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Zeros of the partial sums of cos(z) and sin(z). III

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ABSTRACT

In this paper, we give new extensions of results by G. Szegő, in 1924, on the interrelationships between the zeros of the partial sums of e^z, and those of the partial sums of cos(z) and sin(z).

We also include numerical results and figures which illustrate our new results.

1. Introduction

With $\mathbb{N}$ denoting the set of all positive integers n, and with z any complex number (written $z \in \mathbb{C}$), then

$$s_n(z) := \sum_{j=0}^{n} z^j/(j)! \quad (n \in \mathbb{N}) \tag{1}$$

is the familiar n-th partial sum of the Maclauren expansion of e^z about $z = 0$, and for any *even* positive integer n (written $n \in 2\mathbb{N}$),

$$\cos_n(z) := \sum_{j=0}^{n/2} (-1)^j z^{2j}/(2j)! \quad (n \in 2\mathbb{N}) \tag{2}$$

is similarly the n-th partial sum of cos(z); then, for any *odd* positive integer m (written $m \in 2\mathbb{N} - 1$),

$$\sin_m(z) := \sum_{j=0}^{(m-1)/2} (-1)^j z^{2j+1}/(2j+1)! \quad (m \in 2\mathbb{N} - 1) \tag{3}$$

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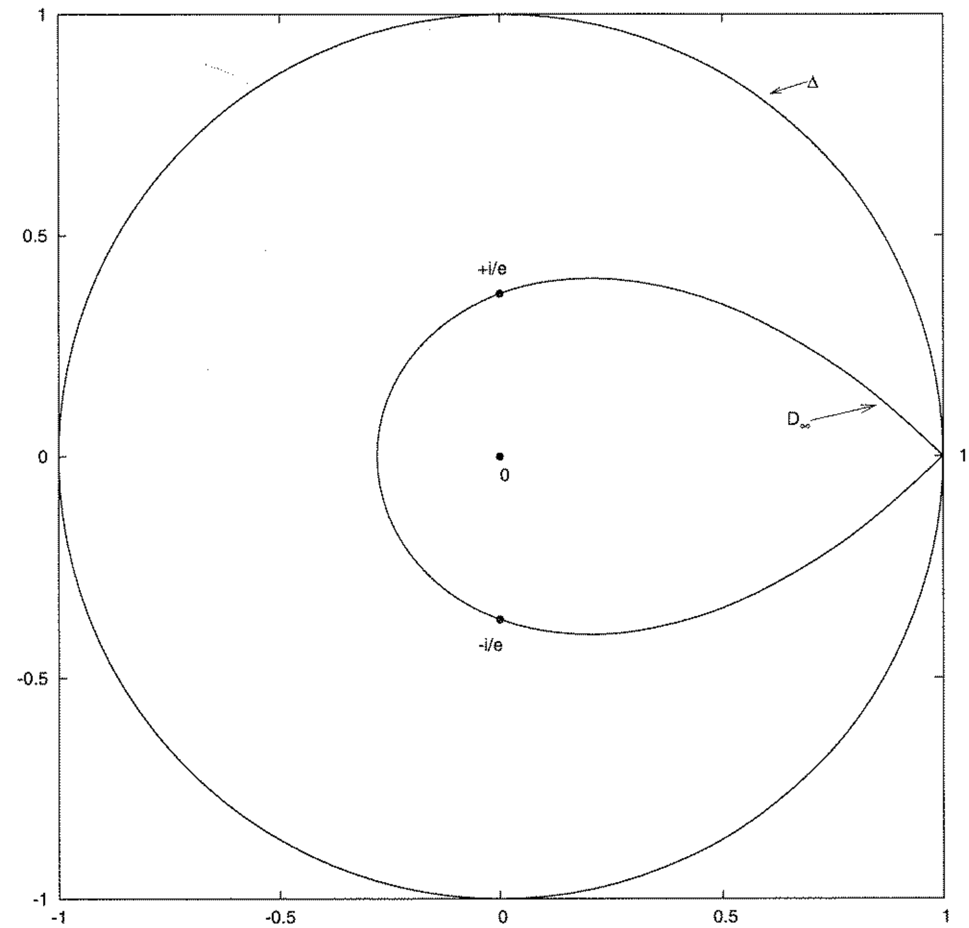

Fig. 1. The Szegő curve D_∞ of (5).

is the m-th partial sum of $\sin(z)$. The relationship between the zeros of $s_n(z)$ with those of the zeros of $\cos_n(z)$, for n even, and those of $s_m(z)$ and $\sin_m(z)$, for m odd, was studied in Szegő [5], followed by results of Kappert [3], and Varga and Carpenter [6] and [7]. In this paper, we further sharpen these results.

For added notation and background information, the well-known Eneström–Kakeya Theorem (cf. Marden [4, p. 137, Exercise 2]) states, for any polynomial $p_n(z) = \sum_{j=0}^{n} a_j z^j$ with $a_j > 0$ for all $0 \leqslant j \leqslant n$, that all the zeros of $p_n(z)$ necessarily lie in the annulus

$$\min_{0 \leqslant i < n} \left(\frac{a_i}{a_{i+1}} \right) \leqslant |z| \leqslant \max_{0 \leqslant i < n} \left(\frac{a_i}{a_{i+1}} \right). \tag{4}$$

Applying the final inequality of (4) to the partial sum $s_n(z)$ of (1) gives that all zeros of $s_n(z)$ satisfy $|z| \leqslant n$, for each $n \geqslant 1$. Thus, with $s_n(nz)$ denoting the n-th *normalized partial sum* of e^z, the zeros of $s_n(nz)$, for *all* $n \geqslant 1$, lie in the closed unit disk, denoted by

$$\Delta := \{ z \in \mathbb{C} \colon |z| \leqslant 1 \}.$$

Consequently, compactness considerations guarantee that the set, of *all* zeros of *all* normalized partial sums $\{s_n(nz)\}_{n=1}^{\infty}$, has at least one accumulation point in Δ. Szegő [5] showed that the set of *all* such accumulation points is exactly the points of the closed curve in Δ, given by

$$D_\infty := \{ z \in \Delta \colon |ze^{1-z}| = 1 \}, \tag{5}$$

which is now called the *Szegő curve* in the literature. This closed curve D_∞ will play an important role in what is to follow, and it is shown in Fig. 1, where we note, from (5), that the points $\pm i/e$ and 1 are points of D_∞.

Next, from his Hilfsatz 1, Szegő [5] showed that

$$e^{-nz} s_n(nz) = 1 - \frac{(ze^{1-z})^n}{\sqrt{2\pi n}} \left(\frac{z}{1-z} \right) \{ 1 + \varepsilon_n(z) \} \quad (n \in \mathbb{N}), \tag{6}$$

where

$$\lim_{n\to\infty} \varepsilon_n(z) = 0, \tag{7}$$

uniformly on any compact subset of $\Delta \setminus \{1\}$. The results of (6) and (7) can be interpreted as saying that the n solutions of

$$\frac{(ze^{1-z})^n}{\sqrt{2\pi n}}\left(\frac{z}{1-z}\right) = 1, \tag{8}$$

approximate the n zeros of $s_n(nz)$ in Δ, for all $n \geqslant 1$. Later, more refined versions of (6) were obtained. For example, (6) was sharpened in Carpenter, Varga, and Waldvogel [2] to

$$e^{-nz}s_n(nz) = 1 - \frac{(ze^{1-z})^n}{\tau_n\sqrt{2\pi n}}\left(\frac{z}{1-z}\right)\left\{1 - \frac{1}{(n+1)(1-z)^2} + \frac{z(4-z)}{(n+1)(n+2)(1-z)^4} + \cdots\right\}, \tag{9}$$

uniformly on any compact subset of $\Delta \setminus \{1\}$, where τ_n is the exact error in Stirling's formula, i.e.,

$$\tau_n := \frac{n!}{n^n e^{-n}\sqrt{2\pi n}}, \tag{10}$$

which has the following known asymptotic series expansion:

$$\tau_n = 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots, \quad \text{as } n \to \infty, \tag{11}$$

so that

$$\tau_n = 1 + O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty. \tag{12}$$

It then follows, using (12), that (9) can be expressed as

$$e^{-nz}s_n(nz) = 1 - \frac{(ze^{1-z})^n}{\sqrt{2\pi n}}\left(\frac{z}{1-z}\right)\left\{1 + O\left(\frac{1}{n}\right)\right\}, \quad \text{as } n \to \infty, \tag{13}$$

uniformly on any compact subset of $\Delta \setminus \{1\}$. If only the first term, namely unity, of the terms in the braces of (13) is used, then the n solutions $\{\hat{z}_{k,n}\}_{k=1}^n$ of

$$\frac{(\hat{z}e^{1-\hat{z}})^n}{\sqrt{2\pi n}}\left(\frac{\hat{z}}{1-\hat{z}}\right) = 1 \tag{14}$$

approximate the n zeros $\{z_{k,n}\}_{k=1}^n$ of $s_n(nz)$. The accuracy of the solutions $\{\hat{z}_{k,n}\}_{k=1}^n$ of (14), with respect to the actual zeros $\{z_{k,n}\}_{k=1}^n$ of $s_n(nz)$, has been studied in the literature, and it is known from [2] and [8] that, given a compact subset of $\Delta \setminus \{1\}$, then

$$|z_{k,n} - \hat{z}_{k,n}| = O\left(\frac{1}{n^2}\right), \quad \text{as } n \to \infty, \tag{15}$$

holds for all the $z_{k,n}$ in this given compact set. As can be imagined, taking more terms, in the expansion in braces in (9), gives rise to higher-order approximations of the actual zeros of $s_n(nz)$ (cf. [2, p. 119]).

