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DISCRETIZATION ERRORS FOR WELL-SET CAUCHY PROBLEMS. I.*

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1. Introduction. It is generally believed that, given a well-set Cauchy problem, there exist accurate and numerically "stable" *difference approximations* to the differential equations (DE's) defining the problem, from which (in principle) the solution can be computed to any desired degree of accuracy in finite time. However, to do this in practice is a very different matter.

Specifically, one must construct a difference approximation which:

(i) simulates the DE's to a reasonably high *order of accuracy*, and

(ii) is *numerically stable* to small errors (e.g., of roundoff).

As is well known, condition (ii) may fail if too large time-steps are taken. The most practical test for (ii) is provided by the *von Neumann condition* (§7).

An attempt to treat the problem from a rigorous modern standpoint naturally leads one to interpret solutions $\mathbf{u}(\mathbf{x}, t)$ of well-set problems as orbits $u(t)$ in suitable function spaces. For example, if these orbits lie in a *Banach space B*, then one can define the *discretization error* at time t (following Kantorovich [9]) as the norm $\| v(t) - u(t) \|$ in B of the difference between the approximate solution $v(t)$ and the exact solution $u(t)$. One's objective is really to *minimize* this error, in a given amount of computing time.

There are few systematic general theorems which one can apply to this question. The most important of these is due to Lax and Richtmyer [11]. It applies to general systems of first-order *linear* partial DE's with *constant coefficients*, in domains without boundaries. It states that a *given consistent* difference scheme for solving such a Cauchy problem, well-set in a *given* Banach space, is *convergent* if and only if it is *uniformly bounded* ("stable"). In special cases, similar results were obtained independently by Douglas [5].

In the present paper, we deal with the same class of linear Cauchy problems with constant coefficients, assuming only the coefficients given. It is shown how, for any such problem which is "well-set" in the sense of Petrowsky, one can obtain *semi-discretizations* having arbitrarily high order of consistency and finite "stability index" (the natural *algebraic* criterion for stability), by using central differences; our main results for semi-discretizations are in §5.

We then study true *discretizations*, and in particular those obtained by combining central space differences with Padé approximations to e^z in time. Perhaps our strongest result (Theorem 5) is that uniform central space semi-discretization, coupled with diagonal Padé approximations, can give arbitrarily high orders ν of consistency. Moreover, for such discretizations, the von Neumann condition is satisfied with $\Delta t = rh^\alpha$ independently of r and α.

Finally, we apply the general machinery developed to an interesting (apparently new) difference approximation to systems equivalent to the telegraph and Klein-Gordon equations.

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In Part II (not yet completed), we will show how one can interpret our conclusions in suitable Banach spaces defined by "Fourier integral norms".

2. Notation: stability index. An important class of Cauchy problems, amenable to Fourier transform methods, refers to systems of linear partial DE's with constant coefficients of the normal form

$$\partial u_j/\partial t = \textstyle\sum_{k=1}^{n} p_{jk}(D_1, \cdots, D_r)u_k ; \qquad D_i = \partial/\partial x_i , \tag{1}$$

where the p_{jk} are polynomials with real or complex coefficients:

$$p_{jk}(\mathbf{D}) = \textstyle\sum_{\ell} \alpha_{jk\ell} D_1^{l_1} \cdots D_r^{l_r}, \qquad \ell = (l_1, \cdots, l_r). \tag{1'}$$

We exclude the trivial case that $\alpha_{jk\ell} = 0$ unless $\ell = 0$.

One can consider the Cauchy ("initial value") problem for (1) on any Cartesian product $X = X_{r,s} = R^s C^{r-s}$ of Euclidean s-space R^s and an $(r - s)$-torus C^{r-s}. The *dual* of X is $Q = X^* = R^s Z^{r-s}$ as in [2, §1]. The x_i and t are all *real*: R is the real line; C the real circle; Z the set of integers.

