponding measures, associated with the weighted approximation problem in G by Theorem 3.1.

4.1 Incomplete Polynomials and Laurent Polynomials

The *incomplete polynomials* of Lorentz [8] are a sequence of polynomials of the form

$$\left\{z^{m(i)}P_{n(i)}(z)\right\}_{i=0}^{\infty}, \quad \deg P_{n(i)} \le n(i), \ (m(i), n(i) \in \mathbb{N}_0), \tag{4.6}$$

where it is assumed that $\lim\limits_{i\to\infty} \dfrac{m(i)}{n(i)} =: \alpha$, where $\alpha > 0$ is a real number. The question of the possibility of approximation by incomplete polynomials is closely connected to that of approximation by the weighted polynomials

$$\{z^{\alpha n}P_n(z)\}_{n=0}^{\infty}, \quad \deg P_n \le n. \tag{4.7}$$

The question of approximation by the incomplete polynomials of (4.6) was completely settled by Saff and Varga [14], and by Golitschek [4] on the interval $[0,1]$ (see Totik [17] and Saff and Totik [13] for the associated history and later developments). We consider now the analogous problem in the complex plane. Since the weight $W(z) := z^{\alpha}$ in (4.7) is multiple-valued in $\mathbb{C}$ if $\alpha \notin \mathbb{N}_0$, we then restrict ourselves to the slit domain $S_1 := \mathbb{C}\backslash(-\infty, 0]$ and the single-valued branch of $W(z)$ in S_1 satisfying $W(1) = 1$.

For the related question of the approximation by the so-called Laurent polynomials

$$\left\{\frac{P_{n(i)}(z)}{z^{m(i)}}\right\}_{i=0}^{\infty}, \deg P_{n(i)} \le n(i), \ (m(i), n(i) \in \mathbb{N}_0), \tag{4.8}$$

where $\lim\limits_{i\to\infty} \dfrac{m(i)}{n(i)} := \alpha, \alpha > 0$, we are similarly led to the question of the approximation by the weighted polynomials

$$\{z^{-\alpha n}P_n(z)\}_{n=0}^{\infty}, \deg P_n \le n, \tag{4.9}$$

with the only difference being in the sign in the exponent of the weight function. Thus, we can give a unified treatment of both problems by considering weighted approximation by $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, $\deg P_n \le n$, with

$$W(z) := z^{\alpha}, \quad z \in S_1 := \mathbb{C}\backslash(-\infty, 0], \tag{4.10}$$

where α is *any* fixed real number and where we choose, as before, the single-valued branch of $W(z)$ in S_1 satisfying $W(1) = 1$.

Theorem 4.2 [12] *Given an open set G as in* (3.2(i)) *with σ finite, such that $\overline{G} \subset S_1$, and given the weight function $W(z)$ of* (4.10), *then the pair (G, W) has the approximation property* (3.3) *if and only if*

$$\mu = (1+\alpha)\omega(\infty, \cdot, \Omega) - \alpha\omega(0, \cdot, \Omega) \tag{4.11}$$

is a positive measure, where $\omega(\infty, \cdot, \Omega)$ and $\omega(0, \cdot, \Omega)$ are, respectively, the harmonic measures with respect to the unbounded component Ω of $\overline{\mathbb{C}}\backslash\overline{G}$, at $z = \infty$ and at $z = 0$.

In some cases, when the geometric shape of G is given explicitly, we can determine the explicit form of the measure of (4.11). This is especially easy to do for disks.

Corollary 4.3 [12] *Given the disk $D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$, where $a \in (0, +\infty)$ and where $\overline{D}_r(a) \subset S_1 = \mathbb{C}\backslash(-\infty, 0]$, i.e., $r < a$, and given the weight function of* (4.10), *then the pair $(D_r(a), W)$ has the approximation property* (3.3) *if and only if*

$$r \le r_{\max}(a,\ \alpha) = \begin{cases} a, & \alpha \in [-1, 0], \\ \dfrac{a}{|2\alpha+1|}, & \alpha \in (-\infty, -1) \cup (0, \infty). \end{cases} \tag{4.12}$$

Furthermore, if (4.12) *is satisfied, then the associated measure $\mu(D_r(a), z^{\alpha})$* (see Theorem 3.1) *is given by*

$$d\mu(D_r(a), z^{\alpha}) = \left(1 + \alpha - \alpha\frac{a^2 - r^2}{|z|^2}\right)\frac{ds}{2\pi r}, \tag{4.13}$$

where ds is the arclength measure on the circle $|z - a| = r$.