2. Connections with $\cos_n(nz)$ and $\sin_n(nz)$

Since

$$2\cos(z) = e^{iz} + e^{-iz}$$

for any complex number z, it similarly follows, from the definitions in (1) and (2), that

$$2\cos_n(nz) = s_n(inz) + s_n(-inz) \quad (z \in \mathbb{C}, n \in 2\mathbb{N}), \tag{16}$$

which can be equivalently expressed as

$$2\cos_n(nz) = e^{inz}\left(e^{-inz}s_n(inz)\right) + e^{-inz}\left(e^{inz}s_n(-inz)\right) \quad (n \in 2\mathbb{N}). \tag{17}$$

Noting that $\cos(z)$ has only real zeros in the entire complex plane, then, with the open upper half of the unit disk denoted by

$$\Delta^+ := \{z = x + iy \in \Delta\colon\ y > 0\},$$

an application of the expansion of (9) to (17), gives, from Varga and Carpenter [6, Eq. (3.7)], that

$$\frac{\cos_n(nz)}{\cos(nz)} = 1 - \frac{(-ize^{1+iz})^n}{\tau_n\sqrt{2\pi n}}\left(\frac{-2z^2}{1+z^2}\right)\left\{1 - \frac{3-z^2}{(n+1)(1+z^2)^2} + O\left(\frac{1}{n^2}\right)\right\}, \tag{18}$$

as $n \to \infty$, uniformly on any compact subset of $\Delta^+ \setminus \{i\}$, with a similar statement holding for the *reflection*, in the real axis, of the set $\Delta^+ \setminus \{i\}$. For our needs here, using (12) allows us to express (18) in the slightly weaker form

$$\frac{\cos_n(nz)}{\cos(nz)} = 1 - \frac{(-ize^{1+iz})^n}{\sqrt{2\pi n}}\left(\frac{-2z^2}{1+z^2}\right)\left\{1 + O\left(\frac{1}{n}\right)\right\} \quad (n \in 2\mathbb{N}), \tag{19}$$

as $n \to \infty$, uniformly on any compact subset of $\Delta^+ \setminus \{i\}$. On fixing a compact subset of $\Delta^+ \setminus \{i\}$, it can be shown, using the techniques of [8], that the non-real solutions of

$$\frac{(-ize^{1+iz})^n}{\sqrt{2\pi n}}\left(\frac{-2z^2}{1+z^2}\right) = 1 \quad (n \in 2\mathbb{N}), \tag{20}$$

in this compact subset of $\Delta^+ \setminus \{i\}$, similarly produce approximations of the actual non-real zeros of $\cos_n(nz)$ in this subset, with an accuracy of $O(\frac{1}{n^2})$, as $n \to \infty$, as in (15). Similar results hold for the reflection, in the real axis, of this compact set in $\Delta^+ \setminus \{i\}$.

In analogy with (16), we also have from (1) and (3) that

$$2i\sin_m(mz) = s_m(imz) - s_m(-imz) \quad (z \in \mathbb{C}, m \in 2\mathbb{N} - 1), \tag{21}$$

which can be equivalently expressed as

$$2i\sin_m(mz) = e^{imz}\big(e^{-imz}s_m(imz)\big) - e^{-imz}\big(e^{imz}s_m(-imz)\big), \tag{22}$$

for all $z \in \mathbb{C}$ and all $m \in 2\mathbb{N} - 1$. Noting that $\sin(z)$ also has only real zeros in the entire complex plane, then applying (9) to (22) gives (cf. Varga and Carpenter [6, Eq. (3.10)])

$$\frac{\sin_m(mz)}{\sin(mz)} = 1 - \frac{(-ize^{1+iz})^m}{\tau_n\sqrt{2\pi n}}\left(\frac{-2z^2}{1+z^2}\right)\left\{1 - \frac{3-z^2}{(m+1)(1+z^2)^2} + O\left(\frac{1}{m^2}\right)\right\}, \tag{23}$$

as $m \to \infty$, uniformly on any compact set in $\Delta^+ \setminus \{i\}$, with similar statements holding for the reflection, in the real axis, of the set $\Delta^+ \setminus \{i\}$. On comparing the equations of (23) with (18), we see that the right sides are *identical*, when $m \in 2\mathbb{N} - 1$ is replaced by $n \in 2\mathbb{N}$, and vice versa. This has the effect that, henceforth, it suffices to work *only* with $\cos_n(nz)$, $n \in 2\mathbb{N}$, as analogous results for $\sin_m(mz)$ trivially follow by simply replacing $n \in 2\mathbb{N}$ by $m \in 2\mathbb{N} - 1$ in what follows.

Next, on comparing the equation in (8) with that of (20), we see that if z is replaced by iz in (20), i.e., a rotation of z by $\pi/2$, we obtain

$$\left(ze^{1-z}\right)^n = \sqrt{2\pi n}\left(\frac{1-z^2}{2z^2}\right), \tag{24}$$

which is quite similar to the following equivalent form of (8), i.e.,

$$\left(ze^{1-z}\right)^n = \sqrt{2\pi n}\left(\frac{1-z}{z}\right). \tag{25}$$

Because the first terms of (24) and (25) are the same and are dominant, for n large, this would suggest that the zeros of the normalized partial sums $s_n(nz)$ of e^z, in the right half-plane, when properly rotated into the open upper half-plane or into the open lower half-plane, would give excellent approximations to the actual *non-real* zeros of $\cos_n(nz)$ or $\sin_m(mz)$, as suggested by the original work of Szegő [5].

To illustrate this, consider Fig. 2, where the zeros of $\cos_{12}(12z)$ are shown as "×"'s, while the zeros of the normalized partial sum $s_{12}(12z)$ of e^z, rotated by $\pi/2$, are shown as "•"'s. From this figure, we see that the *open* upper half-plane contains 4 rotated zeros of $s_{12}(12z)$, which are nicely approximated by the 4 non-real zeros of $\cos_{12}(12z)$ in this open upper half-plane. The remaining zeros of $\cos_{12}(12z)$ in Fig. 2 consist of 4 real zeros, which are nearly uniformly spaced in a symmetric real interval which contains $[-1/e, +1/e]$, and 4 non-real zeros which are the reflection, in the real axis, of the non-real zeros in the upper half-plane. (Also shown in this figure is the dashed boundary of the circle $\{z \in \Delta\colon |z - \frac{i}{3}| = \frac{2}{3}\}$, which will be used later, and also the solid boundary of the Szegő curve from the right half-plane, which has been rotated both $+\pi/2$, and $-\pi/2$, in this figure.) Fig. 3 shows similar results for the zeros of $\sin_{13}(13z)$ and $s_{13}(13z)$.

Looking very carefully at Figs. 2 and 3, we notice not only that the non-real zeros of $\cos_{12}(12z)$ or of $\sin_{13}(13z)$ are *very well approximated* by the appropriately rotated zeros of the normalized partial sums $s_{12}(12z)$ or $s_{13}(13z)$, but also that there is a *definite pattern* to the difference of these zeros. Specifically, on considering the open first quadrant, we notice from Figs. 2 and 3 that

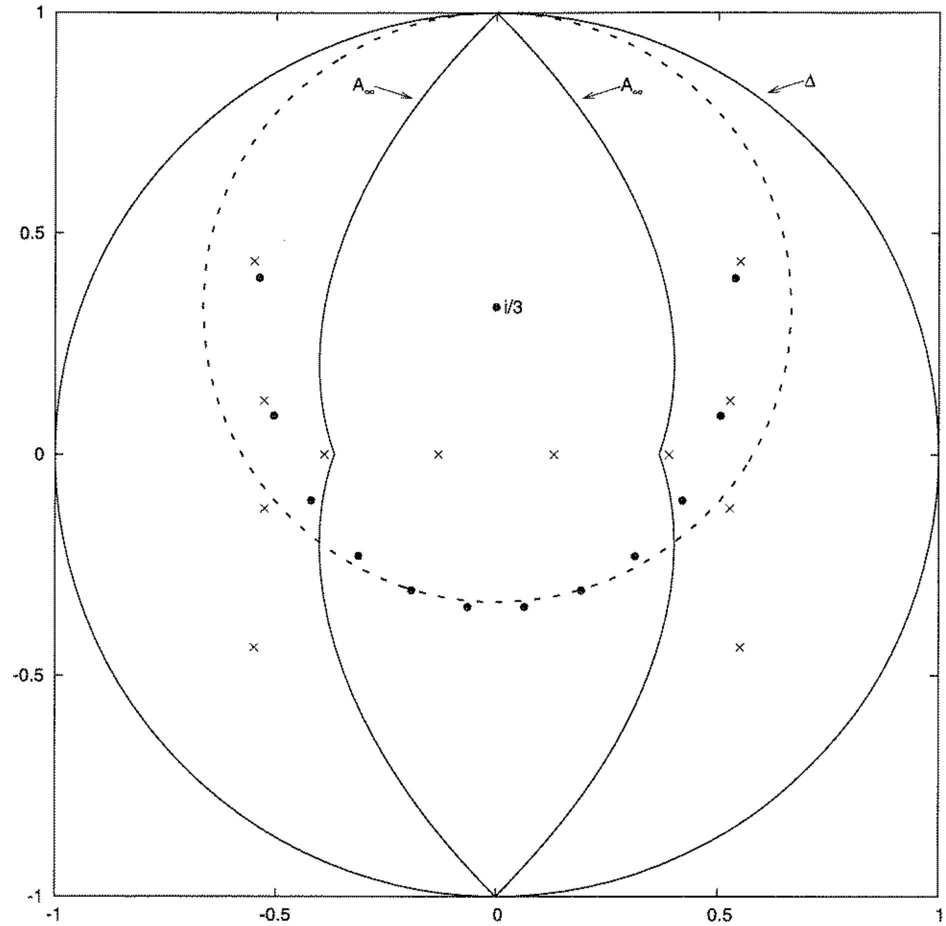

Fig. 2. The rotated zeros of $s_{12}(12z)$ (dots) and the zeros of $\cos_{12}(12z)$ (crosses).

