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Angular Distribution of Zeros of the Partial Sums of e^z via the Solution of Inverse Logarithmic Potential Problem

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Abstract. We continue the work of Szegő [18] on describing the angular distribution of the zeros of the normalized partial sum $s_n(nz)$ of e^z, where $s_n(z) := \sum_{k=0}^{n} z^k/k!$. We imbed this problem in some inverse problem of potential theory and prove a so-called Erdős-Turán-type theorem, which is of interest in itself.

Keywords. Szegő curve, logarithmic potential, harmonic measure.

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1. Introduction

Let

$$s_n(z) := \sum_{k=0}^{n} \frac{z^k}{k!}, \qquad n \in \mathbb{N} := \{1, 2, \ldots\},$$

denote the partial sums of the exponential function e^z. This paper is devoted to the investigation of the angular distribution of the zeros of $s_n(z)$, or, what is the same, of zeros $Z_n := \bigcup_{k=1}^{n}\{z_{k,n}\}$ of the normalized partial sums $s_n(nz)$.

In 1924, Szegő [18] showed that the set of accumulation points of $\cup_{n=1}^{\infty} Z_n$ coincides with what is now called the Szegő curve

$$S := \{z \in \mathbb{C} : |ze^{1-z}| = 1 \text{ and } |z| \le 1\},$$

where $\mathbb{C}$ is the complex plane. What is remarkable is that $\phi(z) = ze^{1-z}$ maps the interior of S conformally and univalently onto the unit disk $\mathbb{D} := \{w : |w| < 1\}$.

Subsequently, Buckholtz [7] established the results that all $z_{k,n}$ lie outside the curve S, and that

$$\text{dist}(z_{k,n}, S) \le \frac{2e}{\sqrt{n}}, \tag{1.1}$$

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where $\operatorname{dist}(A,B)$ denotes the distance between sets A and B in the complex plane $\mathbb{C}$, i.e.

$$\operatorname{dist}(A,B) := \inf_{z\in A,\zeta\in B} |z-\zeta|.$$

For refinements of Buckholtz's result, see [8].

Szegő [18] also showed that the asymptotic angular distribution of the zeros of $s_n(nz)$ is governed by the mapping $w=\phi(z)$, in the following sense: let θ_1 and θ_2 be any real numbers with $0<\theta_1<\theta_2<2\pi$, and let $z_j := \psi(e^{i\theta_j})$, $j=1,2$, where $\psi := \phi^{-1}$ is the inverse mapping, so that z_1 and z_2 are points of S. Let W be the sector defined by

$$W = W(\theta_1,\theta_2) := \{z\in\mathbb{C} : \arg z_1 \le \arg z \le \arg z_2\}.$$

Then,

$$\lim_{n\to\infty} \frac{\#(W\cap Z_n)}{n} = \frac{\theta_2-\theta_1}{2\pi}, \tag{1.2}$$

where $\#(W\cap Z_n)$ denotes the number of zeros of $s_n(nz)$ in W.

A restatement of this result, in terms of the weak*-convergence of measures, was done in [16, Theorem 2.1].

A sharper form of (1.2) was also proved by Szegő in [18], namely that

$$\left|\frac{\#(W\cap Z_n)}{n} - \frac{\theta_2-\theta_1}{2\pi}\right| \le \frac{C_1}{n}, \qquad \text{as } n\to\infty, \tag{1.3}$$

where $C_1 = C_1(\theta_1,\theta_2) > 0$ is a constant which depends on θ_1 and θ_2, for $0<\theta_1<\theta_2<2\pi$. The main purpose of this paper is to obtain a related result, in Theorem 2 below, which states that for any choice of θ_1 and θ_2 with $0<\theta_2-\theta_1<2\pi$ and for any positive integer n,

$$\left|\frac{\#(W\cap Z_n)}{n} - \frac{\theta_2-\theta_1}{2\pi}\right| \le \frac{C_2}{n^\alpha}, \tag{1.4}$$

where C_2 and α are absolute positive constants.