4.2 Jacobi and Jacobi-Type Weights

We continue along the same lines by considering weighted approximation with Jacobi weights, i.e., we set

$$W(z) := (1-z)^{\alpha}(1+z)^{\beta}, \quad z \in S_2 := \mathbb{C}\backslash\{(-\infty, -1] \cup [1, \infty)\}, \tag{4.14}$$

where $\alpha, \beta \in \mathbb{R}$ are any numbers, and where we choose the branch of weight function in (4.14) such that $W(0) = 1$.

An analogue of Theorem 4.2 in this case is the following result:

Theorem 4.4 [12] *Given an open set G as in (3.2(i)) with σ finite, such that $\overline{G} \subset S_2$, and given the weight function $W(z)$ of (4.14), then the pair (G, W) has the approximation property (3.3) if and only if*

$$\mu = (1+\alpha+\beta)\omega(\infty, \cdot, \Omega) - \alpha\omega(1, \cdot, \Omega) - \beta\omega(-1, \cdot, \Omega) \tag{4.15}$$

is a positive measure, where Ω is the unbounded component of $\overline{\mathbb{C}} \backslash \overline{G}$.

We next state a corollary of Theorem 4.4, which deals with the explicit formula for the radius of a *largest* disk $D_r(a)$, centered at $a \in (-1,1)$, for which $(D_r(a), W)$ has the approximation property.

Corollary 4.5 [12] *Given the disk $D_r(a) := \{z \in \mathbb{C} : |z-a| < r\}$, with $a \in (-1,1)$ and with $\overline{D}_r(a) \subset S_2$, and given the Jacobi weight function $W(z)$ of (4.14), then the pair $(D_r(a), W)$ has the approximation property (3.3) if and only if*

$$1+\alpha+\beta - \alpha\frac{(1-a)^2 - r^2}{|z-1|^2} - \beta\frac{(1+a)^2 - r^2}{|z+1|^2} \geq 0 \text{ on } |z-a| = r. \tag{4.16}$$

In particular, if $\alpha \geq 0$ and $\beta \geq 0$, then the approximation property (3.3) holds if and only if

$$r \leq r_{\max}(a, \alpha, \beta) :=$$
$$\frac{\sqrt{[\alpha - \beta + a(1+\alpha+\beta)]^2 + (1-a^2)(1+2\alpha+2\beta)} - |\alpha-\beta+a(1+\alpha+\beta)|}{1+2\alpha+2\beta}. \tag{4.17}$$

Furthermore, if (4.16) is valid, then

$$\begin{aligned} &d\mu\left(D_r(a), (1-z)^\alpha(1+z)^\beta\right) \\ &= \left(1+\alpha+\beta - \alpha\frac{(1-a)^2 - r^2}{|z-1|^2} - \beta\frac{(1+a)^2 - r^2}{|z+1|^2}\right)\frac{ds}{2\pi r}, \end{aligned} \tag{4.18}$$

where ds is the arclength measure on $|z-a| = r$.

Both weight functions introduced in (4.10) and (4.14) are special cases of the following Jacobi-type weight function:

$$W(z) := \prod_{i=1}^{p} (z - t_i)^{\alpha_i}, \tag{4.19}$$

where $\{\alpha_i\}_{i=1}^p$ are real numbers and where $\{t_i\}_{i=1}^p \subset \mathbb{C}$ is a fixed set of distinct points. For a given open set G (as in (3.2(i)) with σ finite) such that $t_i \notin \overline{G}$, $i = 1, ..., p$, we assume that there exist p cuts, connecting each t_i with ∞. Then, we can define a single-valued branch of $W(z)$ in the p-slit complex plane which contains $\overline{G}$ in its interior. (It is not possible to specify in advance these cuts, as they necessarily depend on each preassigned open set G.)

Theorem 4.6 [12] *The pair (G, W), defined in the previous paragraph, has the approximation property* (3.3) *if and only if*

$$\mu = \left(1 + \sum_{i=1}^{p} \alpha_i\right) \omega(\infty, \cdot, \Omega) - \sum_{i=1}^{p} \alpha_i \omega(t_i, \cdot, \Omega) \tag{4.20}$$

is a positive measure, where Ω is the unbounded component of $\overline{\mathbb{C}} \backslash \overline{G}$.