(i) for $n = 12$ and $n = 13$, the rotated zeros of $s_n(nz)$ in this *open* first quadrant (denoted by "•"'s), all have *smaller moduli* than those of the corresponding non-real zeros of $\cos_n(nz)$ or $\sin_n(nz)$ (denoted by "×"'s), and similarly,
(ii) for $n = 12$ and $n = 13$, the arguments of the rotated zeros of $s_n(nz)$ in this *open* first quadrant, all have *smaller arguments* than those of the corresponding non-real zeros of $\cos_n(nz)$ or $\sin_n(nz)$.

This "definite pattern" is something which has not been observed in the literature.

One of the goals of this paper is to theoretically show that these observations are indeed valid, for all $n \geqslant 6$.

3. Main results

Following the notation of [6], consider the closed set

$$A_\infty^+ := \{z \in \Delta:\ |-ize^{1+iz}| = 1, \text{ and } \operatorname{Im} z \geqslant 0\}, \tag{26}$$

which is simply the restriction, of the Szegő curve of (5), to the right half-plane, which is then *rotated* by $\pi/2$ to the upper half-plane. Then, with the closed set

$$A_\infty := A_\infty^+ \cup \{z \in \Delta:\ \bar{z} \in A_\infty^+\}, \tag{27}$$

which is the union of A_∞^+ of (26) with its reflection in the real axis, Szegő [5] showed that the accumulation points of *all* zeros of $\cos_n(nz)$, for any even n, and *all* zeros of $\sin_m(mz)$, for any odd m, is *precisely* given by the union of the closed set A_∞, with the real interval $[-\frac{1}{e}, +\frac{1}{e}]$:

$$A_\infty \cup \left[-\frac{1}{e}, +\frac{1}{e}\right]. \tag{28}$$

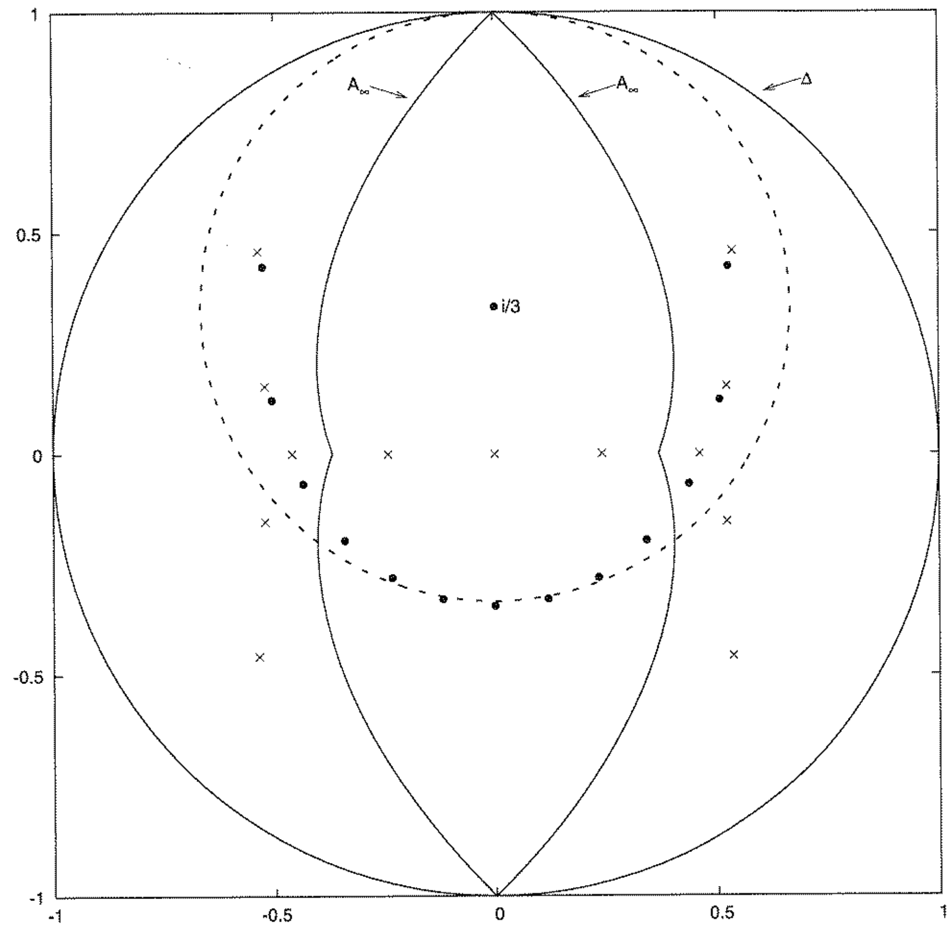

Fig. 3. The rotated zeros of $s_{13}(13z)$ (dots) and the zeros of $\sin_{13}(13z)$ (crosses).

That the accumulation points of all these zeros is given by (28) is certainly suggested by the results of Fig. 4, where all zeros of $\cos_n(nz)$, $n = 2, 4, \ldots, 100$, and all zeros of $\sin_n(nz)$, $n = 1, 3, \ldots, 101$, are shown by "•"'s. We also remark that $\cos_n(nz)$, for n even, has *some* non-real zeros if and only if $n \geqslant 6$, and similarly, $\sin_m(mz)$, for m odd, has *some* non-real zeros if and only if $m \geqslant 5$, as shown by Kappert [3].

Using the notation of [7], we define the closed set K as

$$K := A_\infty \cup \operatorname{int} A_\infty, \tag{29}$$

where the set K is shown in Fig. 5. Then, from Theorem 4.3 of Varga and Carpenter [7], we know that

$$\begin{cases} \text{(i)} \quad \cos_n(nz) \text{ has } \textit{no} \text{ non-real zeros in } K \text{ for any } n \in 2\mathbb{N}, \text{ and} \\ \text{(ii)} \quad \sin_m(mz) \text{ has } \textit{no} \text{ non-real zeros in } K \text{ for any } m \in 2\mathbb{N} - 1, \end{cases} \tag{30}$$

which is surely indicated from Fig. 4. We remark that the results of (30) are a more tedious extension, to $\cos_n(nz)$ and $\sin_m(mz)$, of a result of Buckholtz [1], which showed that *any* zero of *any* normalized partial sum $s_n(nz)$, $n \in \mathbb{N}$, must lie *outside* the Szegő curve D_∞ of (5).

Next, for any $n \in \mathbb{N}$, we define the n complex numbers $\{z_{k,n}\}_{k=1}^{n}$, which satisfy (14), to be *approximate zeros* of the normalized partial sum $s_n(nz)$, where, from [2] and [8], we know that these approximate zeros approximate the actual zeros of $s_n(nz)$ to $O(\frac{1}{n^2})$, as $n \to \infty$, provided that these points belong to a fixed compact subset of $\Delta \setminus \{1\}$. In a similar manner, we define the points $\{\check{z}_{k,n}\}_{k=1}^{p(n)}$, in the open first quadrant, to be *rotated approximate zeros* of $\cos_n(nz)$ if n is even, for any $n \geqslant 6$, or of $\sin_n(nz)$ if n is odd, for any $n \geqslant 5$, if these points satisfy (cf. (24))

$$\left\{\check{z} \in \Delta^+ \colon \frac{(\check{z}e^{1-\check{z}})^n 2\check{z}^2}{\sqrt{2\pi n}(1-\check{z}^2)} = 1, \text{ and } \operatorname{Re} \check{z} > 0\right\}, \tag{31}$$

where the numbers $\{\check{z}_{k,n}\}_{k=1}^{p(n)}$ are arranged in order of increasing arguments, i.e.,

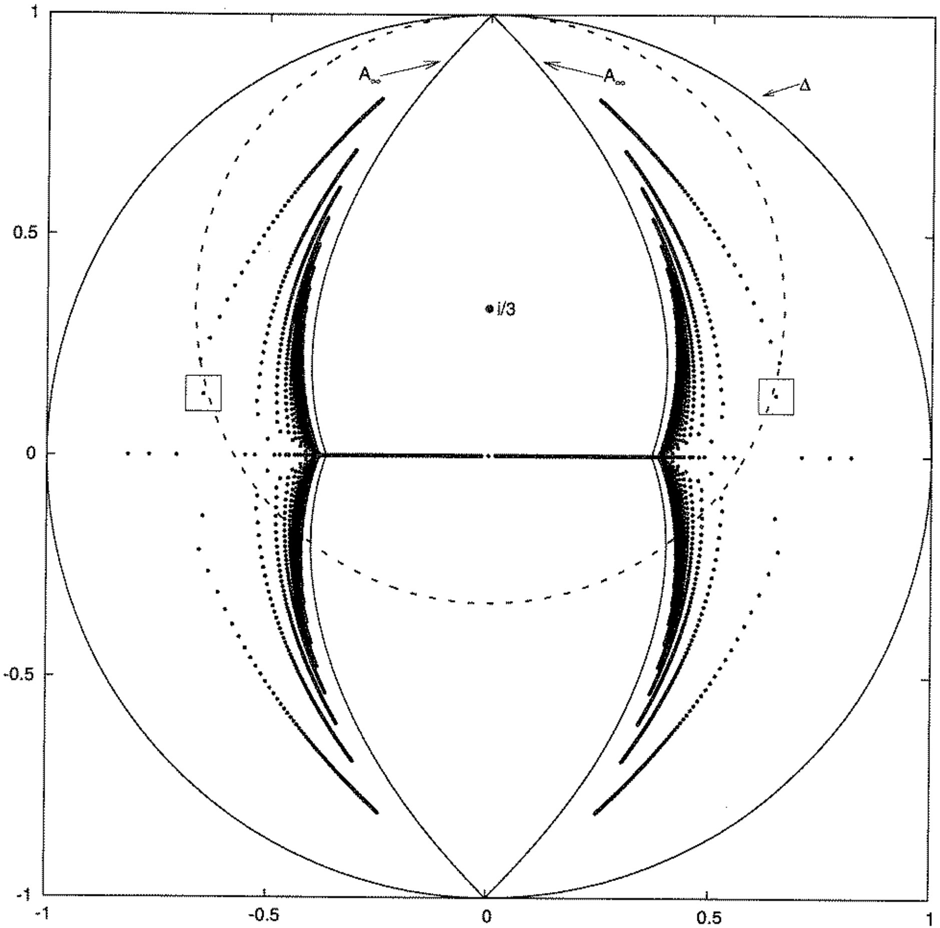

Fig. 4. The zeros of $\cos_n(nz)$ for $n = 2, 4, \ldots, 100$ and the zeros of $\sin_n(nz)$ for $n = 1, 3, \ldots, 101$.