Let $P(i\mathbf{q})$ denote the $n \times n$ matrix with entries $p_{jk}(iq_1, \cdots, iq_r)$ and let $\lambda_l(\mathbf{q})$ denote its eigenvalues. Then the *stability index* for the Cauchy problem (1) on X is defined as in [2, §3] by

$$\Lambda(P) = \sup_{l,\mathbf{q}} \{\mathrm{Re}\, \lambda_l(\mathbf{q})\}, \qquad \mathbf{q} \in Q. \tag{2}$$

If $\rho(T)$ denotes the *spectral radius* [15, p. 9] of a matrix T (i.e., the maximum of the moduli of the eigenvalues of T), then the stability index of (1) can also be described as

$$\Lambda(P) = \sup_{\mathbf{q}} \{\ln \rho[\exp P(i\mathbf{q})]\}, \qquad \mathbf{q} \in Q. \tag{2'}$$

It is bounded if and only if the Cauchy problem for (1) is *well-set* for $t \geqq 0$. More precisely, it was shown in [2] that $\Lambda(P) < +\infty$ if and only if (1) defines a C_0-semigroup on a suitable Banach space.

Following the notation of [2], we can formally write

$$\mathbf{u}(\mathbf{x}; t) = \int_Q \mathbf{f}(\mathbf{q}, t) e^{i\mathbf{q}\cdot\mathbf{x}}\, dQ, \qquad dQ = dq_1 \cdots dq_r . \tag{3}$$

It follows from (1) that

$$d\mathbf{f}(\mathbf{q}, t)/dt = P(i\mathbf{q})\mathbf{f}(\mathbf{q}, t), \tag{4}$$

from which we infer (still formally) that

$$\mathbf{u}(\mathbf{x}, t) = \int_Q \exp\,[tP(i\mathbf{q})]\mathbf{f}(\mathbf{q}, 0) e^{i\mathbf{q}\cdot\mathbf{x}}\, dQ. \tag{5}$$

For any fixed $\mathbf{q}$, the linear system (4) of ordinary DE's is called *strictly stable* when $\mathrm{Lim}_{t\uparrow+\infty}\, e^{tP} = 0$, and *stable* when $\mathrm{Lim\,sup}_{t\uparrow+\infty} \| e^{tP} \| < +\infty$; cf. [3, p. 81]. (These conditions are independent of the matrix norm $\| \quad \|$; cf. §5.) It is well known that these are equivalent to the following conditions on the eigenvalues $\lambda_l(\mathbf{q})$ of $P(i\mathbf{q})$, respectively:

$$\Lambda(P(i\mathbf{q})) = \sup_l \mathrm{Re}\, \{\lambda_l(i\mathbf{q})\} < 0 \qquad \text{(strict stability)}, \tag{6a}$$

and

(6b) $\Lambda(P(i\mathbf{q})) = 0$, and $\operatorname{Re}\{\lambda_l(i\mathbf{q})\} = 0$ implies that

λ_l is a non-degenerate eigenvalue for $P(i\mathbf{q})$. (stability)

We shall therefore call the system (1) *strictly stable* when (6a) holds for all real wave-vectors $\mathbf{q}$, and *stable* when (6b) holds for all such $\mathbf{q}$.

Clearly, the condition $\Lambda(P) < 0$ is *sufficient* for *strict stability*, while the condition $\Lambda(P) \leqq 0$ is *necessary* for *stability*, as defined above. Following [2], we shall call (1) *regular* when $\Lambda(P) < +\infty$. Evidently, one can convert any regular system (1) to a strictly stable system by setting $v = e^{-t[\Lambda(P)+1]}u$. This replaces (1) by a system of the same form with stability index -1.

Analogous notions will be defined in §§3, 7 for discrete and semi-discrete approximations to (1).

3. Semi-discretization. We now consider a *uniform rectangular* spatial mesh H in X. We can reduce to this case (that H is rectangular) without loss of generality, because of the following result.

LEMMA 1. Let H be any discrete r-dimensional mesh in the quotient-group $R^r/S = X_{r,s}$, where $S = Z^s$. Then H has an integral basis of vectors $\mathbf{h}_1, \cdots, \mathbf{h}_r$ in R^r/S, such that $x_1\mathbf{h}_1 + \cdots + x_r\mathbf{h}_r \in S$ if and only if each x_i is an integral multiple of n_i, where n_i is some nonnegative integer.

Proof. Let $\bar{H}$ be the discrete subgroup of R^r consisting of the cosets of S which correspond to mesh-points (we assume one mesh-point at the origin). Then $\bar{H}$ is a free discrete Abelian group with r generators, which contains S as a subgroup. The main theorem about canonical forms of matrices of integers under row- and column-equivalence asserts* that bases can be constructed for $\bar{H}$ and S with the property specified, for which moreover $n_i \mid n_{i+1}$ for $i = 1, \cdots, r-1$. Q. E. D.