Note that the condition in (1.3), that $0<\theta_1<\theta_2<2\pi$, is more restrictive than the condition for (1.4), that $0<\theta_2-\theta_1<2\pi$, as (1.3) cannot directly cover sectors including the positive real axis. In addition, (1.4) holds for all positive integers n, while (1.3) holds for $n\to\infty$. These are important differences since it is known [7] that the convergence rate of the zeros of $s_n(nz)$ to the point $z=1$ of S is $\mathcal{O}(1/\sqrt{n})$, as $n\to\infty$, while the convergence of the zeros of $s_n(nz)$ to any fixed arc of S, not containing $z=1$, is $\mathcal{O}((\log n)/n)$ as $n\to\infty$ (cf. [8]).

At this moment, the new theoretical result of (1.4) gives no indication as to the actual value of α or the constant C_2. It is our hope that further investigations, including numerical calculations, will shed some light on this open problem. Some related results can also be found in [21].

We obtain our result by a generalization of this question to some inverse problem of potential theory and proving an Erdős-Turán-type theorem, which is interesting by itself. For corresponding results and references concerning Erdős-Turán-type theorems, see [2, 3, 4, 5, 6, 9, 10, 11, 14, 19].

For more details concerning potential theoretic notions, such as Borel measure, logarithmic potential, harmonic measure, etc, see [17, 20].

2. Main definitions and results

First, we formulate the inverse logarithmic potential problem which arises naturally from the investigation of the normalized partial sums $s_n(nz)$.

Let $L \subset \mathbb{C}$ be a quasiconformal curve (since S has a corner at point $z = 1$, quasiconformal curves can be regarded as its natural generalization). We recall that, according to Ahlfors' criterion (see [13, Ch. II.8]), a (closed) Jordan curve L is quasiconformal if and only if for any pair of distinct points z_1 and $z_2 \in L$, the inequality

$$\min\{\operatorname{diam}(L'), \operatorname{diam}(L'')\} \le c|z_2 - z_1|$$

holds with some constant $c = c(L) \ge 1$, where L' and L'' are the two arcs which are defined from $L \setminus \{z_1, z_2\}$ and

$$\operatorname{diam}(A) := \sup_{z,\zeta \in A} |z - \zeta|$$

is the diameter of $A \subset \mathbb{C}$.

Using Ahlfors' criterion, one can easily verify that convex curves, smooth and even piecewise smooth curves without cusps (including S) are quasiconformal. At the same time, well-known examples (see [13, Ch. II.8]) show how complicated the behavior of a quasiconformal curve can be.

The curve L divides $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, the extended complex plane, into two Jordan domains, the unbounded domain $\Omega := \operatorname{ext}(L)$, and the bounded domain $G := \operatorname{int}(L)$.

Let $\sigma = \sigma^+ - \sigma^-$ be a signed measure, where $\sigma^\pm$ are arbitrary positive unit Borel measures with a compact support in $\mathbb{C}$. Thus, $\sigma(\mathbb{C}) = 0$. It is usual to estimate the deviation of σ from the "0-measure" in terms of bounds for the logarithmic potential

$$U^\sigma(z) := \int \log \frac{1}{|\zeta - z|}\, d\sigma(\zeta), \qquad z \in \mathbb{C},$$

on subsets of $\mathbb{C}$.

In particular, results of Erdős and Turán [9, 10, 11], devoted to the study of the distribution of zeros of polynomials, can be interpreted in this way; a corresponding bibliography can be found in [2, 3, 4, 5, 6, 14, 19].

We denote by Φ the Riemann function that conformally and univalently maps Ω onto the exterior $\Delta := \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ of the closed unit disk $\overline{\mathbb{D}}$, and which is normalized by the conditions $\Phi(\infty)=\infty, \Phi'(\infty)>0$. Let $z_0 \in G$ be a fixed point. Analogously, we denote by ϕ the conformal mapping of G onto $\mathbb{D}$, with the normalization $\phi(z_0)=0$, $\phi'(z_0)>0$. Set $\Psi := \Phi^{-1}$ and $\psi := \phi^{-1}$. The functions Φ, Ψ, ϕ and ψ can be naturally extended to homeomorphisms between the appropriate closed domains and we keep the previous notation for these extensions. Further, set

$$l_r := \{\zeta : |\phi(\zeta)| = r\}, \qquad 0<r<1,$$

and

$$N(A,\delta) := \{\zeta : \operatorname{dist}(\zeta, A) < \delta\}, \qquad A \subset \mathbb{C},\ \delta > 0.$$