$$0 < \arg \check{z}_{1,n} < \arg \check{z}_{2,n} < \cdots < \arg \check{z}_{p(n),n} < \pi/2. \tag{32}$$

(We remark, from [3], that $p(n) \geqslant 1$ for all $n \in \mathbb{N}$ with $n \geqslant 5$.) Then, because of symmetries in the location of the zeros of $\cos_n(nz)$ and $\sin_n(nz)$, we deduce that if $\check{z}_{k,n}$ is a rotated approximate zero of the normalized partial sum $s_n(nz)$, then $\check{z}_{k,n}e^{i\pi/2} = i\check{z}_{k,n}$ is an *approximate zero* of $\cos_n(nz)$ if n is even, or of $\sin_n(nz)$ if n is odd, in the open second quadrant. Moreover, reflections of these points, in both the real axis or imaginary axis, produce *all* non-real complex zeros of $\cos_n(nz)$, n even, or of $\sin_n(nz)$, n odd.

Returning to the observation that the left sides of the equations of (24) and (25), are identical, we next compare their different right-hand sides by taking their *ratio*, i.e., for $z \neq 0$,

$$\frac{(\frac{1-z}{z})}{(\frac{1-z^2}{2z^2})} = \frac{2z}{1+z}. \tag{33}$$

The *argument* of this ratio then geometrically satisfies

$$\arg\left(\frac{2z}{1+z}\right) = \arg(z) - \arg(1+z) > 0, \tag{34}$$

for any z in Δ^+. Similarly, it can be verified that the *modulus*, of the final ratio of (33), satisfies $|\frac{2z}{1+z}| < 1$ is equivalent to z lying in the open disk $\{z \in \mathbb{C}\colon\ |z - \frac{1}{3}| < \frac{2}{3}\}$, but, because (20) holds only in $\Delta^+ \setminus \{i\}$, then with the rotation by $-\pi/2$ used in Eq. (24), we have that

$$\left|\frac{2z}{1+z}\right| < 1 \quad \text{in } \operatorname{Re} z > 0, \quad \text{if and only if} \quad \left|z - \frac{1}{3}\right| < \frac{2}{3} \quad \text{in } \operatorname{Re} z > 0. \tag{35}$$

The results of (34) and (35) will be used below.

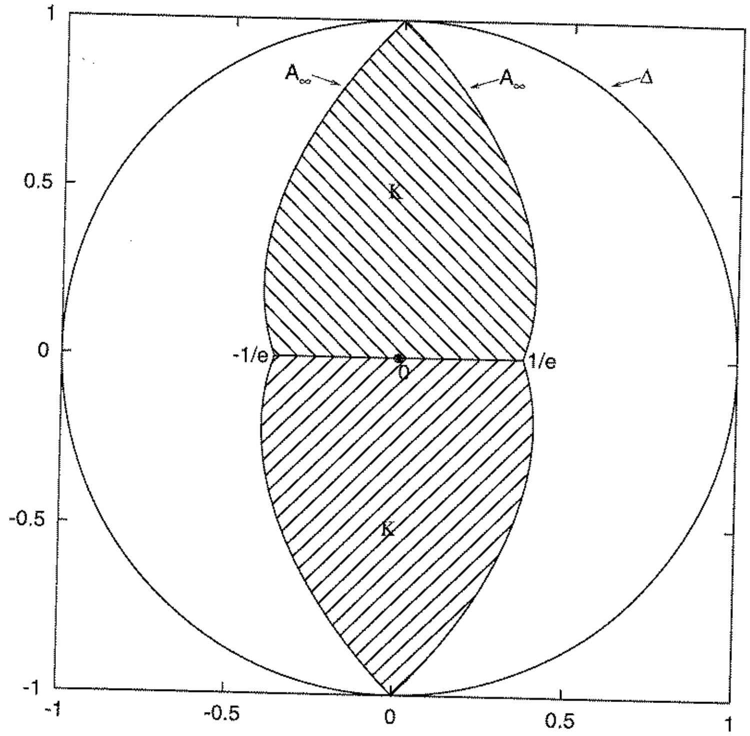

Fig. 5. The set K of (29).

We know from (30) that the accumulation points of the zeros of $\cos_n(nz)$, for any n even, or of $\sin_n(nz)$, for any n odd, must lie on the set of (28). However, Figs. 2–4 *suggest* that most of the non-rotated zeros of $\cos_n(nz)$, for n even, or of $\sin_n(nz)$, for n odd, in the open upper half-plane, lie in the shaded open set G, defined by

$$G := \left\{ z \in \Delta^+ : \left| z - \frac{i}{3} \right| < \frac{2}{3} \right\} \setminus K, \tag{36}$$

where the portion of the open disk, in the definition of G, comes from the rotation of the open disk of (35). The set G is shown in Fig. 6.

A more exact location of the non-real zeros of $\cos_n(nz)$, for n even, and of $\sin_n(nz)$, for n odd, is the set G of (36), as shown below in

Theorem 3.1. *For all $n \geqslant 6$, the non-real zeros of $\cos_n(nz)$, for n even, and of $\sin_n(nz)$, for n odd, in the open upper half-plane, all lie in the set G of* (36)*, with a similar result holding in the reflection of the set G in the real axis.*

Proof. As previously stated, it is known from [3] that $\cos_n(nz)$, for n even, has some non-real zeros only if $n \geqslant 6$, and that $\sin_n(nz)$, for n odd, has some non-real zeros only if $n \geqslant 5$, but on determining the actual zeros of $\sin_5(5z)$, it follows that $\sin_5(5z)$ has *exactly* one non-real zero in each quadrant of Δ. Two particular non-real zeros of $\sin_5(5z)$ are shown in small squares in the upper half-plane in Fig. 4. Note, however, that these two non-real zeros of $\sin_5(5z)$, in $\operatorname{Im} z > 0$, lie just *outside* the set G of (36), showing that Theorem 3.1 can hold only for $n \geqslant 6$.

Continuing, for convenience, let G_R denote the restriction of the set G of (36) to the *first* quadrant. For $n = 6, 7, 8$, and 9, there is a single zero, of each $\cos_n(nz)$ or $\sin_n(nz)$, in G_R, which have increasing arguments for increasing n, while for $n = 10, 11, 12, 13, 14$, and 15, there are exactly two zeros, of each of these polynomials, in G_R, etc. Then, if we connect, by straight-line segments, the successive zeros of $\cos_n(nz)$ or $\sin_n(nz)$, having the *largest* arguments in the open first quadrant, we will have a path which stays in G_R, and tends to the point $z = i$, as shown by the clearly visible "outside" path of "•"'s in the first quadrant of Fig. 4. Moreover, this path *separates* G_R into two disjoint parts, where the lower part contains *all* remaining zeros of $\cos_n(nz)$ and $\sin_n(nz)$ in this open first quadrant. The reason that this separation takes place is that, when a non-real zero of $\cos_n(nz)$ or $\sin_n(nz)$, for larger values of n, enters G_R, it enters from the real axis with a *smaller* modulus, as these zeros, by necessity, must tend to the *boundary* of K in the first quadrant, as this is the *only* place in the first quadrant where non-real zeros can accumulate. In a similar fashion, for $n \geqslant 10$, one can join, again by straight-line segments, the successive zeros of $\cos_n(nz)$ or $\sin_n(nz)$, having the *second-largest* arguments, thereby generating a second path, in the first quadrant, which also tends to the point $z = i$, but, as can be seen from Fig. 4, this path is *inside* the previous outside path in G_R. (Higher-order paths can also be defined, but they are more difficult to see in Fig. 4.) □

