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Alternating Direction Iteration Methods For n Space Variables

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We consider the iterative solution of the system of linear equations

$$(X_1 + X_2 + \cdots + X_n)z = f, \qquad n \geqq 2, \tag{1}$$

where each X_j, $1 \leqq j \leqq n$, is a Hermitian and positive definite $N \times N$ matrix. If $n = 2$, the iterative methods of Peaceman-Rachford [1, Chapter 7], or D'yakonov [2] and Kellogg [3], may be used to solve (1). In this paper these methods are generalized to $n \geqq 2$, and are shown, in a sense, to be dual to one another.

Let $\rho > 0$ be fixed, and define $z_j = (\rho I + X_j)z$. From (1) we get the compound $nN \times nN$ matrix equation

$$\begin{bmatrix} I & -W_2(\rho) & \cdots & -W_n(\rho) \\ -W_1(\rho) & I & & -W_n(\rho) \\ \vdots & & & \vdots \\ -W_1(\rho) & -W_2(\rho) & \cdots & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} f \\ f \\ \vdots \\ f \end{bmatrix}, \tag{2}$$

where

$$W_j(\rho) = (\rho I + X_j)^{-1}\left(\frac{\rho}{n-1} I - X_j\right). \tag{3}$$

Our first set of alternating direction iterative methods will be the block Jacobi and block Gauss-Seidel iterative methods applied to (2), namely

$$(\rho I + X_j)u_j^{(m+1)} = \sum_{k \neq j} \left(\frac{\rho}{n-1} I - X_k\right) u_k^{(m)} + f, \qquad 1 \leqq j \leqq n, \tag{4}$$

and

$$(\rho I + X_j)u_j^{(m+1)} = \sum_{k<j} \left(\frac{\rho}{n-1} I - X_k\right) u_k^{(m+1)} + \sum_{k>j} \left(\frac{\rho}{n-1} I - X_k\right) u_k^{(m)} + f. \tag{5}$$

If $n = 2$, (5) is the Peaceman-Rachford method.

We now form the transpose of the matrix of (2), and consider the compound matrix equation

$$\begin{bmatrix} I & -W_1(\rho) & \cdots & -W_1(\rho) \\ -W_2(\rho) & I & & -W_2(\rho) \\ \vdots & \vdots & & \vdots \\ -W_n(\rho) & -W_n(\rho) & & I \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}. \tag{6}$$

If the block Jacobi and block Gauss-Seidel iterative methods are applied to (6),

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one obtains the alternating direction iterative methods

$$(7)\qquad (\rho I + X_j)y_j^{(m+1)} = \left(\frac{\rho}{n-1} I - X_j\right) \sum_{k\neq j} y_k^{(m)} + f_j, \qquad 1 \leqq j \leqq n,$$

and

$$(8)\qquad (\rho I + X_j)y_j^{(m+1)} = \left(\frac{\rho}{n-1} I - X_j\right) \{\sum_{k<j} y_k^{(m+1)} + \sum_{k>j} y_k^{(m)}\} + f_j, \qquad 1 \leqq j \leqq n.$$

Here $f_j = (\rho I + X_j)g_j$, and it is assumed that

$$(9)\qquad f_1 + \cdots + f_n = f.$$

When $n = 2$, (8) is the method of D'yakonov. Thus, the Peaceman-Rachford iterative method and D'yakonov's method (and their generalization) are dual to one another in the sense that either can be viewed as the Gauss-Seidel iterative method applied to a particular composite matrix or its transpose.

Since each matrix X_j is Hermitian and positive definite, let the eigenvalues $\lambda_i(j)$ of X_j satisfy

$$0 < a \leqq \lambda_i(j) \leqq b, \qquad 1 \leqq i \leqq N, \qquad 1 \leqq j \leqq n.$$

THEOREM. *If* $\rho > (n-2)b/2$, *and* $\{u_j^{(m)}\}$ *is defined by* (4) *or* (5), *and* $\{y_j^{(m)}\}$ *is defined by* (7) *or* (8), *then*

$$(10)\qquad \lim_{m\to\infty} u_j^{(m)} = z \quad \textit{for each} \quad 1 \leqq j \leqq n,$$

and

$$(11)\qquad \lim_{m\to\infty} (y_1^{(m)} + \cdots + y_n^{(m)}) = z,$$

where z *is the solution of* (1).