Our basic result will be formulated in terms of

$$b_\sigma(h) := \sup_{z \in l_{1-h}} (U^\sigma(z) - U^\sigma(z_0)), \qquad 0<h<1,$$

and

$$b_\sigma^*(h) := \min(b_\sigma(h), b_{-\sigma}(h)), \qquad 0<h<1.$$

Theorem 1. *Let L be a quasiconformal curve, and let $z_0 \in G := \operatorname{int}(L)$ be fixed. Let $\sigma = \sigma^+ - \sigma^-$ be a signed measure, where σ^+ and σ^- are positive unit Borel measures such that* $\operatorname{supp}(\sigma^+) = L$, $\operatorname{supp}(\sigma^-) \subset \overline{\Omega} \cap N(L,\delta)$ *for some $0<\delta<1$, where $\Omega = \operatorname{ext}(L)$. Moreover, let $c>0$ and $\beta>0$ be constants such that for all subarcs $J \subset L$,*

$$\sigma^+(J) \le c(\operatorname{diam}(J))^\beta,$$

and let $\varepsilon > 0$ be a sufficiently small fixed number. Then, there exist positive constants γ, κ and μ, depending only on L and β, as well as a positive constant c_1 depending upon $L, c, \beta, z_0, \varepsilon$, such that for any $m \in \mathbb{N}, 0<h<\frac{1}{3}$ and a subarc $J \subset L$, the inequality

$$(2.1) \qquad |\sigma^+(J) - \sigma^-(N(J,\delta))| \le c_1 \left(\frac{b_\sigma^*(h)}{h(1-3h)^m} + \delta^{\kappa\varepsilon} + \frac{1}{\delta^{\mu\varepsilon} m^\gamma} \right)$$

holds.

The proof of this theorem is given in Section 5.

As an immediate application we have the following assertion for the angular distribution of Z_n, the zeros of $s_n(nz)$.

We associate with Z_n its normalized counting measure, i.e.

$$\nu_n := \frac{1}{n} \sum_{z \in Z_n} \delta_z,$$

where δ_z is the unit point mass at z and where all zeros are counted according to their multiplicities.

Let $\omega(\cdot) = \omega(0, \cdot, G)$, where $G = \operatorname{int}(S)$ is the interior of the Szegő curve S, denote the harmonic measure at the point $z=0$ with respect to G. That is, for any Borel set $B \subset \mathbb{C}$,

$$\omega(B) = m(\phi(B \cap S)),$$

where $m(\cdot)$ is the normalized arc length measure on $\mathbb{T} := \{w : |w| = 1\}$, the unit circle.

For any subarc $J \subset S$, we introduce a sector

$$W(J) := \{z \in \mathbb{C} : \text{ there exists } \zeta \in J \text{ with } \arg\zeta = \arg z\}.$$

Theorem 2. *For any $n \in \mathbb{N}$ and any subarc $J \subset S$,*

$$|\nu_n(W(J)) - \omega(J)| \leq \frac{c_2}{n^\alpha},$$

with some absolute positive constants c_2 and α (i.e. c_2 and α are independent of n and J).

The proof of this theorem is given in Section 3.

In what follows we denote by $\alpha, \beta, \gamma, \kappa, \mu, c, c_1, \ldots$ positive constants (different each time, in general) that either are absolute or depend on parameters not essential for the arguments.

3. Proof of Theorem 2

Let n be sufficiently large. We set in Theorem 1, $L = S, z_0 = 0, \sigma^+ = \omega, \sigma^- = \nu_n$, i.e. the point 0 takes over the role of z_0 in the application of Theorem 1. From the boundary behavior of the conformal mapping ϕ (cf. [15, Ch. 3]), we obtain, for any subarc $J \subset S$, that

$$\omega(J) \leq c\,\mathrm{diam}(J).$$

Next we put $\delta = (3e)/\sqrt{n}$, so that by the result of Buckholtz [7] (cf. (1.1)) $\operatorname{supp}\nu_n \subset (\mathrm{ext}(S) \cap N(S, \delta))$.