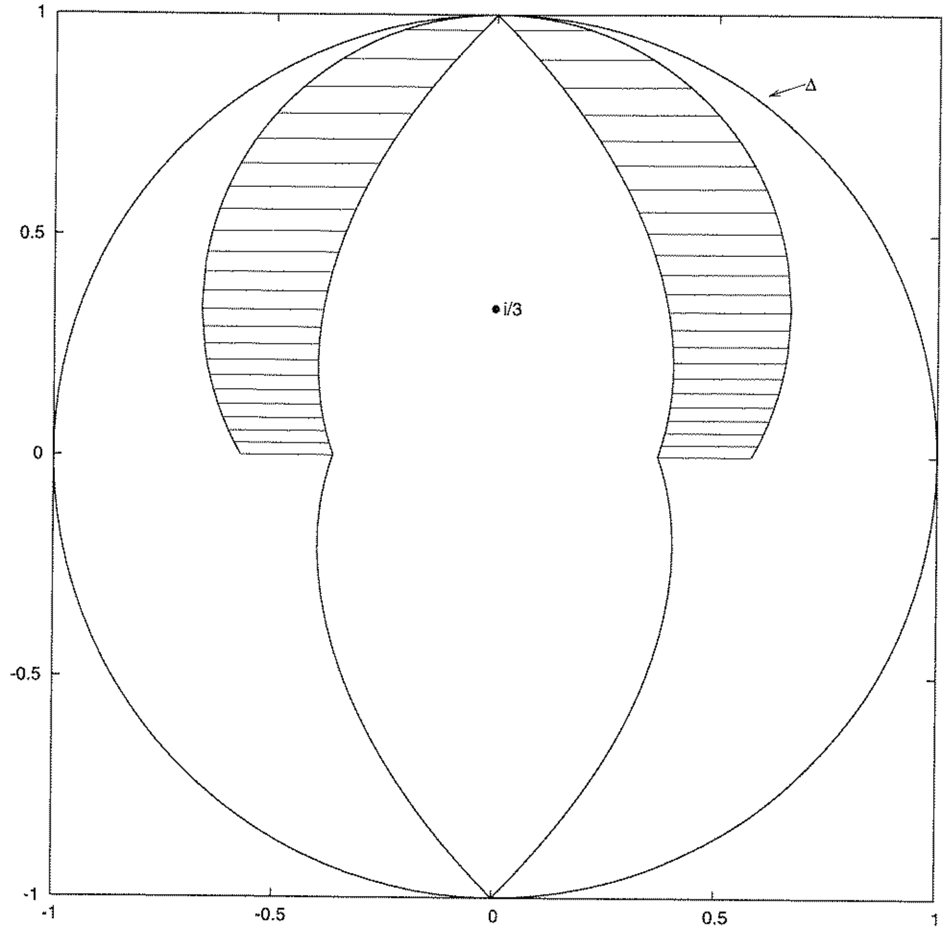

Fig. 6. The set G of (36) is shaded above.

We next return to the question, from Section 2, concerning the "definite pattern" relating the zeros of $\cos_n(nz)$ or $\sin_n(nz)$ in the upper half-plane with the rotated zeros of the normalized partial sum $s_n(nz)$. For our purposes here, it is more convenient to consider the *unrotated* zeros of the normalized partial sums $s_n(nz)$, in the first quadrant of Δ, corresponding to the solutions of Eq. (25), and to consider the zeros of $\cos_n(nz)$ or $\sin_n(nz)$, in the second quadrant of Δ, which are then rotated by $-\pi/2$ to the first quadrant of Δ. For the unrotated zeros of the normalized partial sums $s_n(nz)$ in the first quadrant, we consider the following curve D_n, a variant of the closed curve $\tilde{D}_n$ of [8]. For each positive integer n, set

$$\lambda_n := \cos^{-1}\left\{1 - \frac{v}{2n}\right\}, \tag{37}$$

where v is the largest positive solution of $e^v = 2\pi v$, that is, $v := 2.90394$.[1] The numbers $\{\lambda_n\}_{n=1}^{\infty}$ are strictly decreasing to zero, with $\lambda_1 = 2.03976$ radians. Then for each $n \geqslant 1$, the closed curve $\tilde{D}_n$ in Δ is defined (cf. [8, Eq. (2.4)]) as

$$\tilde{D}_n := \left\{z \in \Delta\colon \frac{|ze^{1-z}|^n}{\sqrt{2\pi n}\,|\frac{1-z}{z}|} = 1, \text{ and } \lambda_n \leqslant \arg z \leqslant 2\pi - \lambda_n\right\} \cup \left\{e^{i\sigma}\colon -\lambda_n \leqslant \sigma \leqslant +\lambda_n\right\}, \tag{38}$$

which comes from taking moduli in (25). It is further known from [8, Proposition 2.1], that the closed curve $\tilde{D}_n$ of (38) is *star-shaped*, with respect to $z = 0$, for any $n \geqslant 1$. Then, we consider the following intersection, which defines the closed arc D_n:

$$D_n := \tilde{D}_n \cap \left\{z \in \Delta\colon \left|z - \frac{1}{3}\right| \leqslant \frac{2}{3},\ z \neq 0,\ \operatorname{Re} z \geqslant 0, \text{ and } \operatorname{Im} z > 0\right\}. \tag{39}$$

[1] We truncate, throughout, any non-integer real number to five decimal digits.

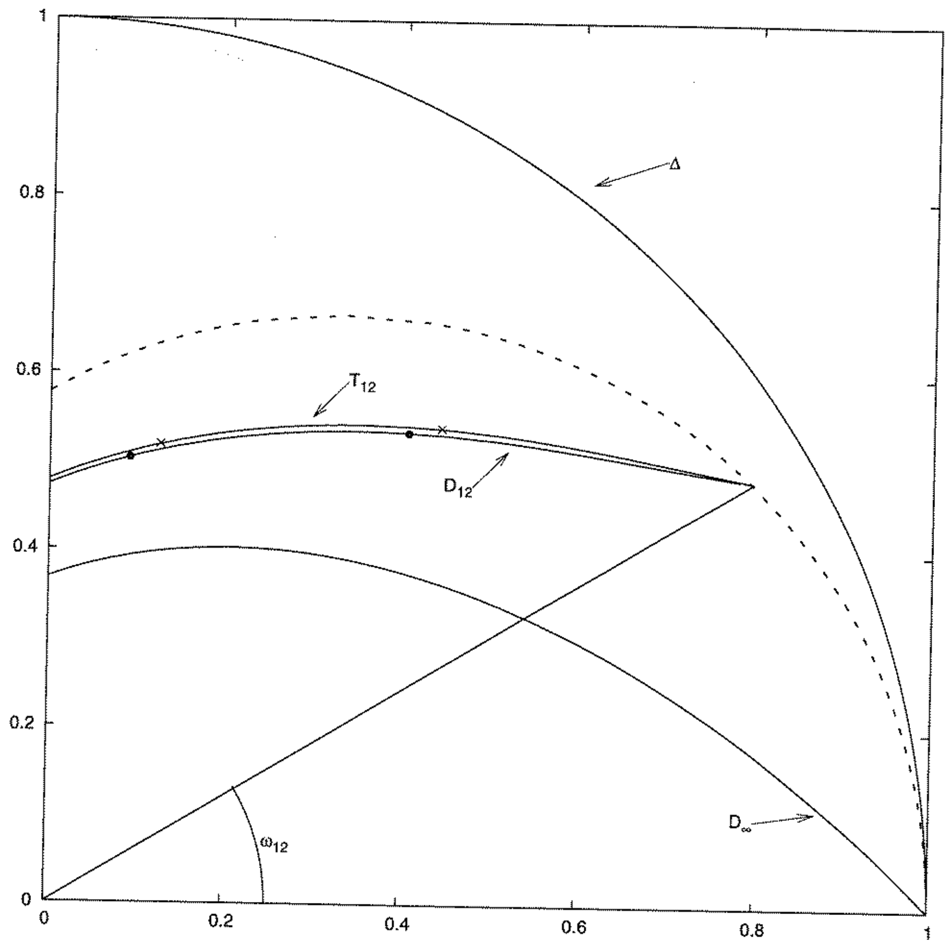

Fig. 7. The curves D_{12} and T_{12}, along with the zeros of $s_{12}(12z)$ (dots) and the rotated zeros of $\cos_{12}(12z)$ (crosses), in the first quadrant of Δ.

For the arc D_n, defined in (39), we are interested in those points z, of D_n, which correspond to *approximate* zeros of $s_n(nz)$ in the first quadrant, i.e., points z which satisfy (14). It can be verified that such points of D_n exist in the first quadrant, *only* if $n \geqslant 6$. We remark that D_n of (39), for each $n \geqslant 6$, is an arc in the first quadrant, as shown in Fig. 7 for $n = 12$.

Continuing, for any $n \geqslant 6$ and for any point $z = r_n(\theta)e^{i\theta}$ of the arc $\tilde{D}_n$ of (38), where $\lambda_n \leqslant \theta \leqslant 2\pi - \lambda_n$, we define the function

$$\Psi_n(z) := \arg\left(\frac{(ze^{1-z})^n}{\sqrt{2\pi n}(\frac{1-z}{z})}\right) = \arg\left((ze^{1-z})^n\right) - \arg\left(\frac{1-z}{z}\right), \tag{40}$$

which can also be expressed as

$$\Psi_n(z) = n\left(\theta - r_n(\theta)\sin\theta\right) + \theta + \tan^{-1}\left(\frac{r_n(\theta)\sin\theta}{1 - r_n(\theta)\cos\theta}\right),$$

so that

$$\frac{(ze^{1-z})^n}{\sqrt{2\pi n}(\frac{1-z}{z})} = e^{i\Psi_n(z)}. \tag{41}$$

From [2, p. 118], we have that $\Psi_n(r_n(\theta)e^{i\theta})$ is a *strictly increasing function* of θ on $[\lambda_n, 2\pi - \lambda_n]$. In addition, for each $n \geqslant 6$, we note from (38) that the non-circular portion of $\tilde{D}_n$ extends to the boundary point $e^{i\lambda_n}$ of the unit disk. As λ_n is positive, from (37), for every positive integer n, it follows that $e^{i\lambda_n}$ lies *outside* the disk $\{z \in \Delta\colon |z - \frac{1}{3}| \leqslant \frac{2}{3}\}$, for any $n \geqslant 6$. Then, the subarc D_n of $\tilde{D}_n$ intersects the circle $\{z \in \Delta\colon |z - \frac{1}{3}| = \frac{2}{3}\}$, in a *unique point*, as $\tilde{D}_n$ is star-shaped with respect to $z = 0$. We define the *angle* of this unique intersection as ω_n, and this is shown in Fig. 7, for the case $n = 12$. It also follows, from this definition of ω_n, that the ray $\{z = re^{i\theta}\colon r \geqslant 0\}$ intersects the arc D_n in a unique point for any θ with $\omega_n \leqslant \theta \leqslant \pi/2$, and that (cf. (37))

$$\lambda_n < \omega_n < \pi/2 \quad (\text{all } n \geqslant 6).$$

(For example, in Fig. 7, we have that $\omega_{12} \doteq 0.54661$ radians, while $\lambda_{12} \doteq 0.49703$ radians from (37).)

