2-4

J. Soc. Indust. Appl. Math.
Vol. 11, No. 4, December, 1963
Printed in U.S.A.

NUMERICAL SOLUTION OF PLANE ELASTICITY PROBLEMS

D. S. GRIFFIN AND R. S. VARGA

J. Soc. Indust. Appl. Math.
Vol. 11, No. 4, December, 1963
Printed in U.S.A.

NUMERICAL SOLUTION OF PLANE ELASTICITY PROBLEMS*

D. S. GRIFFIN† and R. S. VARGA‡

1. Introduction. It has been known for some time that the stress distribution in a plane bounded region with an irregularly shaped boundary may be accurately approximated by finite difference methods. Indeed, some twenty years ago, Southwell and his co-workers [1, 2] utilized finite difference approximations, and applied Southwell's "relaxation" method, a *noncyclic* or *free-steering*[1] iterative procedure, to the approximate solution of interesting engineering problems. Though this "relaxation" method has been known to work well when human insight guides the course of calculations on a desk calculator, such noncyclic iterative techniques become extremely laborious on desk calculators when results of high precision are required. More damaging is the fact that such noncyclic iterative techniques can rarely be adapted efficiently to high-speed digital computers.

On the other hand, newer *cyclic*[1] iterative techniques, such as variants of the successive overrelaxation (SOR) and alternating direction implicit (ADI) methods, are readily adapted to high-speed digital computers, and recently have been applied to the numerical solution of biharmonic stress problems. Motivated by work of Heller [4], Varga [5] and Parter [6, 7] have applied the 2-line variant of the SOR method to the numerical solution of the biharmonic equation with clamped boundary conditions for a rectangle with uniform mesh spacings. Similarly, Conte and Dames [8, 9] have applied the Douglas-Rachford variant of the ADI method to the same problem with simply supported boundary conditions. Unfortunately, as shown by Birkhoff and Varga [10], the results of Conte and Dames are theoretically restricted to rectangular domains. Stiefel and his coworkers [11] have recently investigated the practical application of particular variants of the SOR method, as well as "direct" methods such as the Gaussian elimination and conjugate gradient methods to more typical biharmonic problems. They conclude that overrelaxation can be promising, if relaxation factors can be appropriately chosen.

The main purpose of this paper is to present a general method for the efficient numerical solution of plane elasticity problems which cannot

---

* Received by the editors April 18, 1963.

† Bettis Atomic Power Laboratory, Pittsburgh, Pennsylvania. The Laboratory is operated for the U. S. Atomic Energy Commission by Westinghouse Electric Corporation.

‡ Case Institute of Technology, Cleveland, Ohio.

[1] For a precise definition of cyclic and noncyclic or free-steering iterative methods, see for example Ostrowski [3].

be solved practically by either classical analytical or relaxation methods. Using nonuniform meshes, difference equations are derived which approximate the differential equation and boundary conditions suitably for general irregularly-shaped simply-connected[2] regions. It is shown that the matrix of coefficients is real, symmetric, and positive definite, so that a newer 2-line variant of the SOR method, using the cyclic Chebyshev semi-iterative technique of Golub and Varga [13], can be rigorously applied to the solution of these difference equations. Finally, numerical results are presented for an example which demonstrate the applicability and efficiency of the method.

2. The boundary value problem. Considering a simply connected plane region R with boundary Γ, assume a body force potential function $V(x, y)$ and a temperature distribution $T(x, y)$ are given on $\bar{R}$, the union of R and Γ. When the stresses in the plane are expressed in terms of the Airy stress function $\phi(x, y)$ and $V(x, y)$ by

$$(1) \qquad \sigma_x = \frac{\partial^2\phi}{\partial y^2} + V, \qquad \sigma_y = \frac{\partial^2\phi}{\partial x^2} + V, \qquad \tau_{xy} = -\frac{\partial^2\phi}{\partial x \partial y},$$

the equilibrium equations of plane elasticity are identically satisfied, and the compatibility of strains requires that

$$(2) \quad \nabla^4\phi(x, y) = -\beta\nabla^2 V(x, y) - \gamma\nabla^2 T(x, y) = q(x, y), \quad (x, y) \in R,$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$$\nabla^4 = \nabla^2\nabla^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4},$$

$\beta = 1 - \nu, \qquad \gamma = \alpha E$ for plane stress, or
$\beta = (1 - 2\nu)/(1 - \nu), \qquad \gamma = \alpha E/(1 - \nu)$ for plane strain,
α is the coefficient of thermal expansion,
ν is Poisson's ratio,
E is Young's modulus of elasticity.

The boundary conditions for $\phi(x, y)$ are assumed to take one of the following forms on each portion of Γ:

$$(3a) \qquad \frac{\partial\phi(x, y)}{\partial x} = g_1(x, y), \qquad \frac{\partial\phi(x, y)}{\partial y} = g_2(x, y), \qquad (x, y) \in \Gamma,$$

[2] A treatment for multiply-connected regions is described in [12].

$$\text{(3b)} \qquad \phi(x, y) = g_3(x, y), \qquad \frac{\partial \phi(x, y)}{\partial n} = g_4(x, y), \qquad (x, y) \in \Gamma,$$

$$\text{(4)} \qquad \frac{\partial \phi(x, y)}{\partial n} = 0, \qquad \frac{\partial}{\partial n} \nabla^2 \phi(x, y) = 0, \qquad (x, y) \in \Gamma,$$

where the functions $g_i(x, y)$ are given and n refers to the outward pointing normal.