Accordingly, let $\mathbf{h} = (h_1, \cdots, h_r)$, $h_j > 0$, be the vector whose components specify the mesh spacings in each coordinate direction of X, and let H be the set of all mesh-vectors, $\mathbf{mh} = (m_1h_1, \cdots, m_rh_r)$, where the m_i are integers.

On any such rectangular spatial mesh H, there are many ways in which one can approximate each partial derivative $D^l u_j = D_1^{l_1} \cdots D_r^{l_r} u_j$ of any $u_j(\mathbf{x})$, by a divided difference of the form

$$(7) \qquad \left(\prod\nolimits_k h_k^{-l_k}\right) \sum\nolimits_{\mathbf{m}} \mu_{\mathbf{m}}^{(l)} u_j(x_1 + m_1h_1, \cdots, x_r + m_rh_r) = \delta^l[u_j],$$

where the $\mu_{\mathbf{m}}^{(l)}$ are *fixed*, finite in number, and *independent* of h. The assignment of a particular choice of a set of such divided difference approximations (7) to each term $\alpha_{j,k,l}D^{(l)}$ in (1′) will be called a *semi-discrete*† *finite difference scheme* Δ. When both the scheme Δ and the mesh-vector $\mathbf{h}$ are specified, we will speak of a *semi-discretization* of a system (1), and denote the resulting approximation to $P(D_1, \cdots, D_r)$ by $\Pi(\Delta, \mathbf{h})$.

* As in [14]. The idea of semi-discretization goes back to Lagrange. See also D. R. Hartree and J. Wormersley, Proc. Roy. Soc. A161 (1937), 353–66.

† See for example R. Thrall and L. Tornheim, "Vector spaces and matrices", Wiley, 1957, p. 241.

EXAMPLE 1. Let (1) be the *convection equation* $u_t + u_x = 0$, whose general exact solution is $u(x, t) = u(x - t, 0)$. Then $r = n = \ell$, $P(iq) = -iq$, and $\Lambda(P) = 0$. Consider the semi-discretization scheme (7) defined by the backward divided difference $u_x \doteqdot [u(x) - u(x - h)]/h$. For given h, this replaces $u_t + u_x = 0$ on the mesh of mh by

$$dy_m/dt = [y_{m-1} - y_m]/h, \qquad y_m(t) = u(mh, t). \tag{8}$$

For the initial values $y_m(0) = \delta_m^0$ (delta-function), the solution is given explicitly by

$$y_m(\tau h) = \begin{cases} e^{-\tau}\tau^m/(m!), & m \geqq 0, \\ 0 & m < 0; \end{cases} \tag{8'}$$

the semi-discretization (8) defines a semigroup of nonnegative linear transformations for $t > 0$. Clearly, the $y_m(t)$ are the expectations of the Poisson process*. For fixed $t = \tau h$, the maximum of the approximate solution occurs near $m = \tau$. Using Stirling's formula to approximate $m!$, we see that this maximum is $y_m(mh) \doteqdot 1/\sqrt{2\pi m}$. Qualitatively, whereas any exact solution propagates along the characteristics $x - t =$ const. without change of form, the semi-discrete approximation (8) produces considerable *dispersion* over a belt of width $O(\sqrt{m})$ in distance mh.

In general, each difference operator (7) carries $e^{i\mathbf{q}\cdot\mathbf{x}}$ into a constant multiple of itself. Therefore, any semi-discrete approximation $\Pi = \Pi(\Delta, \mathbf{h})$ to the operator $P(D_1, \cdots, D_r)$ in (1) has the property that, for any complex n-vector $\mathbf{b}$ and any real n-vector $\mathbf{q}$

$$\Pi[\mathbf{b}e^{i\mathbf{q}\cdot\mathbf{x}}] = A(\mathbf{q}, \Pi)\mathbf{b}e^{i\mathbf{q}\cdot\mathbf{x}}. \tag{9}$$

Here, $A(\mathbf{q}, \Pi)$ is an $n \times n$ complex matrix defined from Π by (9). We call $A(\mathbf{q}, \Pi)$ the *infinitesimal amplification matrix* associated with (1) and (7) (as in [13, p. 55]).