Furthermore, if $G = D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$ where $a \in \mathbb{C}$, then the pair $(D_r(a), W)$ has the approximation property (3.3) *if and only if*

$$1 + \sum_{i=1}^{p} \alpha_i - \sum_{i=1}^{p} \alpha_i \frac{|t_i - a|^2 - r^2}{|z - t_i|^2} \geq 0, \quad |z - a| = r. \tag{4.21}$$

4.3 Exponential Weights

Let

$$W(z) := e^{-z^m}, \quad m \in \mathbb{N}. \tag{4.22}$$

The special case $m = 1$ of the weight function (4.22) was considered in Section 2. To avoid technical complications, we shall study only the weighted approximation, with respect to the weight function $W(z) = e^{-z^m}$, in disks centered at the origin. Our next result generalizes Theorem 2.9 of Section 2.

Theorem 4.7 [12] *Given $D_r(0) := \{z \in \mathbb{C} : |z| < r\}$ and given the weight function $W(z)$ of* (4.22)*, then the pair $(D_r(0), W)$ has the approximation property* (3.3) *if and only if*

$$r \leq r_{\max}(m) := (2m)^{-1/m}, \quad m \in \mathbb{N}. \tag{4.23}$$

Moreover, if (4.23) *holds, then*

$$d\mu\left(D_r(0), e^{-z^m}\right) = (1 - 2mr^m \cos m\theta)\, \frac{d\theta}{2\pi}, \tag{4.24}$$

where $d\theta$ is the angular measure on $|z| = r$ and where $z = re^{i\theta}$.

5 Some Proofs

Proof of Proposition 2.1 Let $z \in \overline{G} \backslash \{1\}$, where G is defined as the interior of the Szegő curve S of (1.2). Following Szegő [16], we make substitution $w = \zeta e^{1-\zeta}$ in (2.1), which gives

$$1 - e^{-nz} s_n(nz) = \frac{\sqrt{n}}{\tau_n \sqrt{2\pi}} \int_0^{ze^{1-z}} w^{n-1} \, \frac{\zeta(w)}{1 - \zeta(w)} \, dw. \tag{5.1}$$

Our goal is to bound above the modulus of the right side of (5.1). Denoting the integral in (5.1) by I, then an integration by parts and use of the relation $w(\zeta) = \zeta e^{1-\zeta}$ give that

$$\begin{aligned} I &:= \int_0^{ze^{1-z}} w^{n-1} \frac{\zeta(w)dw}{1-\zeta(w)} \\ &= \frac{w^n}{n} \frac{\zeta(w)}{1-\zeta(w)}\Bigg|_0^{ze^{1-z}} - \int_0^{ze^{1-z}} \frac{w^n}{n} \frac{\zeta'(w)dw}{(1-\zeta(w))^2} \\ &= \frac{(ze^{1-z})^n}{n} \frac{z}{1-z} - \frac{1}{n} \int_0^z \frac{(\zeta e^{1-\zeta})^n}{(1-\zeta)^2}\, d\zeta \\ &= \frac{1}{n(1-z)} \left(z(ze^{1-z})^n + (z-1) \int_0^z \frac{(\zeta e^{1-\zeta})^n}{(1-\zeta)^2} d\zeta \right). \end{aligned}$$

To bound above $|I|$, assume that $z = (x+iy) \in \overline{G} \setminus \{1\}$, and choose the path of integration in the integral of I to consist of the two intervals $[0,x]$ and $[x, x+iy]$. Then, as $|z| \leq 1$ and $|ze^{1-z}| \leq 1$ for all points of $\overline{G}$,

$$\begin{aligned} |I| &\leq \frac{1}{n|1-z|} \left\{ 1 + |z-1| \left(\int_0^{|x|} \frac{dt}{(1-t)^2} + \int_0^{|y|} \frac{ds}{(1-|x|)^2} \right) \right\} \\ &= \frac{1}{n|1-z|} \left\{ 1 + |z-1| \left(\frac{|x|}{1-|x|} + \frac{|y|}{(1-|x|)^2} \right) \right\} \\ &\leq \frac{1}{n|1-z|} \left\{ 1 + |z-1| \left(\frac{1}{1-|x|} + \frac{|y|}{(1-|x|)^2} \right) \right\}. \end{aligned}$$

Then, it can be verified that the square, with vertices ± 1 and $\pm i$, contains $\overline{G}$. (This is shown in Fig. 4 below.) This geometrically implies that

$$|y| \leq 1 - |x| \text{ and } 1 - |x| \leq |1-z| \leq \sqrt{2}(1-|x|).$$

Inserting these inequalities into the upper bound above for $|I|$ yields

$$|I| \leq \frac{1}{n|1-z|} \left\{ 1 + 2\frac{|z-1|}{1-|x|} \right\} \leq \frac{1}{n|1-z|} \left\{ 1 + 2\sqrt{2} \right\} \leq \frac{4}{n|1-z|},$$

for any $z = x + iy \in \overline{G}$. This bound, applied to (5.1), then gives the desired result of (2.4) of Proposition 2.1. □