The boundary conditions (3a, b) are those which arise when forces are specified on Γ. To see this, let s and n be boundary coordinates tangent and normal to Γ, and let θ be the angle between the x and n coordinates. If $X(x, y)$ and $Y(x, y)$ are components of the boundary force in the directions of x and y, equilibrium of forces on Γ requires that

$$\text{(5)} \qquad -\frac{\partial}{\partial s}\left(\frac{\partial \phi}{\partial x}\right) = Y - V \sin \theta, \qquad \frac{\partial}{\partial s}\left(\frac{\partial \phi}{\partial y}\right) = X - V \cos \theta.$$

Integrating these equations along Γ, we obtain the boundary condition of (3a):

$$\text{(6)} \qquad \begin{aligned} \frac{\partial \phi}{\partial x} &= -\int_0^s (Y - V \sin \theta)\, ds' + \left(\frac{\partial \phi}{\partial x}\right)_0 \equiv g_1, \\ \frac{\partial \phi}{\partial y} &= \int_0^s (X - V \cos \theta)\, ds' + \left(\frac{\partial \phi}{\partial y}\right)_0 \equiv g_2, \end{aligned}$$

where $(\partial\phi/\partial x)_0$, and $(\partial\phi/\partial y)_0$ are constants of integration arising from the fact that only second derivatives of ϕ are specified on Γ. Since R is simply connected, these constants have no effect on the stress solution (1) and may thus be chosen arbitrarily. Finally, the equations of (6) can be combined and integrated to obtain the boundary conditions (3b) which specify $\partial\phi/\partial n$ and ϕ on Γ.

The boundary conditions of (4) are obtained for a line of physical symmetry if the constants of integration are chosen in such a manner that ϕ is also symmetric about the same line.

The determination of the stresses σ_x, σ_y, and τ_{xy} in $\bar{R}$ due to a given temperature distribution, body force potential, and boundary forces has been reduced to the determination of a function $\phi(x, y)$ which satisfies (2) in R, and appropriate boundary conditions of the form (3) and (4) on Γ. Because of this, we note that the results to follow apply equally well to the bending of a clamped plate. The Airy stress function is analogous to the deflection of a plate loaded transversely with edge deflection and slope specified.

3. The derivation of difference equations. In order to derive linear difference equations which approximate the differential equation of (2) and

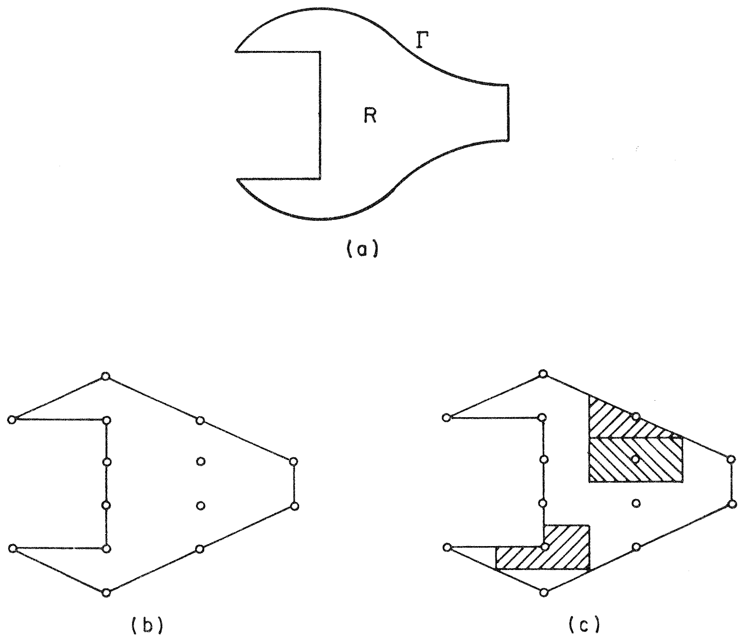

FIG. 1.

the various boundary conditions of (3) and (4), the boundary Γ of the region R is first approximated by a polygonal boundary Γ_h. Then, a rectangular mesh of horizontal and vertical mesh lines, in general not uniformly spaced, is imposed on R in such a manner that the intersections of the mesh lines coincide with the intersections of the boundary segments of Γ_h. The intersections of the mesh lines are called *mesh points* and give rise to a finite point set S. Associated with each mesh point is a *mesh region*, defined so that the boundaries of each mesh region fall halfway between the mesh lines. The transition from the original boundary Γ to the polygonal boundary Γ_h, and then to the set S of mesh points (denoted by small circles) is illustrated in Figs. 1(a) and 1(b). Representative mesh regions (denoted by shaded areas) are illustrated in Fig. 1(c).

The derivation of difference equations which approximate the differential equation of (2) can be carried out in a number of ways, the most familiar being based on Taylor's series expansions. Less familiar, perhaps, is a derivation based on an integration technique which is very convenient for approximating differential equations for irregularly shaped regions using a nonuniform mesh. It might be mentioned that this integration technique[3]

[3] For related treatments of this integration technique for deriving difference equations, see also [15] and [16].

[14, Ch. 6] has been successfully employed for a number of years in computer programs for the design of nuclear reactors, and is closely related [14, Ch. 6] to the derivation of difference approximations using the variational formulation of (2) and (3). Equation (2) is integrated over each mesh region r_i, giving

$$\iint_{r_i} \nabla^4\phi \, dx \, dy = -\iint_{r_i} (\beta\nabla^2 V + \gamma\nabla^2 T) \, dx \, dy, \tag{7}$$

which is equivalent to (2) if such integrations are taken over every possible subregion in R. Using Green's theorem, the area integrals can be replaced by line integrals

$$\oint_{\gamma_i} \frac{\partial}{\partial n} (\nabla^2\phi) \, d\ell = -\oint_{\gamma_i} \left(\beta \frac{\partial V}{\partial n} + \gamma \frac{\partial T}{\partial n}\right) d\ell \tag{8}$$

taken about the boundary γ_i of each subregion r_i.