Evidently, $A(\mathbf{q}, \Pi)$ is an approximation to $P(i\mathbf{q})$, whose entries are *continuous* in $\mathbf{q}$. Denoting the eigenvalues of $A(\mathbf{q}, \Pi)$ by $\lambda_l(\mathbf{q}, \Pi)$, we define, in analogy to (2), the *stability index* of Π as

$$\Lambda(\Pi) = \sup_{l,\mathbf{q}} \{\mathrm{Re}\, \lambda_l(\mathbf{q}, \Pi)\}. \tag{10}$$

A semi-discretization scheme Δ will be called *uniformly regular* if, as $\mathbf{h}$ varies,

$$\sup_{\mathbf{h}} \Lambda(\Pi(\Delta, \mathbf{h})) < +\infty. \tag{10'}$$

To compute the (j, k) entry $a_{jk}(\mathbf{q}, \mathbf{h})$ of $A(\mathbf{q}, \Pi)$, consider again the partial derivative $D^{(l)}u_j = D_1^{l_1} \cdots D_r^{l_r} u_j$. Inserting the finite difference approximation (7) of this partial derivative into (9) contributes

$$\left(\prod_k h_k^{-l_k}\right) \sum_{\mathbf{m}} \mu_{\mathbf{m}}^{(l)} e^{i\mathbf{q}\cdot\mathbf{mh}}$$

* W. Feller, "Probability Theory", 2d ed., Wiley, 1950, p. 401. To derive (8′), one can write $\mathbf{y}(t) = \exp[t(\sigma - I)]\mathbf{y}(0)$, where σ is the *shift* operator $\sigma[y_m] = y_{m-1}$.

to the entry $a_{jk}(\mathbf{q}, \mathbf{h})$. Hence, from (1′), we have

$$(9') \qquad a_{jk}(\mathbf{q}, \mathbf{h}) = \textstyle\sum_{\mathbf{l}} \alpha_{jkl}\{(\prod_k h_k^{-l_k}) \sum_{\mathbf{m}} \mu_{\mathbf{m}}^{(l)} e^{i\mathbf{q}\cdot\mathbf{m}\mathbf{h}}\}.$$

Any such semi-discrete approximation defines, in analogy to (3)–(5):

$$(11) \qquad \mathbf{w}(\mathbf{x}; t) = \int_Q \mathbf{f}(\mathbf{q}, t) e^{i\mathbf{q}\cdot\mathbf{x}}\, dQ,$$

$$(12) \qquad \partial\mathbf{f}(\mathbf{q}, t)/\partial t = A(\mathbf{q}, \Pi)\mathbf{f}(\mathbf{q}, t), \qquad \text{and}$$

$$(13) \qquad \mathbf{w}(\mathbf{x}, t) = \int_Q \exp\,[tA(\mathbf{q}, \Pi)]\mathbf{f}(\mathbf{q}, 0) e^{i\mathbf{q}\cdot\mathbf{x}}\, dQ.$$

EXAMPLE 2. For the pair of equations $u_t = v_x$, $v_t = u_x$, arising from the *wave equation* $u_{tt} = u_{xx}$, the functions $p_{jk}(D)$ of (1) are simply $p_{11} = 0 = p_{22}$, $p_{12} = D_1 = p_{21}$. The finite difference scheme (cf. (7))

$$\Delta\colon p_{12} = v_x \doteqdot \frac{v(x + h) - v(x)}{h}; \qquad p_{21} u = u_x \doteqdot \frac{u(x) - u(x - h)}{h}$$

gives rise for each h to the 2×2 infinitesimal amplification matrix

$$A(q, \Pi) = \begin{bmatrix} 0 & (e^{iqh} - 1)/h \\ (1 - e^{-iqh})/h & 0 \end{bmatrix}, \qquad \Pi = \Pi(\Delta, h),$$

whose entries are continuous in q. Note for comparison that, for any fixed q, $P(iq) = \begin{bmatrix} 0 & iq \\ iq & 0 \end{bmatrix}$ satisfies $|\, a_{jk}(iq) - p_{jk}(iq)| = 0(h)$: in the terminology to be introduced in §5, the difference approximations of Examples 1–2 are consistent to order one.