Let $z \in G = \mathrm{int}(S)$. We consider the logarithmic potentials of the measures ω and ν_n,

$$U^\omega(z) = -\int \log|z-\zeta|\,d\omega(\zeta) = 1 - \mathrm{Re}\,z = 1 + \log|e^{-z}|$$

(see [16, Thm. 4.1]), and

$$U^{\nu_n}(z) = -\int \log|z-\zeta|\,d\nu_n(\zeta) = \frac{1}{n}\log\frac{\gamma_n}{|s_n(nz)|},$$

where $\gamma_n := n^n/n!$ is the highest coefficient of $s_n(nz)$, as well as their difference

$$U^\sigma(z) = U^{\omega-\nu_n}(z) = U^\omega(z) - U^{\nu_n}(z) = 1 - \frac{1}{n}\log\gamma_n + \frac{1}{n}\log|e^{-nz}s_n(nz)|.$$

According to [18],

$$e^{-nz}s_n(nz) = 1 - \frac{\sqrt{n}}{\tau_n\sqrt{2\pi}}\int_0^z (\phi(\zeta))^n\,d\zeta =: 1 - v_n(z),$$

where

$$\tau_n := \frac{n!}{n^n e^{-n}\sqrt{2\pi n}},$$

and Stirling's asymptotic formula gives

$$\lim_{n\to\infty} \tau_n = 1.$$

Therefore,

$$U^\sigma(z) - U^\sigma(0) = \frac{1}{n}\log|1 - v_n(z)|.$$

We note one general distortion property of ψ, which follows easily from the Koebe one-quarter-theorem. Namely, for $w \in \mathbb{D}$, we have

$$|\psi'(w)| \le 4\frac{\operatorname{dist}(\psi(w), S)}{1-|w|};$$

(see, for example, [1, p. 58]).

Further, we set $h = 1/\sqrt{n}$. For $z \in l_{1-h}$ and $w = \phi(z)$, we have

$$\left|\int_0^z \phi(\zeta)^n \, d\zeta\right| = \left|\int_0^w \tau^n \psi'(\tau)\, d\tau\right| \le 4\sqrt{n}\int_0^{1-1/\sqrt{n}} r^n \, dr \le e^{-\beta\sqrt{n}}.$$

Thus, by our choice of parameters, we obtain

$$b_\sigma(h) \le e^{-\gamma\sqrt{n}}.$$

Finally, setting $m = \lfloor\sqrt{n}\rfloor$, the integer part of $\sqrt{n}$, and applying Theorem 1 for sufficiently small ε, we have, for any subarc $J \subset S$, that

$$|\nu_n(N(S,\delta)) - \omega(J)| \le \frac{c_2}{n^\alpha},$$

from which the statement of Theorem 2 directly follows.

4. Some auxiliary facts

Let L be a quasiconformal curve, with $G := \operatorname{int}(L)$, $\Omega := \operatorname{ext}(L)$. It is known (see [13, Ch. II.8]) that the conformal mappings Φ, ϕ, Ψ and ψ can be extended to quasiconformal mappings of the whole plane onto itself, with ∞ as a fixed point. Since such homeomorphisms are Hölder continuous on compact sets in $\mathbb{C}$ (see, for example, [1, p. 97]), we have

$$\text{(4.1)} \quad \frac{1}{c_1}|z_2 - z_1|^{1/\alpha} \le |\Phi(z_2) - \Phi(z_1)| \le c_1|z_2 - z_1|^\alpha, \qquad z_1, z_2 \in \overline{\Omega} \cap N(L,1)$$

and

$$\text{(4.2)} \qquad \frac{1}{c_2}|z_2 - z_1|^{1/\beta} \le |\phi(z_2) - \phi(z_1)| \le c_2|z_2 - z_1|^\beta, \qquad z_1, z_2 \in \overline{G}.$$

Further, let $f(z)$ be analytic in Ω (including ∞) and continuous on $\overline{\Omega}$. Moreover, we assume that f is Hölder-continuous, i.e. there is a positive constant γ such that

$$|f(z_2) - f(z_1)| \le |z_2 - z_1|^\gamma, \qquad z_1, z_2 \in \overline{\Omega}.$$

Replacing in the Belyi Theorem (see [1, p. 119]) z by $1/(z - z_0)$, we obtain the following assertion: for any $n \in \mathbb{N}$, $n > 1$ there exists a rational function R_n of the form

$$R_n(z) = \sum_{j=0}^{n} \frac{c_j}{(z - z_0)^j}, \qquad c_j \in \mathbb{C}, \tag{4.3}$$

such that

$$|f(z) - R_n(z)| \le c(\operatorname{dist}(z, l_{1-1/n}))^\gamma, \qquad z \in L,$$

where c depends only on L and γ.