Ultimately, we seek approximations to the Airy stress function $\phi(x, y)$ only at the mesh points i of the finite set S, $1 \leqq i \leqq n$. Denoting the approximations to ϕ at these points by ϕ_i, some of the values ϕ_i at boundary points may be known directly from the boundary conditions of (3b). At interior points, or boundary points where ϕ is unknown such as in (4), the values of ϕ_i are related by a set of n linear difference equations deduced from (8) by approximating the portions of the line integrals as follows. With reference to Fig. 2, the line integral from a to c is approximated by

$$\int_a^c \frac{\partial}{\partial n} \nabla^2\phi \, d\ell \doteq \left(\frac{\partial}{\partial n} \nabla^2\phi\right)_b \cdot \ell_{ac}$$

and the normal derivative at b is approximated via central differences by

$$\left(\frac{\partial}{\partial n} \nabla^2\phi\right)_b \doteq \frac{\delta^2\phi_1 - \delta^2\phi_0}{h_{01}}.$$

In this way, the difference approximation for (8) at a typical point 0 of Fig. 2 becomes

$$\sum_{i=1}^{4} (\delta^2\phi_i - \delta^2\phi_0) \frac{\ell_{0i}}{h_{0i}} = -\sum_{i=1}^{4} [\beta(V_i - V_0) + \gamma(T_i - T_0)] \frac{\ell_{0i}}{h_{0i}} \tag{9}$$

where

$\delta^2\phi_i$ is a difference approximation of $\nabla^2\phi$ at i,
h_{0i} is the mesh length between points 0 and i,
ℓ_{0i} is the length of the side of the mesh region between points 0 and i, i.e., in Fig. 2, $\ell_{01} = \ell_{ac}$.

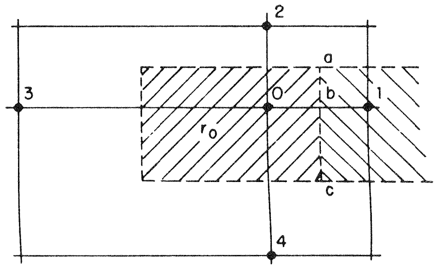

FIG. 2.

It remains now to specify the differences $\delta^2\phi_i$. At any point 0, we have

$$(10) \qquad (\nabla^2\phi)_0 \doteq \frac{1}{A_0}\iint_{r_0} \nabla^2\phi \, dx\, dy = \frac{1}{A_0}\oint_{\gamma_0} \frac{\partial \phi}{\partial n}\, d\ell,$$

where A_0 is the area of the mesh region r_0. Again referring to Fig. 2 we define

$$(11) \qquad \delta^2\phi_0 = \frac{1}{A_0}\sum_{i=1}^{4} (\phi_i - \phi_0)\frac{\ell_{0i}}{h_{0i}},$$

in analogy with (9), as the difference approximation to $(\nabla^2\phi)_0$.

It is obvious that these simple approximations based on mesh lengths and areas can be readily calculated in a program for a digital computer. It is less obvious that for the case of uniform mesh spacings, the simple approximations of (11), when coupled with (9), reduce to the well known [17, p. 267] 13 point biharmonic star

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1\,. \\ & 2 & -8 & 2 & \\ & & 1 & & \end{array}$$

This is the unique point approximation with the least number of couplings, namely 13, which, from Taylor's series considerations, approximates the biharmonic equations to h^2 accuracy on a square mesh.

The derivation of difference equations at boundary points proceeds in a manner similar to that just described for interior points. For example, when the point 0 is on a line of symmetry, as indicated in Fig. 3, the line integrals in (8) are taken around the half mesh region associated with the point 0. Since $(\partial/\partial n)\nabla^2\phi = 0$ on the line of symmetry from (4), $\int(\partial/\partial n)\nabla^2\phi d\ell = 0$ on

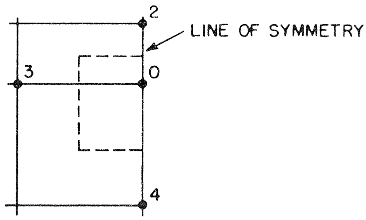

FIG. 3.

the side of the mesh region coinciding with the line of symmetry. Numbering the mesh points as in Fig. 3, the difference approximation for (8) becomes

$$(12)\quad \sum_{i=2}^{4} (\delta^2\phi_i - \delta^2\phi_0)\frac{\ell_{0i}}{h_{0i}} = -\sum_{i=2}^{4} [\beta(V_i - V_0) + \gamma(T_i - T_0)]\frac{\ell_{0i}}{h_{0i}},$$

where it is understood that ℓ_{02} and ℓ_{04} refer to sides of the half mesh region, i.e., $\ell_{02} = \ell_{04} = h_{03}/2$. Similar modifications in (11), i.e., integrations are again carried out over partial mesh regions, give rise to linear approximations to $\delta^2\phi_i$ at a point on a line of symmetry.

This method applies equally well at points on Γ at which the boundary conditions of (3b) are specified. An illustrative example is given in Appendix 1. We might remark that, in the proposed method of derivation, boundary conditions are always treated in conjunction with the differential equation, and no exterior ("imaginary") points are necessary.