Proof of Theorem 2.3 First, fix a small δ with $1 > \delta > 0$ and consider the domain $G_\delta := G \setminus \{z : |z-1| \leq \delta\}$. It follows immediately from (2.4) that

$$\|e^{-nz} s_n(nz) - 1\|_{\overline{G}_\delta} \leq \frac{4}{\sqrt{2\pi n}} \left\| \frac{1}{z-1} \right\|_{\overline{G}_\delta} = \frac{4}{\sqrt{2\pi n}\delta} \quad (n \in \mathbb{N}),$$

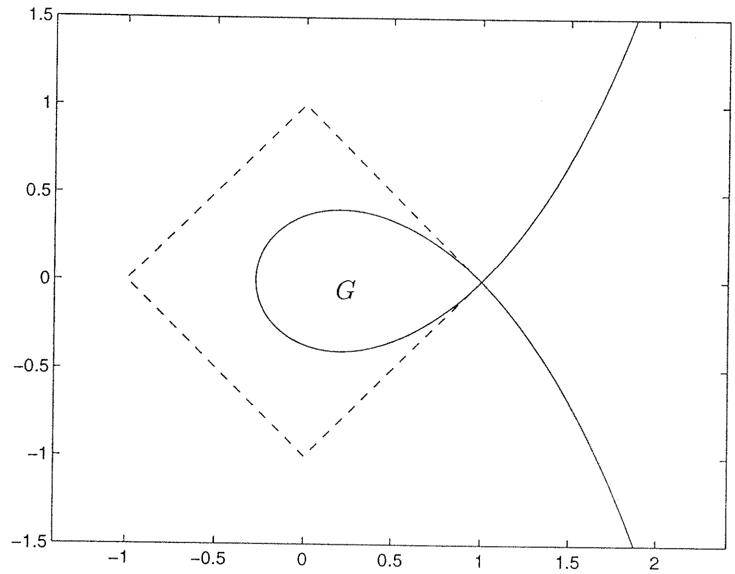

FIG. 4. The Szegő domain G and the covering square

so that

$$\|e^{-nz}s_n(nz) - 1\|_{\overline{G}_\delta} \to 0, \text{ as } n \to \infty. \tag{5.2}$$

Multiplying (2.4) by e^{-kz}, we similarly observe that, for *any* fixed $k = 0, 1, 2, \ldots$,

$$\|e^{-(n+k)z}s_n(nz) - e^{-kz}\|_{\overline{G}_\delta} \le \frac{4}{\sqrt{2\pi n}\delta}\|e^{-kz}\|_{\overline{G}_\delta} \quad (n \in \mathbb{N}),$$

so that

$$\|e^{-(n+k)z}s_n(nz) - e^{-kz}\|_{\overline{G}_\delta} \to 0, \text{ as } n \to \infty. \tag{5.3}$$

This means that e^{-kz}, for any $k \ge 0$, can be uniformly approximated on $\overline{G}_\delta$ by the weighted polynomials $\{e^{-nz}s_{n-k}((n-k)z)\}_{n=k}^{\infty}$. Therefore, any complex polynomial in e^{-z}, say $Q_m(e^{-z}) = \sum_{j=0}^{m} c_j e^{-jz}$, can also be uniformly approximated on $\overline{G}_\delta$ by such weighted polynomials $e^{-nz}P_n(z)$. In fact, it is easy to see from (5.3) that these polynomials $P_n(z)$, in this case, can be chosen to be

$$P_n(z) := \sum_{j=0}^{m} c_j s_{n-j}((n-j)z), \quad n \ge m.$$

If we show that any function $f(z)$, which is analytic in G and continuous on compact subsets of $\overline{G}\setminus\{1\}$, is uniformly approximable on $\overline{G}_\delta$ by polynomials

$Q_m(e^{-z})$, then (2.6) of Theorem 2.3 will follow. Indeed, as ∂G_δ is a Jordan curve (with interior G_δ), so is its image, ∂H_δ (with interior H_δ) in the t-plane, under the conformal mapping $\Phi(z) := e^{-z} = t$. As can be readily verified, for *any* δ with $0 \le \delta \le 1$, the image of $\overline{G}_\delta$, under $\Phi(z) = t$, lies in the open right-half plane of the t-plane, and is symmetric about the positive real axis. Hence, on cutting the t-plane along the negative real axis, then $f(-\log t)$ is analytic and single-valued in H_δ and continuous on ∂H_δ. Thus, by Mergelyan's Theorem (cf. Gaier [3, p. 97]), $f(-\log t)$ can be uniformly approximated on $\overline{H}_\delta$ by the polynomials $Q_m(t)$. But this means that $f(z)$ can be uniformly approximated by $Q_m(e^{-z})$ on $\overline{G}_\delta$.