Therefore, by (4.2) and the maximum modulus principle, we obtain

$$|f(z) - R_n(z)| \le cn^{-\gamma\beta}, \qquad z \in \overline{\Omega}. \tag{4.4}$$

Finally, we cite for convenience an obvious analogue of the Bernstein-Walsh lemma for rational functions (cf. [17, p. 153]). Namely, for any $R_n(z)$ of the form (4.3) and any $z \in G$,

$$|R_n(z)| \le |\phi(z)|^{-n} \sup_{z\in L} |R_n(z)|. \tag{4.5}$$

5. Proof of Theorem 1

We use in our construction below some ideas from [10, 2].

Without loss of generality, we can assume that $\operatorname{diam}(J') \le 1$, where $J' = \Phi(J)$. Let $t := \delta^\varepsilon$, where $0 < \varepsilon < \alpha$ is an arbitrary sufficiently small fixed constant. We assume that $\delta \le 1$, so $t \le 1$. Writing

$$J' = \{\theta : \theta_1 \le \theta \le \theta_2\}, \qquad \theta_1 < \theta_2 < \theta_1 + \frac{\pi}{2},$$

we consider the continuous function $h(e^{i\theta})$ on the unit circle $\mathbb{T}$ which is 1 if $\theta_1 \le \theta \le \theta_2$, 0 if $\theta_2 + \sqrt{t} \le \theta \le 2\pi + \theta_2 - \theta_1 - \sqrt{t}$ and linear otherwise. We denote by the same symbol h the harmonic extension of h onto Δ, i.e. the solution of the Dirichlet problem with corresponding data on the unit circle $\mathbb{T}$. Let $H(w)$, $w \in \Delta$, denote the completion of $h(w)$, i.e. $H(w)$ is an analytic function satisfying $\operatorname{Im} H(\infty) = 0$ and

$$\operatorname{Re} H(w) = h(w), \quad w \in \Delta.$$

Since for any $0 < \eta_1 < \eta_2 < \eta_1 + \pi$,

$$|h(e^{i\eta_1}) - h(e^{i\eta_2})| \le \frac{1}{\sqrt{t}}(\eta_2 - \eta_1),$$

by Privalov's Theorem (see [12, p. 400]) for any $w_1, w_2 \in \overline{\Delta}$ we have

$$|H(w_2) - H(w_1)| \le c_1 \frac{1}{\sqrt{t}} |w_2 - w_1|^{3/4}.$$

Furthermore, H is bounded on $\overline{\Delta}$.

Next, we introduce the functions

$$\begin{aligned} f(z) = f(z, J, \delta) &:= h(\Phi(z)), \quad z \in \overline{\Omega}, \\ F(z) = F(z, J, \delta) &:= H(\Phi(z)), \quad z \in \overline{\Omega}, \end{aligned}$$

and arcs

$$\begin{aligned} J_1 &:= \{z = \Psi(e^{i\theta}) : \theta_1 - \sqrt{t} \le \theta \le \theta_1\}, \\ J_2 &:= \{z = \Psi(e^{i\theta}) : \theta_2 \le \theta \le \theta_2 + \sqrt{t}\}, \\ J_3 &:= L \setminus (J \cup J_1 \cup J_2). \end{aligned}$$

Since Φ and Ψ are Hölder-continuous (cf. (4.1) and (4.2)), the function f has the following properties:

$$\begin{aligned} 0 \le f(z) &\le 1, \quad z \in L, \\ 1 - f(z) &\le c\delta^{\alpha/4}, \quad z \in N(J, \delta) \cap \overline{\Omega}, \\ f(z) &\le c\delta^{\alpha/4}, \quad z \in N(J_3, \delta) \cap \overline{\Omega}. \end{aligned}$$

Our next aim is to approximate function $F(z)$ by rational functions of the form (4.3).