Proof. Using spectral (L_2) norms, it is easy to see that there exists a $q < 1$ such that

$$\| W_j(\rho) \| = \max_{1\leqq i\leqq N} \left| \frac{\frac{\rho}{n-1} - \lambda_i(j)}{\rho + \lambda_i(j)} \right| \leqq \frac{q}{n-1} < \frac{1}{n-1}, \qquad 1 \leqq j \leqq n$$

for $\rho > \left(\frac{n-2}{2}\right) b$. Letting

$$\zeta = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

denote a column vector with nN components it is readily verified that the quantity

$$(12)\qquad \| \zeta \| = \max_{1\leqq j\leqq n} \| v_j \|$$

satisfies all the axioms for a vector norm, where in particular we are using Euclidean norms for the subvectors v_i of ζ. Let us denote the compound matrix of (2) by

$I - W$. It follows that for all ζ and all $\rho > \left(\frac{n-2}{2}\right) b$,

$$\| W\zeta \| \leqq q \| \zeta \| < \| \zeta \| , \tag{13}$$

so W is a convergent matrix. But, as W is just the block Jacobi iteration matrix derived from (2), the block Jacobi iterative method of (4) is convergent. If W^* is the conjugate transpose of W, the same argument shows that $\| W^*\zeta \| < \| \zeta \|$ so the iterative method of (7) is also convergent. A similar argument shows that the block Gauss-Seidel methods (5) and (8) are convergent.

If $u_j = \lim_{m \to \infty} u_j^{(m)}$ in (4) or (5), the u_j satisfy the system of equations

$$(\rho I + X_j)u_j = \sum_{k \neq j} \left(\frac{\rho}{n-1} I - X_k \right) u_k + f, \qquad 1 \leqq j \leqq n.$$

Using (13), it may be seen that this system has a unique solution. Since $u_j = z$, $1 \leqq j \leqq n$, is a solution, (10) is obtained.

If $y_j = \lim_{m \to \infty} y_j^{(m)}$ in (7) or (8), the y_j satisfy the system of equations

$$(\rho I + X_j)y_j = \left(\frac{\rho}{n-1} I - X_j \right) \sum_{k \neq j} y_k + f_j, \qquad 1 \leqq j \leqq n.$$

Adding these, one obtains $(X_1 + \cdots + X_n)(y_1 + \cdots + y_n) = f$, so that (11) is obtained, proving the theorem.

We remark that this theorem can also be deduced as an application of a generalization [4] of the well known result of Collatz [5], viz., that a strictly diagonally dominant matrix gives rise to convergent Jacobi and Gauss-Seidel iterative methods. For the norms of (12), the partitioned matrix of (2) or (6) is block strictly diagonally dominant in the sense of [6].

Because of the restriction $\rho > (n-2)b/2$, it is doubtful that this procedure converges very rapidly, and for this reason, no estimates of rates of convergence are included. (This restriction on ρ is necessary even in the favorable case when the X_j all commute with one another.) We stress, however, that the main point of this paper is the theoretical result of *convergence* without commutativity assumptions on the matrices X_j. To our knowledge, similar results have not been proved for other alternating direction methods applied to n-dimensional problems, $n \geqq 3$. Complementary to this is the fact that three-dimensional matrix problems have been constructed* for which the Douglas-Rachford method [7] and the generalized Peaceman-Rachford method of Douglas [8] each *diverge* for a suitable single positive parameter ρ.

Finally, it is worth mentioning that our generalization of the Peaceman-Rachford iterative method (5) is computationally more attractive than our generalization of the method of D'yakonov, since the latter requires, from (11), more vector storage in practical applications.

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