4. Iterative solution of the difference equations. With (9) and (11) for each interior mesh point, and similar expressions for boundary points, a set of n linear equations in the unknowns ϕ_i is obtained. Numbering the unknowns ϕ_i in some systematic way, the resulting system of linear equations can be written in the form

$$(13)\quad A\boldsymbol{\phi} = \mathbf{p},$$

where A is an $n \times n$ real coefficient matrix, $\boldsymbol{\phi}$ is a column vector with components ϕ_i, and $\mathbf{p}$ is a known column vector which represents the effects of the boundary forces X and Y, the body force potential function V, and the temperature T at each mesh point i. The coefficient matrix A can be assumed in cases of practical interest to be of large order, i.e., n may be several thousand, but each ϕ_i is coupled to at most twelve adjacent ϕ_i's. In other words, most of the entries of A are zero, and A is said to be *sparse*.

In order to rigorously apply the cyclic Chebyshev semi-iterative method [13] to the solution of the matrix problem of (13), it must be shown that the coefficient matrix A is symmetric and positive definite. It is almost obvious from the derivation of the difference equations that A is *sym-*

metric; this is basically a consequence of the fact that, at each interior point 0 (Fig. 2), each line integral (8) occurs again, with a different orientation, in the derivation of difference equations at the adjacent points 1, 2, 3, and 4. It is less obvious that the matrix A is positive definite, but this is intuitively plausible since the method of deriving difference equations based on integration is analogous to the derivation of difference equations from a variational formulation. The details showing that A is positive definite are given in Appendix 2.

With the coefficient matrix A of (13) real, symmetric, and positive definite, we now number the unknowns i of the set S consecutively by double horizontal rows, counting from left to right and alternating between rows. This is illustrated in Fig. 4 for our basic mesh problem of Fig. 1. Since the points in any pair of rows are coupled only to points in adjacent pairs of rows, the matrix A can be partitioned in the form

$$(14) \qquad A = \begin{bmatrix} A_{1,1} & A_{1,2} & & & & \\ A_{1,2}^T & A_{2,2} & A_{2,3} & & & \boldsymbol{O} \\ & A_{2,3}^T & A_{3,3} & A_{3,4} & & \\ & & \ddots & \ddots & \ddots & \\ & \boldsymbol{O} & & & & A_{t-1,t} \\ & & & & A_{t-1,t}^T & A_{t,t} \end{bmatrix},$$

where t is the number of double rows of mesh points, $A_{j,j}$ is an $n_j \times n_j$ symmetric and positive definite matrix, and n_j is the number of mesh points in the jth block of two mesh lines. Specifically, for the problem in Fig. 4, t is three and $A_{1,1}$ is a 4×4 matrix, $A_{2,2}$ is a 6×6 matrix, and $A_{3,3}$ is a 4×4 matrix.

Partitioning the vectors $\boldsymbol{\phi}$ and $\mathbf{p}$ in a manner compatible with (14), the difference equations (13) can be written in the form

$$(15a) \qquad A_{j-1,j}^T \boldsymbol{\Phi}_{j-1} + A_{j,j}\boldsymbol{\Phi}_j + A_{j,j+1}\boldsymbol{\Phi}_{j+1} = \mathbf{P}_j, \qquad j = 1, 2, \cdots, t,$$

or equivalently

$$(15b) \qquad A_{j,j}\boldsymbol{\Phi}_j = \mathbf{P}_j - A_{j-1,j}^T \boldsymbol{\Phi}_{j-1} - A_{j,j+1}\boldsymbol{\Phi}_{j+1}, \qquad j = 1, 2, \cdots, t,$$

where $A_{0,1}^T$ and $A_{t,t+1}$ are null matrices and $\boldsymbol{\Phi}_j$ and $\mathbf{P}_j$ are column vectors with n_j components. The nature of the derivation of difference equations, i.e., each point is coupled to at most twelve of its neighbors, is such that each point is coupled to at most nine neighbors in the same pair of hori-

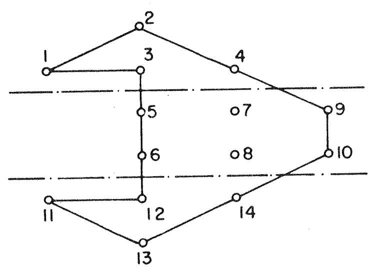

FIG. 4.

zontal mesh lines, and the particular ordering of mesh points is such that each of the diagonal submatrices $A_{j,j}$ is a nine-diagonal matrix, i.e., if $A_{j,j} = (a_{k,\ell}^{(j)})$, then $a_{k,\ell}^{(j)} = 0$ for $|k - \ell| > 4$. Moreover, as the $A_{j,j}$ are principal submatrices of the positive definite symmetric matrix A, they are also symmetric, positive definite and nonsingular. Hence, matrix equations of the form

$$A_{j,j}\mathbf{\Phi}_j = \mathbf{g}_j \tag{16}$$

where $\mathbf{g}_j$ is given, can be solved directly by means of Gaussian elimination with little accumulation of rounding errors [18].

The fact that matrix equations such as (16) can be solved efficiently serves to define the cyclic Chebyshev semi-iterative method applied to (15a). The two line mesh blocks are numbered consecutively from 1 to t, and (15b) is used for all odd j, followed by all even j. The iterative procedure is

$$\begin{aligned} A_{2j+1,2j+1}\,\mathbf{\Phi}_{2j+1}^{*(2m+1)} &= \mathbf{P}_{2j+1} - A_{2j,2j+1}^{T}\,\mathbf{\Phi}_{2j}^{(2m)} - A_{2j+1,2j+2}\,\mathbf{\Phi}_{2j+2}^{(2m)}, \\ \mathbf{\Phi}_{2j+1}^{(2m+1)} &= \mathbf{\Phi}_{2j+1}^{(2m-1)} + \omega_{2m+1}[\mathbf{\Phi}_{2j+1}^{*(2m+1)} - \mathbf{\Phi}_{2j+1}^{(2m-1)}], \\ & 0 \leqq j \leqq \left[\frac{t-1}{2}\right], \qquad m \geqq 0, \end{aligned} \tag{17}$$