To prove the second assertion of Theorem 2.3, we note, on multiplying (2.4) by $(z-1)e^{-kz}$, that

$$\|e^{-(n+k)z}((z-1)s_n(nz)) - (z-1)e^{-kz}\|_{\overline{G}} \le \frac{4\|e^{-kz}\|_{\overline{G}}}{\sqrt{2\pi n}},$$

which implies that any function of the form $(z-1)e^{-kz}, k \ge 1$, can be approximated by weighted polynomials $e^{-nz}P_n(z)$, uniformly on $\overline{G}$. It follows, for any polynomial $Q_m(t)$ with $Q_m(0) = 0$, that $(z-1)Q_m(e^{-z})$ is uniformly approximable by weighted polynomials $e^{-nz}P_n(z)$ on $\overline{G}$. Now, assume in addition that $f(z)$ is analytic at $z = 1$. On defining the function

$$v(z) := \frac{f(z)e^z}{z-1},$$

it follows, since $f(1) = 0$ by hypothesis, that $v(z)$ is analytic in G and continuous on $\overline{G}$. Then, by the previous argument (with G_δ and H_δ being replaced, respectively by G and H) and with the same mapping $\Phi(z) := e^{-z} = t$, it similarly follows that $v(-\log t)$ can be uniformly approximated on $\overline{H}$ by the polynomials $\tilde{Q}_{m-1}(t)$, so that

$$\left\|\frac{f(z)e^z}{z-1} - \tilde{Q}_{m-1}(e^{-z})\right\|_{\overline{G}} = \|v(-\log t) - \tilde{Q}_{m-1}(t)\|_{\overline{H}} \to 0,$$

as $m \to \infty$. On setting $Q_m(t) := t\tilde{Q}_{m-1}(t)$ so that $Q_m(0) = 0$, the above display, after multiplying through by $(z-1)e^{-z}$, gives that

$$\|f(z) - (z-1)Q_m(e^{-z})\|_{\overline{G}} \to 0 \text{ as } m \to \infty,$$

which shows that $f(z)$ is uniformly approximable on $\overline{G}$ by weighted polynomials $e^{-nz}P_n(z)$. To complete the proof, we now drop the hypothesis that $f(z)$ is analytic at $z = 1$. Let $P_n(z)$ be the best uniform approximation from π_n to $f(z)$ on $\overline{G}$. By Mergelyan's Theorem again,

$$\lim_{n\to\infty} \|f(z) - P_n(z)\|_{\overline{G}} = 0.$$

For each $n \geq 0$, define $\tilde{P}_n(z) := P_n(z) - P_n(1)$, so that $\tilde{P}_n(1) = 0$. Because $f(1) = 0$, we see that

$$|P_n(1)| = |f(1) - P_n(1)| \leq \|f - P_n\|_{\overline{G}},$$

which implies that

$$\|f - \tilde{P}_n\|_{\overline{G}} \leq \|f - P_n\|_{\overline{G}} + |P_n(1)| \leq 2\|f - P_n\|_{\overline{G}},$$

i.e., $f(z)$ can be uniformly approximated in $\overline{G}$ by $\tilde{P}_n(z)$. But as our previous proof can be applied to each $\tilde{P}_n(z)$, it follows that $f(z)$ can be uniformly approximated on $\overline{G}$ by the weighted polynomials. □

Proof of Theorem 2.5 Given the pair of numbers (r, R) with $0 < r < R \leq 1$, suppose that $f(z)$ is analytic in G_R. For each $n \geq 0$, let $\left\{z_k^{(n+1)}\right\}_{k=1}^{n+1}$ be $n+1$ points (to be specified below) such that $\left\{z_k^{(n+1)}\right\}_{k=1}^{n+1} \subset G_R$. Then, from the Hermite interpolation formula, the polynomial $P_n(z)$, which interpolates $e^{nz}f(z)$ in the $n+1$ points $\left\{z_k^{(n+1)}\right\}_{k=1}^{n+1}$, is given (cf. [19, p. 50]) by

$$e^{nz}f(z) - P_n(z) = \frac{\omega_{n+1}(z)}{2\pi i} \int_{S_{R-\epsilon}} \frac{f(t)e^{nt}dt}{(t-z)\omega_{n+1}(t)}, \tag{5.4}$$