Next, we define the related arc T_n ("T" for trigonometric), in the open first quadrant, which is derived from rotated non-real zeros of $\cos_n(nz)$, for n even, or of $\sin_n(nz)$, for n odd, by first taking moduli in (24):

$$T_n := \left\{ z \in \Delta\colon \frac{|ze^{1-z}|^n}{\sqrt{2\pi n}\,|\frac{1-z^2}{2z^2}|} = 1, \text{ and } \left|z - \frac{1}{3}\right| \leqslant \frac{2}{3},\ z \neq 0,\ \operatorname{Re} z \geqslant 0,\ \operatorname{Im} z > 0 \right\}. \tag{42}$$

The arc T_n, in the open first quadrant of Δ, is again of interest to us, only for $n \geqslant 6$. Then, in a similar fashion to (40), we define, for $z = R_n(\theta)e^{i\theta}$ a point of T_n, the function

$$\Psi_n^T(z) := \arg\left(\frac{(ze^{1-z})^n}{\sqrt{2\pi n}(\frac{1-z^2}{2z^2})}\right) = \arg\{(ze^{1-z})^n\} - \arg\left(\frac{1-z^2}{2z^2}\right), \tag{43}$$

which can also be expressed as

$$\Psi_n^T(z) = n\big(\theta - R_n(\theta)\sin\theta\big) + 2\theta + \tan^{-1}\left(\frac{R_n^2(\theta)\sin 2\theta}{1 - R_n^2(\theta)\cos 2\theta}\right), \tag{44}$$

so that

$$\frac{(ze^{1-z})^n}{\sqrt{2\pi n}(\frac{1-z^2}{2z^2})} = e^{i\Psi_n^T(z)}.$$

Continuing, for any $n \geqslant 6$ and for any fixed θ with $\omega_n < \theta \leqslant \pi/2$, it can be directly verified (via differentiation) that the following functions of r,

$$\begin{aligned}
&\frac{(re^{1-r\cos\theta})^n}{\{2\pi n(\frac{1-2r\cos\theta+r^2}{r^2})\}^{1/2}} = \frac{|ze^{1-z}|^n}{\sqrt{2\pi n}\,|\frac{1-z}{z}|}, \quad \text{and} \\
&\frac{(re^{1-r\cos\theta})^n}{\{2\pi n(\frac{1-2r^2\cos 2\theta+r^4}{4r^4})\}^{1/2}} = \frac{|ze^{1-z}|^n}{\sqrt{2\pi n}\,|\frac{1-z^2}{2z^2}|},
\end{aligned} \tag{45}$$

are both *strictly increasing* in r for $r \in (0, 1]$. Then, with Eqs. (33) and (35) and the definitions of D_n and T_n in (39) and (42), the two functions in (45) necessarily satisfy

$$\frac{|ze^{1-z}|^n}{\sqrt{2\pi n}\,|\frac{1-z}{z}|} > \frac{|ze^{1-z}|^n}{\sqrt{2\pi n}\,|\frac{1-z^2}{2z^2}|}, \quad \text{for any } z \text{ on the ray } \{z = re^{i\theta}\colon r > 0\}, \tag{46}$$

with $\omega_n < \theta \leqslant \pi/2$. Hence, for any $n \geqslant 6$, the strictly increasing nature of the functions of r in (45) gives us that the ray $\{z = re^{i\theta}\colon r > 0\}$ *first* intersects the arc D_n, in a unique point $r_n(\theta)e^{i\theta}$, and then later similarly intersects the arc T_n in a unique point $R_n(\theta)e^{i\theta}$, where, from (46), we deduce that

$$R_n(\theta) > r_n(\theta), \quad \text{for } \omega_n < \theta \leqslant \pi/2. \tag{47}$$

The above inequality gives us that the *arc* of T_n, in the first quadrant and in the open disk $\{z \in \mathbb{C}\colon |z - \frac{1}{3}| < \frac{2}{3}\}$, is always *outside* the associated arc of D_n, in the same region, and these paths are shown in the first quadrant of Fig. 7, for the case $n = 12$. We note that arcs D_n and T_n are only defined, from (39) and (42), up to the circle $\{z \in \mathbb{C}\colon |z - \frac{1}{3}| = \frac{2}{3}\}$, where these arcs approach a common point of this circle. This can be seen in Fig. 7 for the case $n = 12$.

Next, noting that the definitions, of $\Psi_n(z)$ of (40) and $\Psi_n^T(z)$ of (43), can be applied to *any* nonzero point z in the first quadrant with $n \geqslant 6$ and with $\omega_n \leqslant \arg z \leqslant \pi/2$, it follows from (40) and (43) that we can express the difference $\Psi_n^T(z) - \Psi_n(z)$ as

$$\Psi_n^T(z) - \Psi_n(z) = \arg\left(\frac{1-z}{z}\right) - \arg\left(\frac{1-z^2}{2z^2}\right) = \arg\left\{\frac{(\frac{1-z}{z})}{(\frac{1-z^2}{2z^2})}\right\},$$

so, that from (33) and (34),

$$\Psi_n^T(z) - \Psi_n(z) = \arg\left(\frac{2z}{1+z}\right) = \arg z - \arg(1+z) > 0, \tag{48}$$

in the open first quadrant. This gives us that

Table 1
$\theta = \frac{\pi}{2}$.

n	$R_n(\frac{\pi}{2})$	$r_n(\frac{\pi}{2})$	$R_n(\frac{\pi}{2}) - r_n(\frac{\pi}{2})$
10	0.49647	0.49112	+0.00535
20	0.43923	0.43466	+0.00457
40	0.40679	0.40398	+0.00281
80	0.38890	0.38736	+0.00154
160	0.37917	0.37836	+0.00081
320	0.37391	0.37350	+0.00041

Table 2
$\theta = \frac{\pi}{3}$.

n	$R_n(\frac{\pi}{3})$	$r_n(\frac{\pi}{3})$	$R_n(\frac{\pi}{3}) - r_n(\frac{\pi}{3})$
10	0.65250	0.64405	+0.00845
20	0.56890	0.56188	+0.00702
40	0.52137	0.51713	+0.00424
80	0.49509	0.49278	+0.00231
160	0.48074	0.47953	+0.00121
320	0.47296	0.47234	+0.00062

Table 3
$\theta = \frac{\pi}{4}$.

n	$R_n(\frac{\pi}{4})$	$r_n(\frac{\pi}{4})$	$R_n(\frac{\pi}{4}) - r_n(\frac{\pi}{4})$
10	0.78415	0.77674	+0.00741
20	0.67560	0.66751	+0.00809
40	0.61372	0.60861	+0.00511
80	0.57938	0.57656	+0.00282
160	0.56055	0.55907	+0.00148
320	0.55030	0.54954	+0.00076

$$\Psi_n^T(z) > \Psi_n(z), \tag{49}$$

where, for θ fixed with $\omega_n < \arg z \leqslant \pi/2$, the ray $\{z = re^{i\theta}\colon r > 0\}$ intersects both arcs D_n and T_n in the first quadrant. The result of (48) will be used below.

Next, we consider more carefully the arcs D_n and T_n in the first quadrant, for $n \geqslant 6$. As can be seen from Fig. 7, the particular arcs D_{12} and T_{12} are *very close*, and we explore this closeness in the following way. Fixing first the angle $\theta = \frac{\pi}{2}$, the ray $\{z = it\colon t \geqslant 0\}$ intersects the arcs D_n and T_n in the points $r_n(\frac{\pi}{2})i$ and $R_n(\frac{\pi}{2})i$, respectively. The moduli of these intersections and their differences are given in Table 1 for different values of n, with similar data being given in Table 2 for $\theta = \frac{\pi}{3}$, and Table 3 for $\theta = \frac{\pi}{4}$.

The results of Tables 1–3 show that the differences $R_n(\theta) - r_n(\theta)$, in the last columns, are all positive, in agreement with (47), but what is more interesting is that, the terms, in the last columns of Tables 1–3, tend to zero, since these terms behave as

$$\frac{E(\theta)}{n}, \quad \text{as } n \to \infty, \text{ where } E\left(\frac{\pi}{2}\right) \doteq 0.136, \tag{50}$$

for example. With this in mind, we next consider the following differences of arguments of

$$\delta_n(z) := \Psi_n^T(z) - \Psi_n(z) = \arg(z) - \arg(1+z) > 0. \tag{51}$$

If $z = R_n(\theta)e^{i\theta}$, then z is a point of the arc T_n, so that if

$$\Psi_n^T\left(R_n(\theta)e^{i\theta}\right) = 2\pi k, \quad k \text{ a positive integer},$$

then, by definition, $R_n(\theta)e^{i\theta}$ is an approximate rotated zero of $\cos_n(nz)$, n even, or of $\sin_n(nz)$, n odd; similarly, $\Psi_n(r_n(\theta)e^{i\theta})$ is, by definition, a point of the arc D_n.