and

$$\begin{aligned} A_{2j,2j}\,\mathbf{\Phi}_{2j}^{*(2m+2)} &= \mathbf{P}_{2j} - A_{2j-1,2j}^{T}\,\mathbf{\Phi}_{2j-1}^{(2m+1)} - A_{2j,2j+1}\,\mathbf{\Phi}_{2j+1}^{(2m+1)}, \\ \mathbf{\Phi}_{2j}^{(2m+2)} &= \mathbf{\Phi}_{2j}^{(2m)} + \omega_{2m+2}[\mathbf{\Phi}_{2j}^{*(2m+2)} - \mathbf{\Phi}_{2j}^{(2m)}], \\ & 1 \leqq j \leqq \left[\frac{t}{2}\right], \qquad m \geqq 0. \end{aligned} \tag{18}$$

Here, $[m]$ denotes the greatest integer less than or equal to m. The numbers

ω_j are computed recursively from

$$(19)\qquad \omega_1 = 1, \qquad \omega_2 = \frac{2}{(2 - \rho^2)}, \qquad \omega_{j+1} = \frac{1}{1 - (\rho^2\omega_j/4)}, \qquad j \geqq 2,$$

where ρ is an estimate of the spectral radius of an associated iteration matrix ($0 \leqq \rho < 1$). An initial guess is required for the subvectors $\mathbf{\Phi}_{2j}^{(0)}$ from which $\mathbf{\Phi}_{2j+1}^{(1)}$, $\mathbf{\Phi}_{2j}^{(2)}$, etc. are calculated. From the results of [13], the iterative procedure of (17) and (18) will converge for any set of initial subvectors $\mathbf{\Phi}_j^{(0)}$ and any ρ with $0 < \rho < 1$.

The iterations are terminated when

$$(20)\qquad \max\{R_1^{(2m+1)}, R_0^{(2m+2)}\} \leqq \delta,$$

where

$$(21)\qquad \begin{aligned} R_1^{(2m+1)} &= \sum_j \|\mathbf{\Phi}_{2j+1}^{(2m+1)} - \mathbf{\Phi}_{2j+1}^{(2m-1)}\|, & m \geqq 1, \\ R_0^{(2m)} &= \sum_j \|\mathbf{\Phi}_{2j}^{(2m)} - \mathbf{\Phi}_{2j}^{(2m-2)}\|, & m \geqq 1, \end{aligned}$$

and it is understood that $\| \mathbf{v} \| \equiv \sum_{i=1}^{n} | v_i |$. The quantity δ is an input parameter which measures the difference between successive vector iterates. Of course, smaller values of δ give rise to increasingly accurate approximations of the solution of (13).

The convergence rate of the process described in (17) and (18) is highly dependent upon the choice of ρ, especially when the coefficient matrix A is almost singular. In general, it is difficult to obtain a good estimate of the spectral radius without actually solving a related eigenvalue problem. But for the case of a rectangular region with uniform mesh spacing, a good working estimate of ρ is given in [19]. If a and b are lengths of the sides and h the mesh spacing,

$$(22)\qquad \rho = \frac{1}{1 + (\lambda^2 h^4/2)},$$

where

$$\lambda^2 = (\pi/a)^4[5.144(1 + a^4/b^4) + 3.115\, a^2/b^2].$$

The effect of estimates of ρ and the accuracy δ on the rate of convergence of the iterative procedure are discussed in some detail in [20].

5. A numerical example. In order to demonstrate the usefulness of this method for determining the stresses in irregularly shaped plane regions, it has been applied to the tension member with semi-circular notches shown in Fig. 6. This problem has been treated by Southwell using the relaxation technique [2, p. 284], and an estimate of the stress at the notch can be obtained using series methods (see, e.g., [21]). For this problem, $T = V = 0$ on $\bar{R}$.

By symmetry, only a quarter of the plate need be considered. A relatively coarse nonuniform mesh spacing was used so the difference equations could be solved exactly as well as iteratively. However, finer mesh spacings were used near the root of the notch where the maximum stress occurs. The mesh layout, shown in Fig. 5, contains only 66 interior mesh points. An iterative solution using the method described here was obtained with the aid of an experimental program written for the Philco 2000 computer. The iterative solution required 239 iterations, taking 25.6 seconds on the Philco 2000. An exact solution of the difference equations was also obtained using the Gaussian elimination method. Comparison with the iterative solution showed agreement to four significant figures.

With a solution for ϕ, the stress σ_x can be calculated at an interior point (Fig. 2) using the difference approximation

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \doteq \frac{1}{A_0} \sum_{i=2,4} (\phi_i - \phi_0) \frac{\ell_{0i}}{h_{0i}} .$$

Values of σ_x at each mesh point are shown in Fig. 5. They compare well with those obtained by Southwell using a uniform mesh length of $a/4$ with a total of 116 interior mesh points. A comparison of Southwell's results with those presented here is shown in Fig. 6 for the stress distribution across the neck of the plate. The maximum stress at the notch in an infinitely long strip, according to [21], is 3.08. For the finite length strip shown in Fig. 6, the maximum stress should be somewhat larger, so the value of 3.15 obtained here appears to be quite satisfactory. It seems remarkable to us