4. Central difference schemes. An important class of one-parameter semi-discrete difference schemes is defined by letting each spatial derivative $D_k u_j$ be approximated by the simplest *central difference* quotient:

$$(14) \qquad D_k u_j \doteqdot \frac{u_j(\mathbf{x} + h_k \mathbf{e}_k) - u_j(\mathbf{x} - h_k \mathbf{e}_k)}{2h_k} = \frac{\delta u_j}{\delta x_k};$$

here, $\mathbf{e}_k$ is the k-th unit vector, and we have used the central difference symbol δ usually reserved for $[u_j(x + h/2) - u_j(x - h/2)]/h$. Higher order derivatives can be approximated by simply iterating (14). In practice, one usually replaces second derivatives by

$$(14') \qquad D_k^2 u_j \doteqdot \delta_k^2 u_j / h_k^2,$$

where

$$\delta_k^2 u_j = [u_j(\mathbf{x} + h_k \mathbf{e}_k) - 2u_j(\mathbf{x}) + u_j(\mathbf{x} - h_k \mathbf{e}_k)].$$

EXAMPLE 3. As in Example 1, let (1) be $u_t + u_x = 0$. Then (14) replaces the ordinary DE (9) by

$$(15) \qquad dy_m/dt = [y_{m-1} - y_{m+1}]/2h.$$

This scheme has higher "order of consistency" $\nu = 2$ than that $\nu = 1$ of Example 1, since we now have

$$| a(q, \Pi) - p(iq)| = 0(h^2) \qquad \text{for any fixed} \quad q,$$

whereas in Example 1 the difference was only $0(h)$.

Nevertheless, the scheme of Example 1 is actually preferable! This can be seen by integrating (15) exactly for $y_m(0) = \delta_m^0$. Again setting $y_m(0) = \delta_m^0$ and $t = \tau h$, we see that since the Bessel functions satisfy [3, p. 58, Ex. 6]

$$2J'_m(x) = J_{m-1}(x) - J_{m+1}(x) \tag{15'}$$

and $J_m(0) = \delta_m^0$, $y_m(h\tau) = J_m(\tau)$ for all m and τ. For any fixed m, $J_m(\tau) = 0(\tau^{\frac{1}{2}})$ as $\tau \to \infty$. Moreover the semi-discretization (15′) makes $J_{-m}(\tau) = (-1)^m J_m(\tau)$, whence

$$\sum_{m=1}^{\infty} | y_m(t) |^2 = \sum_{m=1}^{\infty} | y_{-m}(t) |^2,$$

which does not represent reality at all. Although the semi-discretization (8) involves some unrealistic diffusion, it makes $y_m(t)$ vanish for $m < 0$, $t > 0$, and makes $y_m(t)$ vanish exponentially as $t \to \infty$ for any fixed m, as it should. (Incidentally, the value of $y_m(m)$—which would be 1 if the method were exact—is with (15) asymptotically*

$$J_m(m) = y_m(mh) \doteqdot Km^{-\frac{1}{3}} + 0(m^{-\frac{5}{3}}),$$

where $K = \Gamma(\frac{1}{3})/2^{\frac{2}{3}}3^{\frac{1}{6}}\pi$.)

In general, for $u = e^{i\mathbf{q}\cdot\mathbf{x}}$, the central difference approximation (14) yields

$$\delta u_j/\delta x_k = iq_k[\sin(h_k q_k)/h_k q_k]u = i\tilde{q}_k u. \tag{16}$$

Formulas (16) and (9) together show that, for any real vector $\mathbf{q}$, the entries of the matrix $A(\mathbf{q}, \Pi)$ are exactly the same as the corresponding ones of the matrix $P(i\tilde{\mathbf{q}})$ for suitable $\tilde{\mathbf{q}}$. That is, we have the following

LEMMA 2. For the semi-discrete difference scheme Δ_c based on the central difference approximations (14),

$$A(\mathbf{q}, \Pi) = P(i\tilde{\mathbf{q}}), \quad \text{where} \quad \tilde{q}_k = \left(\frac{\sin(h_k q_k)}{h_k q_k}\right) q_k. \tag{17}$$

Now, returning to the ideas introduced at the end of §2, we make a definition.