Note that F is Hölder-continuous on $\overline{\Omega}$, that is, for $z_1, z_2 \in L$,

$$|F(z_2) - F(z_1)| \le \frac{c_1}{\sqrt{t}} |\Phi(z_2) - \Phi(z_1)|^{3/4} \le \frac{c_2}{\sqrt{t}} |z_2 - z_1|^{3\alpha/4}.$$

By (4.4) for any $m \in \mathbb{N}$, there exist a rational function $R_m(z)$, of the form (4.3), and a constant $\beta > 0$ such that

$$|F(z) - R_m(z)| \le \frac{c_3}{\sqrt{t} m^\beta}, \qquad z \in \overline{\Omega}.$$

Hence, the rational function

$$Q_m(z) := \frac{c_3}{\sqrt{t} m^\beta} + R_m(z) \left(1 - \frac{2c_3}{\sqrt{t} m^\beta}\right),$$

and its real part $q_m(z) := \operatorname{Re} Q_m(z)$ satisfy the following conditions:

$$0 \le q_m(z) \le 1, \quad z \in L, \tag{5.1}$$

$$|Q_m(z)| \le c_4, \quad z \in L. \tag{5.2}$$

For $r_m(z) := \operatorname{Re} R_m(z)$ and $z \in N(J,\delta) \cap \overline{\Omega}$,

$$\begin{aligned} 1 - q_m(z) &= 1 - \frac{c_3}{\sqrt{t}m^\beta} - r_m(z) + \frac{2c_3}{\sqrt{t}m^\beta} r_m(z) \\ &\le 1 - f(z) + f(z) - r_m(z) + \frac{2c_3}{\sqrt{t}m^\beta} r_m(z) \\ &\le c_5 \left(\delta^{\alpha/4} + \frac{1}{\delta^{\varepsilon/2} m^\beta} \right). \end{aligned} \tag{5.3}$$

Analogously, for $z \in N(J_3, \delta) \cap \overline{\Omega}$,

$$q_m(z) \le c_5 \left(\delta^{\alpha/4} + \frac{1}{\delta^{\varepsilon/2} m^\beta} \right). \tag{5.4}$$

Applying the Green formula to the function $q_m(z)$ and the unbounded domain $\operatorname{ext}(l_{1-2h})$, for $0 < h < 1/2$, we obtain, for $z \in \operatorname{ext}(l_{1-2h})$, that

$$q_m(z) = q_m(\infty) + \frac{1}{2\pi} \int_{l_{1-2h}} \left(\frac{\partial q_m(\zeta)}{\partial \mathbf{n}_\zeta} \log|\zeta - z| - q_m(\zeta) \frac{\partial}{\partial \mathbf{n}_\zeta} \log|\zeta - z| \right) |d\zeta|,$$

where $\partial/\partial \mathbf{n}_\zeta$ is the operator of differentiation with respect to the outward normal to the curve l_{1-2h} at the point ζ.

Integrating with respect to $d\sigma$ and applying Fubini's Theorem, we have

$$\int q_m \, d\sigma = -\frac{1}{2\pi} \int_{l_{1-2h}} \left(U^\sigma(\zeta) \frac{\partial q_m(\zeta)}{\partial \mathbf{n}_\zeta} - q_m(\zeta) \frac{\partial}{\partial \mathbf{n}_\zeta} U^\sigma(\zeta) \right) |d\zeta|. \tag{5.5}$$

Our next aim is to derive an upper bound for the expression on the right-hand side of (5.5).

By (5.2) and the analogue of the Bernstein-Walsh lemma for rational functions (cf. (4.5)), we have, for $z \in G \setminus \{z_0\}$, that

$$|q_m(z)| \le |Q_m(z)| \le \frac{c_6}{|\phi(z)|^m}, \tag{5.6}$$

which, in particular, implies, for $z \in l_{1-2h}$, $\tilde{Q}_m(\tau) := Q_m(\psi(\tau))$ and $w = \phi(z)$, that

$$\begin{aligned} |\operatorname{grad} q_m(z)| &= |Q_m'(z)| = |\tilde{Q}_m'(w)||\Phi'(z)| \\ &\le |\Phi'(z)| \frac{1}{2\pi} \int_{|\tau - w| = h} \frac{\tilde{Q}_m(\tau)}{|\tau - w|^2} |d\tau| \\ &\le |\Phi'(z)| \frac{1}{h} \sup_{z \in l_{1-3h}} |Q_m(z)| \le \frac{c_7 |\Phi'(z)|}{h(1-3h)^m}. \end{aligned}$$

Without loss of generality, we assume that

$$b := b_\sigma^*(h) = b_\sigma(h).$$

The reasoning given below shows that the other case, i.e. when $b_\sigma^*(h) = b_{-\sigma}(h)$, can be handled in the same way.