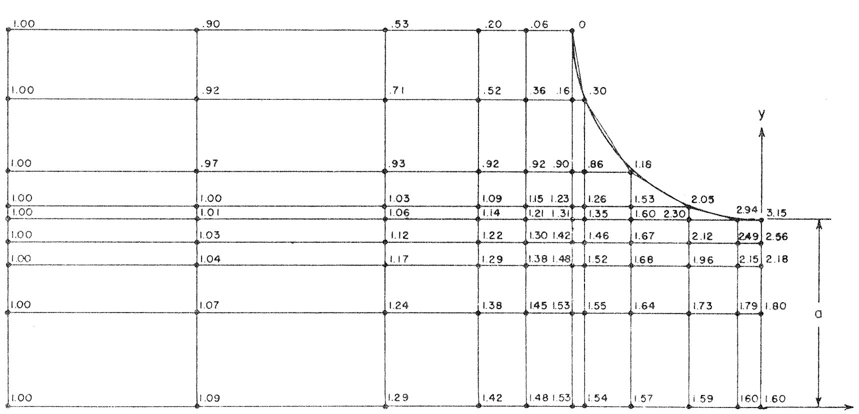

FIG. 5. σ_x *Distribution for the plate shown in Fig. 6.*

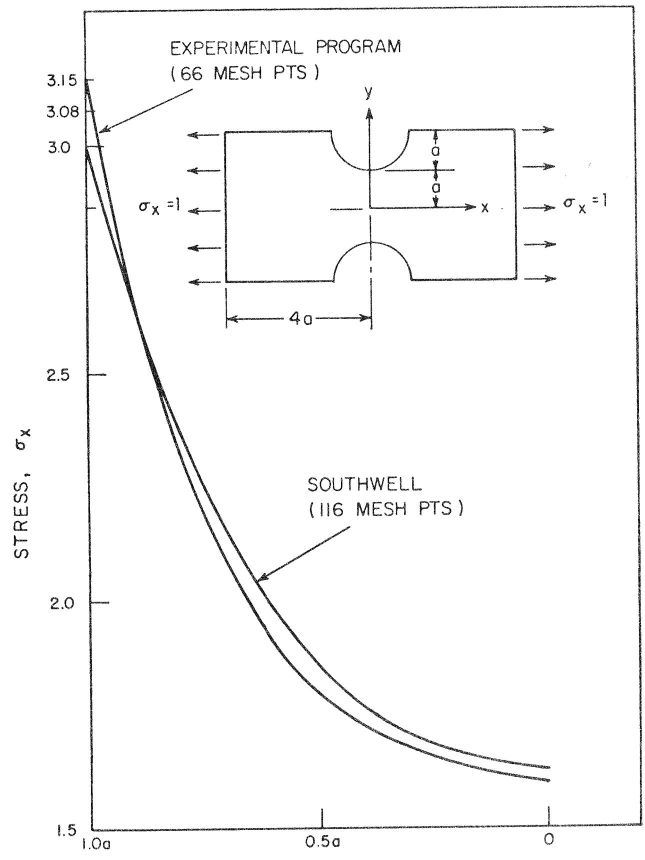

FIG. 6.

that such a close approximation can be obtained for the stress on a curved boundary with so few mesh points.

Appendix 1. Derivation of difference equations for boundary mesh regions. To illustrate the derivation of difference equations at a point on Γ where boundary conditions (3b) are specified, consider the mesh region of Fig. 7, which is outlined by $0abc0$. An approximation for ϕ at point 0 is obtained by integration of (6). Thus, to complete the derivation of difference equations at point 0, we seek an approximation $\delta^2\phi_0$ to $(\nabla^2\phi)_0$, such as is given in (11). Using (10) for the mesh region of Fig. 7, the problem is further reduced to approximating line integrals involving $\partial\phi/\partial n$.

The integral of $\partial\phi/\partial n$ along the sides $c0$ and $0a$ is approximated by $(\partial\phi/\partial n)_{0-}\cdot h_{0c}$ and $(\partial\phi/\partial n)_{0+}\cdot h_{0a}$ where $(\partial\phi/\partial n)_{0-}$ and $(\partial\phi/\partial n)_{0+}$ are ap-

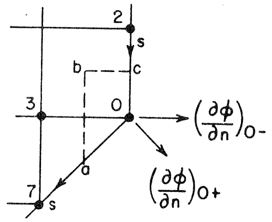

FIG. 7.

proximate values of the normal derivative on the segments 02 and 07, respectively, to be determined shortly. With these approximations, the difference equation for $\delta^2\phi$ at the boundary point 0 becomes

$$\delta^2\phi_0 = \frac{1}{A_0}\left[\sum_{i=2,3}(\phi_i - \phi_0)\frac{\ell_{0i}}{h_{0i}} + \frac{h_{02}}{2}\left(\frac{\partial\phi}{\partial n}\right)_{0-} + \frac{h_{07}}{2}\left(\frac{\partial\phi}{\partial n}\right)_{0+}\right], \tag{23}$$

where

$$A_0 = \frac{h_{02}\,h_{03}}{4} + \frac{h_{03}\,h_{37}}{8}.$$

The functions ϕ_0 and ϕ_2 can be determined directly from the boundary conditions, so the only unknown in this expression is ϕ_3.