where $\omega_{n+1}(z) := \prod_{k=1}^{n+1} \left(z - z_k^{(n+1)}\right)$ and where $z \in G_{R-\epsilon}$; here, $\epsilon > 0$ is chosen sufficiently small so that $\left\{z_k^{(n+1)}\right\}_{k=1}^{n+1} \subset G_{R-\epsilon}$. Dividing by e^{nz} in (5.4) gives

$$f(z) - e^{-nz}P_n(z) = \frac{e^{-nz}\omega_{n+1}(z)}{2\pi i} \int_{S_{R-\epsilon}} \frac{f(t)dt}{(t-z)e^{-nt}\omega_{n+1}(t)}, \tag{5.5}$$

for $z \in G_{R-\epsilon}$.

Let $\nu_n(\omega_n)$ be the normalized counting measure of the zeros of $\omega_n(z)$, i.e.,

$$\nu_n(\omega_n) = \frac{1}{n}\sum_{k=1}^{n} \delta_{z_k^{(n)}} \quad (n \in \mathbb{N}), \tag{5.6}$$

where δ_z is the unit point mass at z and where all zeros are counted according to their multiplicities. Then, from the definition in (3.4),

$$|\omega_n(z)| = \exp\left\{-nU^{\nu_n(\omega_n)}(z)\right\} \quad (n \in \mathbb{N}). \tag{5.7}$$

For each r with $0 < r < R$, we now choose an interpolation scheme in (5.4) which satisfies

$$\left\{z_k^{(n)}\right\}_{k=1}^{n} \subset S_r \quad (n \in \mathbb{N}), \tag{5.8}$$

and for which

$$\nu_n(\omega_n) \xrightarrow{*} \omega(0,\cdot,G_r), \qquad \text{as } n \to \infty. \tag{5.9}$$

(Here, $\omega(0,\cdot,G_r)$ is the harmonic measure; see [10] and [18]. In addition, the convergence in (5.9) of the discrete measures is in terms of weak* convergence of measures, i.e., a sequence of Borel measures $\{\mu_n\}_{n\in\mathbb{N}}$ on $\mathbb{C}$ converges to a measure μ, as $n \to \infty$, in the *weak* topology* (written $\mu_n \xrightarrow{*} \mu$) if

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$$

for any continuous function f on $\mathbb{C}$ having compact support.) As an example of an interpolation where (5.8) and (5.9) are valid, one can take the preimages of equally spaced points on $|w| = r$ under the conformal map $w = \varphi(z) = ze^{1-z}$, i.e., for $\psi := \varphi^{[-1]}$, we define

$$z_k^{(n)} := \psi\left(re^{i\frac{2\pi k}{n}}\right) \quad (1 \le k \le n, \quad n \in \mathbb{N}). \tag{5.10}$$

It follows from (5.7)–(5.9) that

$$\begin{aligned} \lim_{n\to\infty} |\omega_n(z)|^{1/n} &= \lim_{n\to\infty} \exp\left\{-U^{\nu_n(\omega_n)}(z)\right\} \\ &= \exp\left\{-U^{\omega(0,\cdot,G_r)}(z)\right\}, \end{aligned} \tag{5.11}$$

which holds locally uniformly in $\mathbb{C}\backslash\overline{G}_r$. Taking any ϵ small enough so that $r+\epsilon < R-\epsilon$, we estimate the difference in (5.5) by

$$\|f(z) - e^{-nz}P_n(z)\|_{\overline{G}_r} \le \|f(z) - e^{-nz}P_n(z)\|_{\overline{G}_{r+\epsilon}}$$

$$\le \frac{\|e^{-nz}\omega_{n+1}(z)\|_{S_{r+\epsilon}}\|f\|_{S_{R-\epsilon}}}{2\pi \operatorname{dist}(S_{r+\epsilon}, S_{R-\epsilon}) \cdot \min_{t\in S_{R-\epsilon}} |e^{-nt}\omega_{n+1}(t)|}.$$

Thus, we obtain, by (5.11), that (for details, see Pritsker and Varga [11])

$$\limsup_{n\to\infty} \|f(z) - e^{-nz}P_n(z)\|_{S_r}^{1/n} \le \frac{e^{\log(r+\epsilon)-1}}{e^{\log(R-\epsilon)-1}} = \frac{r+\epsilon}{R-\epsilon}.$$

Letting $\epsilon \to 0$, this gives (2.12) of Theorem 2.5.