A routine argument, involving the last estimate and the mean value property of harmonic functions, shows that

$$\left|\int_{l_{1-2h}} U^\sigma(\zeta)\frac{\partial q_m(\zeta)}{\partial \mathbf{n}_\zeta}\,|d\zeta|\right| \le \int_{l_{1-2h}} (b - U^\sigma(\zeta) + U^\sigma(0))\left|\frac{\partial q_m(\zeta)}{\partial \mathbf{n}_\zeta}\right|\,|d\zeta| \le \frac{c_8 b}{h(1-3h)^m}.$$

Further, by (5.6),

$$\left|\int_{l_{1-2h}} q_m(\zeta)\frac{\partial}{\partial \mathbf{n}_\zeta}U^\sigma(\zeta)\,|d\zeta|\right| \le \frac{c_9}{(1-2h)^m}\int_{l_{1-2h}} |\operatorname{grad} U^\sigma(\zeta)|\,|d\zeta| = \frac{c_9}{(1-2h)^m}\int_{|w|=1-2h} |\operatorname{grad} \tilde{U}^\sigma(w)|\,|dw|,$$

where $\tilde{U}^\sigma(w) = U^\sigma(\psi(w))$.

Next we use Schwarz's formula, Fubini's Theorem and the mean value property of harmonic functions to obtain

$$\begin{aligned}\int_{|w|=1-2h} |\operatorname{grad} \tilde{U}^\sigma(w)|\,|dw| &= \int_{|w|=1-2h} |\operatorname{grad}(b - \tilde{U}^\sigma(w) + \tilde{U}^\sigma(0))|\,|dw| \\ &\le \frac{1}{\pi}\int_{|w|=1-2h}\int_{|t|=1-h} \frac{b - \tilde{U}^\sigma(t) + \tilde{U}^\sigma(0)}{|t-w|^2}\,|dt||dw| \\ &= \frac{1}{\pi}\int_{|t|=1-h} (b - \tilde{U}^\sigma(t) + \tilde{U}^\sigma(0))\int_{|w|=1-2h}\frac{|dw|}{|t-w|^2}|dt| \\ &\le \frac{c_{10} b}{h}.\end{aligned}$$

Hence,

$$\left|\int q_m\,d\sigma\right| \le \frac{c_{11} b}{h(1-3h)^m}.$$

Next, we note that by (4.1), (5.1), (5.3), (5.4) and the last inequality,

$$\begin{aligned}
&\sigma^+(J) - \sigma^-(N(J,\delta)) \\
&\quad = \int_J q_m\, d\sigma^+ + \int_J (1-q_m)\, d\sigma^+ - \int_{N(J,\delta)} q_m\, d\sigma^- - \int_{N(J,\delta)} (1-q_m)\, d\sigma^- \\
&\quad \geq \int_{\mathbf{C}} q_m\, (d\sigma^+ - d\sigma^-) - \int_{J_1\cup J_2} d\sigma^+ - \int_{J_3} q_m\, d\sigma^+ - \int_{N(J,\delta)} (1-q_m)\, d\sigma^- \\
&\quad \geq -c_{12}\left(\frac{b}{h(1-3h)^m} + \delta^{\kappa\varepsilon} + \frac{1}{\delta^{\varepsilon/2} m^\beta}\right) := -B.
\end{aligned}$$

The same inequality holds for $L \setminus J$ instead of J, i.e.

$$\sigma^+(L\setminus J) - \sigma^-(N(L\setminus J,\delta)) \geq -B.$$

Since

$$\sigma^+(J) - \sigma^-(N(J,\delta)) + \sigma^+(L\setminus J) - \sigma^-(N(L\setminus J,\delta)) \leq 0,$$

we have

$$\sigma^+(J) - \sigma^-(N(J,\delta)) \leq -(\sigma^+(L\setminus J) - \sigma^-(N(L\setminus J,\delta))) \leq B,$$

which completes the proof of Theorem 2.

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