Next, we give values of $\delta_n(r_n(\theta)e^{i\theta})$ and $\delta_n(R_n(\theta)e^{i\theta})$ in Tables 4 and 5, which show that these differences are *huge*, compared with the differences of $R_n(z) - r_n(z)$ in Tables 1–3.

We remark that, for example, the second and third columns of Table 5 for $\theta = \frac{\pi}{4}$ can be shown to converge to 0.51639 as $n \to \infty$.

The above material can be interpreted as follows: Let $z = R_n(\theta)e^{i\theta}$ be a point of the arc T_n, with $\Psi_n^T(z) = 2\pi k$. Then, a small decrease in $R_n(\theta)$ to $r_n(\theta)$, gives a point, $r_n(\theta)e^{i\theta}$, of the arc D_n, where the difference $R_n(\theta) - r_n(\theta)$ appears in the final columns of Tables 1–3. But then, $\delta_n(r_n(\theta)e^{i\theta})$ of (51) is large and *positive*, from the second columns of Tables 4–5, which means that, while $r_n(\theta)e^{i\theta}$ is a point of D_n, then the inequality of (51) gives us that

Table 4
$\theta = \frac{\pi}{3}$.

n	$\delta_n(r_n(\theta)e^{i\theta})$	$\delta_n(R_n(\theta)e^{i\theta})$	$\delta_n(r_n(\theta)e^{i\theta}) - \delta_n(R_n(\theta)e^{i\theta})$
10	0.48213	0.47958	0.00255
20	0.50836	0.50602	0.00234
40	0.52377	0.52227	0.00150
80	0.53252	0.53167	0.00085
160	0.53738	0.53694	0.00044
320	0.54006	0.53983	0.00023

Table 5
$\theta = \frac{\pi}{4}$.

n	$\delta_n(r_n(\theta)e^{i\theta})$	$\delta_n(R_n(\theta)e^{i\theta})$	$\delta_n(r_n(\theta)e^{i\theta}) - \delta_n(R_n(\theta)e^{i\theta})$
10	0.44470	0.44276	0.00194
20	0.47510	0.47271	0.00239
40	0.49314	0.49152	0.00162
80	0.50349	0.50256	0.00093
160	0.50931	0.50881	0.00050
320	0.51253	0.51227	0.00026

$$\Psi_n\big(r_n(\theta)e^{i\theta}\big) < 2\pi k. \tag{52}$$

But, $\Psi_n(r_n(\theta)e^{i\theta})$ is a *strictly increasing function* of θ from [2, p. 118], so that there is a $\theta' > \theta$ for which

$$\Psi_n\big(r_n(\theta')e^{i\theta'}\big) = 2\pi k. \tag{53}$$

As this argument holds for all $n \geqslant 6$, we have the desired result of

Theorem 3.2. *For any $n \geqslant 6$, assume that θ satisfies $\omega_n < \theta < \pi/2$, and that $R_n(\theta)e^{i\theta}$ is a point of the arc T_n, in the open first quadrant, with*

$$\Psi_n^T\big(R_n(\theta)e^{i\theta}\big) = 2\pi k, \quad k \text{ a positive integer}, \tag{54}$$

so that $R_n(\theta)e^{i\theta}$ is a rotated approximate zero of $\cos_n(nz)$, n even, or of $\sin_n(nz)$, n odd. Then, there is an $r_n(\theta')e^{i\theta'}$, a point of the arc D_n, for which

$$\Psi_n\big(r_n(\theta')e^{i\theta'}\big) = 2\pi k \quad (\text{same } k \text{ as in } (54)),$$

so that $r_n(\theta')e^{i\theta'}$ is a zero of $s_n(nz)$ in the first quadrant, for which

$$R_n(\theta) > r_n(\theta'), \quad \text{and} \quad \theta' > \theta. \tag{55}$$

Remark. The items of (55) *exactly* corresponds to the "definite pattern" noted in (i) and (ii) of Section 2, after one rotates to the first quadrant.

Continuing, we note that $\cos_n(nz)$, n even, and $\sin_n(nz)$, n odd, have no non-real rotated zeros, in the open first quadrant of Δ, for any $1 \leqslant n \leqslant 4$, while for $n = 5$, $\sin_5(5z)$ has one rotated zero and $s_5(5z)$ has one zero in the open first quadrant, where these two zeros possess the "definite pattern"! As the cases for $n \geqslant 6$ are covered in Theorem 3.2, then this "definite pattern" holds for *all* n, as remarked at the end of Section 2.

4. Estimating the number of real and non-real zeros of $\cos_n(nz)$, n even, and of $\sin_n(nz)$, n odd

In this section, we give easily computed estimates of the number of real and non-real zeros of $\cos_n(nz)$, n even, and of $\sin_n(nz)$, n odd.

To begin, we recall that the closed arc T_n of (42), in the first quadrant and for $n \geqslant 6$, is such that $z = R_n(\theta)e^{i\theta}$ is a unique point of the arc T_n for each θ with $\omega_n \leqslant \theta \leqslant \pi/2$. This implies from (42) that $z = R_n(\theta)e^{i\theta}$ satisfies

$$\big(R_n(\theta)e^{1-R_n(\theta)\cos\theta}\big)^n = \left\{\frac{\pi n(1 - 2R_n^2(\theta)\cos(2\theta) + R_n^4(\theta))}{2R_n^4(\theta)}\right\}^{1/2}, \tag{56}$$

where $\omega_n \leqslant \theta \leqslant \pi/2$, and if

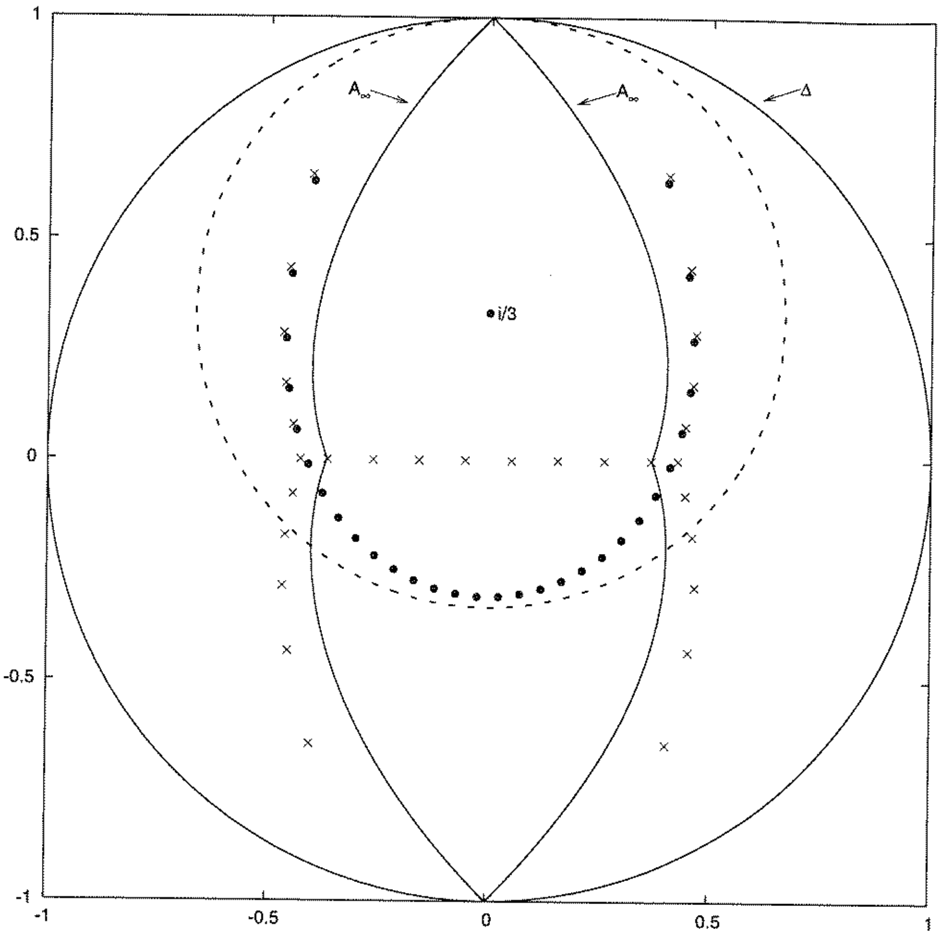

Fig. 8. The rotated zeros of $s_{30}(30z)$ (dots) and the zeros of $\cos_{30}(30z)$ (crosses).

$$\frac{2z^2(ze^{1-z})^n}{\sqrt{2\pi n}(1-z^2)} =: e^{i\Psi_n^T(z)}, \tag{57}$$

then $\Psi_n^T(z)$ is a real-valued strictly increasing function of θ on the interval $\omega_n \leqslant \theta \leqslant \pi/2$, given by

$$\Psi_n^T(z) := n\big[\theta - R_n(\theta)\sin\theta\big] + 2\theta + \tan^{-1}\left\{\frac{R_n^2(\theta)\sin 2\theta}{1 - R_n^2(\theta)\cos 2\theta}\right\}. \tag{58}$$