To show that the converse part of Theorem 2.5 is valid, suppose that (2.12) holds true for r with $0 < r < R \le 1$. Then, the rest of the proof is a classical converse theorem argument (see [19, p. 81], for example). By the uniform convergence on $\overline{G}_r$, the function $f(z)$ can be represented, in a telescopic series, as

$$f(z) = e^{-nz}P_n(z) + \sum_{k=n}^{\infty}\left(e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z)\right), z \in \overline{G}_r. \tag{5.12}$$

Thus,

$$|f(z) - e^{-nz}P_n(z)| \le \sum_{k=n}^{\infty} \left| e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z) \right|. \tag{5.13}$$

For any $\epsilon > 0$, we have from (2.12) that

$$\|f(z) - e^{-kz}P_k(z)\|_{\overline{G}_r} \le \left(\frac{r}{R-\epsilon}\right)^k,$$

if $k \ge n$ is sufficiently large. This gives that

$$\|e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z)\|_{\overline{G}_r}$$

$$\le \|f(z) - e^{-kz}P_k(z)\|_{\overline{G}_r} + \|f(z) - e^{-(k+1)z}P_{k+1}(z)\|_{\overline{G}_r}$$

$$\le C_1 \left(\frac{r}{R-\epsilon}\right)^k, \quad k \ge n,$$

where C_1 is a constant, independent of k. Using the result of [11, Corollary 4.2], it is known, for any polynomial $P_n(z)$ with deg $P_n \le n$, that

$$|e^{-nz}P_n(z)| \le \|e^{-nz}P_n(z)\|_{S_r} \cdot \left(\frac{|\varphi(z)|}{r}\right)^n, \tag{5.14}$$

for any $z \in \mathbb{C}\backslash G_r, n \in \mathbb{N}_0$, and $0 < r \le 1$. Applying this to the previous display gives

$$|e^{-(k+1)z}P_{k+1}(z) - e^{-kz}P_k(z)| \le C_1 \left(\frac{r}{R-\epsilon}\right)^k \left(\frac{|\varphi(z)|}{r}\right)^k$$

$$= C_1 \left(\frac{|\varphi(z)|}{R-\epsilon}\right)^k, \quad k \ge n.$$

If $|\varphi(z)| = R - 2\epsilon$, i.e., $z \in S_{R-2\epsilon}$, then the telescopic series (5.12) converges to the analytic continuation of $f(z)$ in $G_{R-2\epsilon}$. Thus, from (5.13) and the above inequalities,

$$\|f(z) - e^{-nz}P_n(z)\|_{\overline{G}_{R-2\epsilon}} \le C_2 \left(\frac{R-2\epsilon}{R-\epsilon}\right)^n, \tag{5.15}$$

for any sufficiently large n. Hence, the sequence $\{e^{-nz}P_n(z)\}_{n=0}^{\infty}$ converges to the analytic continuation of $f(z)$, uniformly on $\overline{G}_{R-2\epsilon}$. Since $\epsilon > 0$ can be taken arbitrarily small, then $f(z)$ must be analytic in G_R. □

Proof of Corollary 2.6 If $f(z)$ is analytic in G_R, then by Theorem 2.5,

$$\limsup_{n\to\infty} \left[E_n^{\exp(-z)}\left(f, \overline{G}_r\right)\right]^{1/n} \le \frac{r}{R}, \tag{5.16}$$

where $0 < r < R < 1$. However, strict inequality in (5.16) is equivalent to the analyticity of $f(z)$ in G_ρ, for some ρ with $R < \rho < 1$, by virtue of Theorem 2.5. Thus, $f(z)$ has a singularity on S_R if and only if equality holds in (5.16). □

Proof of Theorem 2.7 Suppose that $f(z)$ is analytic in the interior of $\zeta + E$ and continuous on $\{\zeta + E\}$, where E is a compact set. Then, $g(t) := f(t + \zeta)$ is analytic in the interior of E and continuous on E, which implies by hypothesis that $g(t)$ can be uniformly approximated on E by $\{e^{-nt}P_n(t)\}_{n=0}^{\infty}$. Thus, with $z = t + \zeta$, $f(z)$ can be approximated on $\{\zeta + E\}$ by the weighted polynomials

$$e^{-n(z-\zeta)}P_n(z - \zeta) = e^{-nz}\left(e^{n\zeta}P_n(z - \zeta)\right) \quad (n \in \mathbb{N}_0). \quad \square$$

Proof of Theorem 2.8 Because of Theorem 2.7, we may assume that the domain H is such that $\overline{G} \subset H$. Further, assume to the contrary, that, for some $f(z)\not\equiv 0$ which is analytic in H, there exists a sequence of polynomials $\{P_n(z)\}_{n=0}^{\infty}$, $\deg P_n \leq n$, such that

$$\lim_{n\to\infty} \left\| f(z) - e^{-nz}P_n(z)\right\|_{\overline{G}} = 0. \tag{5.17}$$