It remains only to derive approximations for $(\partial\phi/\partial n)_{i-}$, $(\partial\phi/\partial n)_{i+}$, and boundary values of ϕ_i from the boundary conditions (6). The boundary of the plane region is composed of straight line segments which are sufficiently short that it can be assumed X, Y, and V of (5) are constant over the length of each segment. Numbering the boundary points in order $n + 1$, $n + 2, \cdots, k, \cdots, m$ in the direction of s, the line integrations in (6) can be carried out segment by segment going from $n + 1$ to $n + 2$, $n + 2$ to $n + 3$, and so forth to obtain approximations for ϕ, $\partial\phi/\partial x$, and $\partial\phi/\partial y$ at each point in terms of their values at the preceding point. Integrating from the $(k - 1)$th to the kth point, shown in Fig. 8,

$$\begin{aligned}
\left(\frac{\partial\phi}{\partial x}\right)_k &= -(Y_{k-1,k} - V_{k-1,k}\sin\theta_{k-1,k})h_{k-1,k} + \left(\frac{\partial\phi}{\partial x}\right)_{k-1}, \\
\left(\frac{\partial\phi}{\partial y}\right)_k &= (X_{k-1,k} - V_{k-1,k}\cos\theta_{k-1,k})h_{k-1,k} + \left(\frac{\partial\phi}{\partial y}\right)_{k-1}, \\
\phi_k &= (X_{k-1,k}\cos\theta_{k-1,k} + Y_{k-1,k}\sin\theta_{k-1,k} - V_{k-1,k})\frac{h_{k-1,k}^2}{2} \\
&\quad + \left[\left(\frac{\partial\phi}{\partial y}\right)_{k-1}\cos\theta_{k-1,k} + \left(\frac{\partial\phi}{\partial x}\right)_{k-1}\sin\theta_{k-1,k}\right]h_{k-1,k} + \phi_{k-1},
\end{aligned} \tag{24}$$

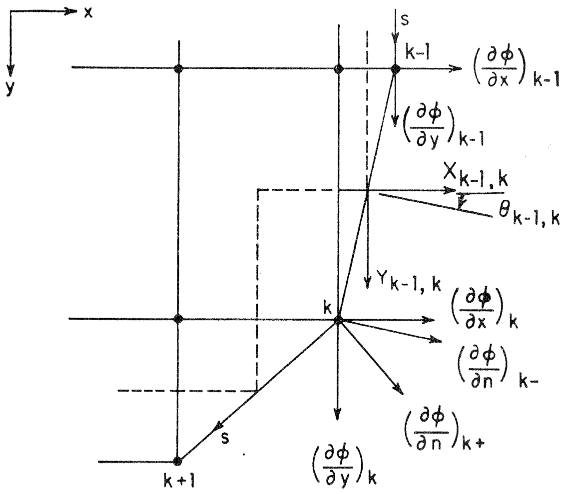

FIG. 8.

where the subscripts $k - 1$, k refer to the value of the function on the segment between points $k - 1$ and k. The normal derivatives on the segments $k - 1$, k and k, $k + 1$ at the point k can similarly be found from

$$(25) \qquad \begin{aligned} \left(\frac{\partial\phi}{\partial n}\right)_{k-} &= \left(\frac{\partial\phi}{\partial x}\right)_k \cos\theta_{k-1,k} + \left(\frac{\partial\phi}{\partial y}\right)_k \sin\theta_{k-1,k}\,, \\ \left(\frac{\partial\phi}{\partial n}\right)_{k+} &= \left(\frac{\partial\phi}{\partial x}\right)_k \cos\theta_{k,k+1} + \left(\frac{\partial\phi}{\partial y}\right)_k \sin\theta_{k,k+1}\,. \end{aligned}$$

Since the integration constants ϕ_{n+1}, $(\partial\phi/\partial x)_{n+1}$, and $(\partial\phi/\partial y)_{n+1}$ may be chosen arbitrarily, ϕ_k, $(\partial\phi/\partial n)_{k-}$, and $(\partial\phi/\partial n)_{k+}$ can be determined recursively by (24) and (25).

Appendix 2. The positive definite nature of the matrix A. As stated in §4, the real square matrix A of (13) is symmetric and positive definite, the symmetry of A being an immediate consequence of the derivation of difference equations based on integration. To show that A is positive definite, consider the vector $\boldsymbol{\psi}$ with components ψ_i which are related (Fig. 2) by

$$(26) \qquad -\sum_{i=1}^{4} (\psi_i - \psi_0) \frac{\ell_{0i}}{h_{0i}} = 0$$

in the region R, where $\boldsymbol{\psi}$ is assumed to be known on the boundary Γ_h. The resulting set of difference equations is of the form

$$(27) \qquad B\boldsymbol{\psi} = \mathbf{q}$$

where B is real and symmetric, and $\mathbf{q}$ is known from the boundary conditions. By derivation, the matrix B is irreducible,[4] its off-diagonal entries are nonpositive, and the sum of its entries in any row is nonnegative, being strictly positive for at least one row. It follows [14, p. 23] that B is positive definite, and det $B \neq 0$.

If, for the same region R, (9) is applied at interior mesh points, the resulting difference equations can be written in the form

$$-B\boldsymbol{\delta}^2\boldsymbol{\phi} + H\boldsymbol{\phi} = \mathbf{f} \tag{28}$$

where B is given in (27), $\boldsymbol{\delta}^2\boldsymbol{\phi}$ is a column vector with components $\delta^2\phi_i$ $(i = 1, 2, \cdots, n)$ and H is an $n \times n$ real symmetric matrix which appears because boundary values of $\delta^2\phi_i$, rather than known, are expressed in terms of the unknown ϕ_i's. Difference approximations for $\delta^2\phi_i$ obtained from (11) are of the form

$$\boldsymbol{\delta}^2\boldsymbol{\phi} = -D^{-1}B\boldsymbol{\phi} + \mathbf{g} \tag{29}$$

where $\mathbf{g}$ is a known vector, and D is a diagonal matrix whose diagonal entries are positive, and equal to the areas of the various mesh regions. Substitution of (29) into (28) gives a set of difference equations of the form (13) with

$$A = BD^{-1}B + H. \tag{30}$$

Since B is symmetric and nonsingular and D^{-1} is symmetric and positive definite, $BD^{-1}B$ is symmetric and

$$\boldsymbol{\phi}^T(BD^{-1}B)\boldsymbol{\phi} = (B\boldsymbol{\phi})^TD^{-1}(B\boldsymbol{\phi}) > 0 \tag{31}$$

for all $\boldsymbol{\phi} \neq 0$, i.e., $BD^{-1}B$ is positive definite.