Our interest now is in the case when $\theta = \pi/2$, so that from (56) and (58),

$$R_n\left(\frac{\pi}{2}\right)e = \left\{\frac{\pi n(1 + R_n^2(\pi/2))^2}{2R_n^4(\pi/2)}\right\}^{1/2n}, \tag{59}$$

and

$$\Psi_n^T\left(iR_n\left(\frac{\pi}{2}\right)\right) = n\left[\frac{\pi}{2} - R_n\left(\frac{\pi}{2}\right)\right] + \pi, \tag{60}$$

where $\Psi_n^T(iR_n(\pi/2))/2\pi$ is a *measure* of the number of zeros of $\cos_n(nz)$, n even, or of $\sin_n(nz)$, n odd, which lie outside of the set K of (29), in the closed first quadrant of Δ. As this number of zeros is necessarily a nonnegative integer, then using the floor function,

$$\lfloor x \rfloor := \text{largest integer} \leqslant x \quad \text{for any real } x,$$

applied to (60), gives us the result of

Theorem 4.1. *For any positive integer $n \geqslant 6$, $\lfloor \Psi_n^T(iR_n(\pi/2))/2\pi \rfloor$ estimates the number of zeros of $\cos_n(nz)$, n even, or of $\sin_n(nz)$, n odd, which lie outside the set K of (29) in the closed first quadrant of Δ.*

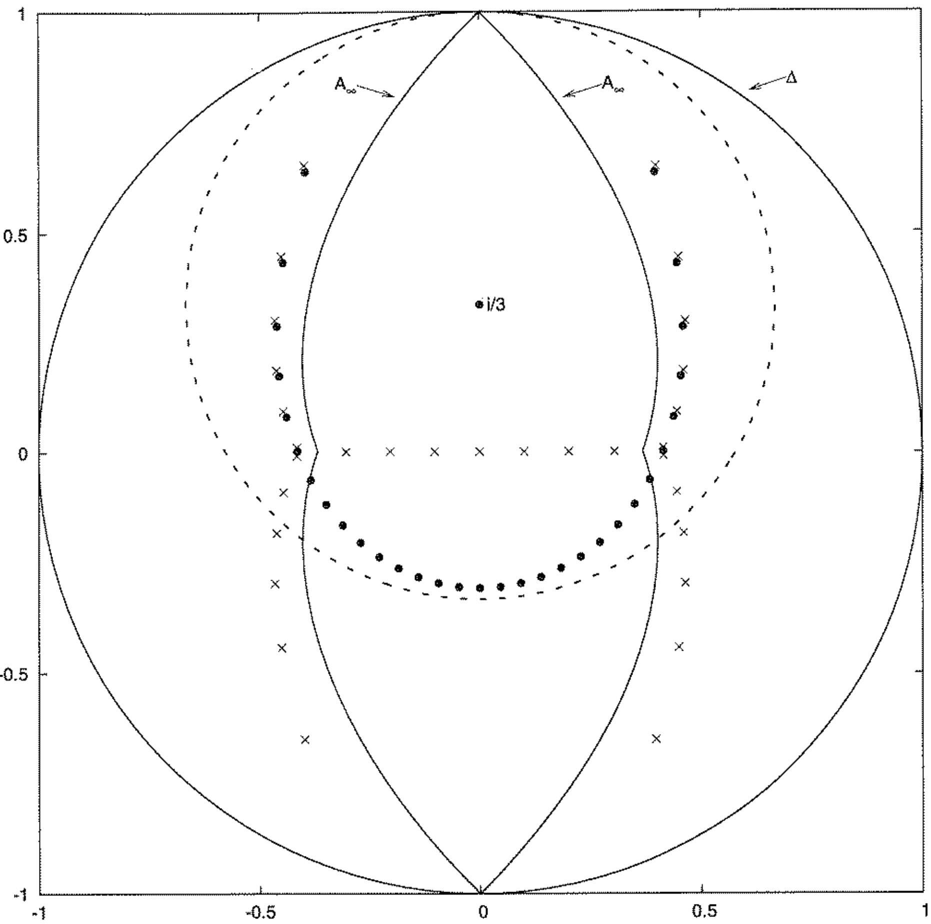

Fig. 9. The rotated zeros of $s_{31}(31z)$ (dots) and the zeros of $\sin_{31}(31z)$ (crosses).

As an example of Theorem 4.1, consider the case of $n = 30$, where we wish to estimate the number of zeros, of $\cos_{30}(30z)$, in the closed first quadrant of Δ, which lie *outside* the set K. It can be verified from (59) and (60) that

$$R_{30}\left(\frac{\pi}{2}\right) = 0.41800, \quad \text{and} \quad \Psi_{30}^T\big(iR_{30}(\pi/2)\big)/2\pi = 6.00418,$$

so that $\lfloor \Psi_{30}^T(iR_{30}(\pi/2))/2\pi \rfloor = 6$, and we see, from Fig. 8, that the number of zeros of $\cos_{30}(30z)$ *outside* the set K, in the closed first quadrant, is exactly 6, where one of these 6 zeros is real and positive. Similarly, for $n = 31$, we have

$$R_{31}\left(\frac{\pi}{2}\right) = 0.41658, \quad \text{and} \quad \Psi_{31}^T\big(iR_{31}(\pi/2)\big)/2\pi = 6.19466,$$

so that $\lfloor \Psi_{31}^T(iR_{31}(\pi/2))/2\pi \rfloor = 6$, which is again correct from Fig. 9, but in this case, there is no real zero in the estimate.

The point of Theorem 4.1 is that estimating the number of zeros of $\cos_n(nz)$, for n even, or of $\sin_n(nz)$, for n odd, in the closed first quadrant, can be easily determined from a calculation of $\lfloor \Psi_n^T(iR_n(\pi/2))/2\pi \rfloor$, which is far simpler than finding *all* the zeros of $\cos_n(nz)$, n even, or of $\sin_n(nz)$, n odd. We further note that the result of Theorem 3.2 easily extends to the estimation of the number of zeros of $\cos_n(nz)$, n even, or of $\sin_n(nz)$, n odd, in *any* sector $0 < \theta_1 \leqslant \theta_2 \leqslant \pi/2$ of Δ, upon determining the two numbers $\Psi_n^T(R_n(\theta_1)e^{i\theta_1})$ and $\Psi_n^T(R_n(\theta_2)e^{i\theta_2})$, from (58), and using $\lfloor (\Psi_n^T(R_n(\theta_2)e^{i\theta_2}) - \Psi_n^T(R_n(\theta_1)e^{i\theta_1}))/2\pi \rfloor$.

The previous paragraphs were aimed at estimating the number of zeros of $\cos_n(nz)$, n even, and of $\sin_n(nz)$, n odd, which lie *outside* of K, which leaves open the estimation of their *real* zeros. To this end, we mention that the results of Ref. [7] can be used, as follows. For any $n \in \mathbb{N}$, set

$$\xi_n^+ := \sin^{-1}\left(\frac{1+e^{-2n}}{4}\right) \quad (\text{all } n \in \mathbb{N}), \tag{61}$$

where $\xi_n^+/\pi = 0.08043$ for *all* $n \geqslant 5$. Then, for n even ($n \in 2\mathbb{N}$), let

$$\frac{n}{e\pi}+\frac{1}{2}=k_n+t_n, \quad \text{where } k_n:=\left\lfloor \frac{n}{e\pi}+\frac{1}{2}\right\rfloor \text{ and } 0<t_n<1. \tag{62}$$

It follows from Lemma 3.1 of [7] that if $\xi_n^+/\pi \leqslant t_n$, then $\cos_n(nz)$ has exactly $2k_n$ simple real zeros in the interval $[-\frac{1}{e},+\frac{1}{e}]$. If $t_n<\xi_n^+/\pi$, then from Lemma 4.1 of [7], there are exactly $2k_n$ simple real zeros of $\cos_n(nz)$ in the slightly larger interval $(-\rho_n,+\rho_n)$, where $\rho_n:=\frac{1}{e}(1+\frac{1}{n})$.

As an example, for $n=30$, we have from (62) that $k_{30}=4$ and $t_{30}=0.01298$, so that $t_{30}<\xi_{30}^+/\pi$. In this case, $\rho_{30}=0.38014>\frac{1}{e}=0.36787$, and $\cos_{30}(30z)$ has 8 simple real zeros in the interval $[-0.38014,+0.38014]$, which is correct from Fig. 8.

Similarly, for m an odd positive integer ($m\in 2\mathbb{N}-1$), let

$$\frac{m}{e\pi}=k_m+t_m, \quad \text{where } k_m:=\left\lfloor \frac{m}{e\pi}\right\rfloor \text{ and } 0<t_m<1. \tag{63}$$

Then, from Lemma 3.2 of [7], if $\xi_m^+/\pi \leqslant t_m$, then $\sin_m(mz)$ has exactly $2k_m+1$ simple real zeros in the interval $[-\frac{1}{e},+\frac{1}{e}]$. If $t_m<\xi_m^+/\pi$, then from Lemma 4.2 of [7], then there are exactly $2k_m+1$ simple real zeros of $\sin_m(mz)$ in the interval $(-\rho_m,\rho_m)$, where $\rho_m:=\frac{1}{e}(1+\frac{1}{m})$.

As a final example, consider the case $m=31$. From (63), we have $k_{31}=3$, $t_{31}=0.63008$, and $\xi_{31}^+/\pi=0.08043$, so that $\xi_{31}^+<t_{31}$. Hence, $\sin_{31}(31z)$ has exactly 7 simple real zeros which all lie in the interval $[-\frac{1}{e},+\frac{1}{e}]$, as can be seen to be true from Fig. 9.

One item we have not discussed here is the occurrence of, and distinction between, so-called *real Hurwitz zeros* of $\cos_n(nz)$ or $\sin_n(nz)$, and *real spurious zeros* of $\cos_n(nz)$ and $\sin_n(nz)$, which has to do with the special behavior of real zeros of $\cos_n(nz)$ or $\sin_n(nz)$, which arise outside the real interval $[-\frac{1}{e},+\frac{1}{e}]$. This we leave for another occasion.

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