It follows that

$$\lim_{n\to\infty} \left\| e^{-nz}P_n(z)\right\|_{\overline{G}} = \|f\|_{\overline{G}} \neq 0. \tag{5.18}$$

But (5.14), for the case $r = 1$, and (5.18) immediately give that

$$\lim_{n\to\infty} \left| e^{-nz}P_n(z)\right| = 0, \qquad \text{for any } z \in \Omega_0, \tag{5.19}$$

where the convergence in (5.19) is locally uniform in Ω_0. Thus, the convergence of $\{e^{-nz}P_n(z)\}_{n=0}^{\infty}$ to $f(z)\not\equiv 0$, locally uniformly in H, is impossible because $H \cap \Omega_0 \neq \emptyset$. □

Proof of the Second Remark after Theorem 3.1 To prove the second statement in this remark, suppose then that $W(z_0) = 0$ with $z_0 \in G_\ell$, where $W(z)\not\equiv 0$ in G_ℓ, and suppose, given an analytic function $f(z)$ in G, that polynomials $\{P_n(z)\}_{n=0}^{\infty}$ can be found such that $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$ converges to $f(z)$, locally uniformly in G. As $W(z)\not\equiv 0$, we can choose $R > 0$ such that $D_R(z_0) := \{z \in \mathbb{C} : |z - z_0| < R\}$ satisfies $\overline{D}_R(z_0) \subset G_\ell$ and that

$$M := \min_{|z-z_0|=R} |W(z)| > 0. \tag{5.20}$$

Then, by the locally uniform convergence of $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$ to $f(z)$,

$$\|P_n\|_{\overline{D}_R(z_0)} = \|P_n\|_{\partial D_R(z_0)} \leq \|W^nP_n\|_{\partial D_R(z_0)}\|W^{-n}\|_{\partial D_R(z_0)} \leq \frac{\|f\|_{\partial D_R(z_0)} + 1}{M^n}, \tag{5.21}$$

for all $n \in \mathbb{N}$ sufficiently large. Since $W(z_0) = 0$, we can find an $r \in (0, R)$ such that

$$m := \|W\|_{\overline{D}_r(z_0)} < M. \tag{5.22}$$

Using (5.21) and (5.22), we obtain

$$\|W^n P_n\|_{\overline{D}_r(z_0)} \le \|W\|^n_{\overline{D}_r(z_0)} \|P_n\|_{\overline{D}_r(z_0)} \le \left(\frac{m}{M}\right)^n \left(\|f\|_{\partial D_R(z_0)} + 1\right) \to 0,$$
as $n \to \infty$.

But because of the locally uniform approximation of $f(z)$ by $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, it follows that $f(z) \equiv 0$ for any $z \in \overline{D}_r(z_0)$, which implies, by the uniqueness theorem, that $f(z) \equiv 0$ in G_ℓ. □

6 Further Remarks and Open Problems

Theorem 3.1 gives a rather complete answer to the question on weighted approximation by $W^n(z)P_n(z)$ in open sets of the complex plane. It is then very natural to consider the uniform approximation by such weighted polynomials on *compact* sets, aiming at an analogue (generalization) of Mergelyan's Theorem (see [19, p. 367]). Let $E \subset \mathbb{C}$ be a compact set with connected complement $\overline{\mathbb{C}}\backslash E$. We denote the set of all functions, analytic interior to E and continuous on E, by $A(E)$. Let $W \in A(E)$, with $W(z) \neq 0$ for any $z \in E$.

Problem *Give a necessary and sufficient condition for the pair (E, W) to have the following approximation property:*

For any $f \in A(E)$, there exist polynomials $\{P_n(z)\}_{n=0}^{\infty}$, with $\deg P_n \le n$, *such that*

$$\lim_{n\to\infty} \|f - W^n P_n\|_E = 0. \tag{6.1}$$

Obviously, the classical uniform approximation by polynomials (Mergelyan's Theorem) corresponds to $W(z) \equiv 1$, $z \in E$. We observe that (3.5) of Theorem 3.1, holding with $G = \text{Int}E$, is a necessary condition for (6.1). Let us also remark that this problem is open even in the case when E is a subset of the real line, such as an interval (see [13, 17] for background and general results).

An even more general approach is to consider the approximation problem in (6.1) with polynomials replaced by rational functions. Certain results concerning such weighted rational approximation have been obtained by Borwein and Chen [1] in the complex plane.

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