The matrix H is of the form

$$H = \sum_{n+1}^{m} H_k \tag{32}$$

where H_k is a matrix which contains all the contributions from the approximation of $\delta^2\phi$ at a boundary mesh point k. Therefore, each H_k is of the form

$$H_k = \frac{1}{A_k}\boldsymbol{\varrho}^T\boldsymbol{\varrho} \tag{33}$$

where A_k is the area of the mesh region for the mesh point k and $\boldsymbol{\varrho}$ is a row vector with at most three nonzero components. If k is a point on a straight

[4] For a definition of irreducibility, see [14, p. 18]. Geometrically, the matrix B has the property of being irreducible basically since the region R is a connected region. For graph theoretic interpretations of this, see [14, p. 20].

boundary, ϱ has only one nonzero component. The nonzero components of ϱ are positive, being just the ratios of the length of the side of the mesh region to the mesh length between the boundary point and each adjacent interior point. From the form of (33), it is obvious that H_k is symmetric and nonnegative definite, i.e.,

$$\phi^T H \phi \geqq 0$$

for any vector ϕ. Thus, it follows from (31) that

$$\phi^T A \phi = \phi^T (BD^{-1}B + H)\phi = \phi^T BD^{-1}B\phi + \phi^T H \phi > 0 \tag{34}$$

for all $\phi \neq 0$, i.e., A is positive definite.

REFERENCES

1. L. Fox and R. V. Southwell, *Biharmonic analysis as applied to the flexure and extension of flat elastic plates*, Philos. Trans. Roy. Soc. Cambridge, 1 (1941), pp. 15–56.
2. R. V. Southwell, *Relaxation Methods in Theoretical Physics*, Oxford University Press, London, 1946.
3. A. M. Ostrowski, *On the linear iteration procedures for symmetric matrices*, Rend. Mat. e Appl., 14 (1954), pp. 140–163.
4. J. Heller, *Simultaneous, successive, and alternating direction iteration schemes*, This Journal, 8 (1960), pp. 150–173.
5. Richard S. Varga, *Factorization and normalized iterative methods*, Boundary Problems in Differential Equations, University of Wisconsin Press, Madison, Wisconsin, 1960, pp. 121–142.
6. Seymour V. Parter, *On 'two-line' iterative methods for the Laplace and biharmonic difference equations*, Numer. Math., 1 (1959), 240–252.
7. ———, *Some computational results on 'two-line' iterative methods for the biharmonic difference equation*, J. Assoc. Comput. Mach., 8 (1961), pp. 359–365.
8. Samuel D. Conte and Ralph T. Dames, *An alternating direction method for solving the biharmonic equation*, Math. Tables Aids Comput., 12 (1958), pp. 198–205.
9. ———, *On an alternating direction method for solving the plate problem with mixed boundary conditions*, J. Assoc. Comput. Mach., 7 (1960), pp. 264–273.
10. Garrett Birkhoff and Richard S. Varga, *Implicit alternating direction methods*, Trans. Amer. Math. Soc., 92 (1959), pp. 13–24.
11. M. Engeli, T. Ginsburg, H. Rutishauser, and E. Stiefel, *Refined Iterative Methods for Computation of the Solution and the Eigenvalues of Self-Adjoint Boundary Value Problems*, Mitteilungen aus dem Institut für angewandte Mathematik Nr. 8, Birkhäuser Verlag, Basel/Stuttgart, 1959.
12. D. S. Griffin and R. S. Varga,[5] *A numerical solution for plane elasticity problems*, WAPD-256, Bettis Atomic Power Laboratory, Pittsburgh, Pennsylvania, (1962).
13. Gene H. Golub and Richard S. Varga, *Chebyshev semi-iterative methods, suc-*

[5] These reports are available from the Office of Technical Services, Department of Commerce, Washington 25, D. C.

cessive overrelaxation iterative methods, and second order Richardson iterative methods, Parts I and II, Numer. Math., 3 (1961), pp. 147–168.

14. RICHARD S. VARGA, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.
15. A. N. TIKHONOV AND A. A. SAMARSKII, *On finite difference methods for equations with discontinuous coefficients*, Dokl. Akad. Nauk SSSR, 108 (1956), pp. 393–396.
16. R. H. MACNEAL, *An asymmetrical finite difference network*, Quart. Appl. Math., 11 (1953), pp. 295–310.
17. L. FOX, *Numerical Solution of Ordinary and Partial Differential Equations*, Addison-Wesley, Reading, Massachusetts, 1962.
18. J. H. WILKINSON, *Error analysis of direct methods of matrix inversion*, J. Assoc. Comput. Mach., 8 (1961), pp. 281–330.
19. J. B. CALLAGHAN, P. H. JARVIS, AND A. K. RIGLER,[5] *A program for the solution of thin elastic plate equations*, WAPD-TM-255, Bettis Atomic Power Laboratory, Pittsburgh, Pennsylvania, (1961).
20. D. S. GRIFFIN,[5] *A numerical solution for plate bending problems*, WAPD-230, Bettis Atomic Power Laboratory, Pittsburgh, Pennsylvania, (1963).
21. R. E. PETERSON, *Stress Concentration Design Factors*, John Wiley and Sons, New York, 1953.