DEFINITION. The *algebraic spectrum* $S(\Pi)$ of Π is the set of all eigenvalues $\lambda_l(\mathbf{q}, \Pi)$ of $A(\mathbf{q}, \Pi)$, for (real) $\mathbf{q} \in Q$.

COROLLARY 1. In R^r, the algebraic spectrum of $\Pi(\Delta_c, \mathbf{h})$ is a subset of that of P, and

$$\Lambda(\Pi) \leqq \Lambda(P), \qquad \text{for any} \quad \mathbf{h}. \tag{18}$$

* Watson, "Bessel functions", p. 232, (2). We note that (15) is a widely recommended recipe for treating hyperbolic equations such as $u_t + u_x = 0$. See K. Friedrichs, Comm. pure appl. math. 7 (1954), 345–92; P. D. Lax, ibid., 159–93; [8, p. 136].

In other words, if $P(i\mathbf{q})$ is regular in R^r, then $\Pi(\Delta_c, \mathbf{h})$ is *uniformly regular* as $\mathbf{h}$ varies*. We therefore have

COROLLARY 2. If (1) is regular in R^r, then the systematic use of central space differences (14) yields a semi-discretization which is uniformly regular (i.e., satisfies $\Lambda(\Pi(\Delta_c, \mathbf{h})) \leqq \Lambda^* < \infty$, where Λ^* is independent of $\mathbf{h}$.

With *hyperbolic* DE's, the spectrum of P is usually concentrated near the imaginary axis, and that of e^{tP} correspondingly is concentrated near the unit circle (especially for small t). In Examples 1–3, for instance, the spectrum is precisely the imaginary axis. Likewise, the spectrum of the (hyperbolic) telegraph equation (53), to be discussed below, consists of the interval $-1 \leqq \lambda \leqq 0$ of the negative real axis, plus the line $\lambda = -1 \pm i\nu$ parallel to the imaginary axis.

The spectrum of the heat conduction equation, to be discussed below, is very different: it consists of the negative real axis $(-\infty, 0]$: $\text{Lim}_{q\to\infty} \text{Re}\{\lambda_l(iq)\} = -\infty$. This behavior is typical of *parabolic*† *DE*'s and provides a fundamental difference between hyperbolic and parabolic systems. As we shall now see, this difference makes it much easier to discretize parabolic systems stably and accurately.

EXAMPLE 4. Let (1) be the *heat conduction equation* $u_t = u_{xx}$. Then (14′) specializes to

$$h^2\, dy_m/dt + 2y_m = y_{m+1} + y_{m-1}. \tag{19}$$

Setting $\tau = 2t/h^2$ and $z_m = e^{\tau} y_m$, this gives

$$2\, dz_m/d\tau = z_{m+1} + z_{m-1}. \tag{19′}$$

For initial values taken as the delta function $y_m = z_m = \delta_m^0$, we obtain a set of defining conditions for the modified Bessel function $I_m(\tau) = z_m$. Hence

$$u(mh, h^2\tau/2) = e^{-\tau} I_m(\tau). \tag{19″}$$

We thus get for $\tau > 0$ a semigroup of *positive* linear transformations. Moreover, as we shall now see, the semi-discretization (19) is very accurate.

Indeed, let $y_m(0) = u(mh, 0) = \delta_m^0$ as is previous examples. According to Whittaker's scheme of *cardinal interpolation* [17], this corresponds to the initial value function

$$u(x, 0) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{iqx}\, dq = \frac{h}{\pi x} \sin \frac{\pi x}{h}. \tag{20}$$

Hence, defining $f(q, 0)$ as the step-function equal to $h/2\pi$ for $|q| < \pi/h$ and zero otherwise, we can compute an expression for the solution of $u_t = u_{xx}$, as in (5). Since $P(iq) = -q^2$, this is given by

$$f(hq, t) = \exp(iqx - tq^2). \tag{20′}$$

* The hypothesis of regularity in R^r is made to cover the (exceptional) case that (1) is regular on some torus but not in R^r, discovered by A. Seidenberg [2, p. 306].

† I. G. Petrowski (Mat. Sbornik 2 (1937), 814–68) and many subsequent writers use this property as their definition of parabolicity.

By comparison, the semi-discretization (19) gives

$$g(hq, t) = \exp\,(iqx - (4t/h^2)\sin^2(hq/2)). \tag{20''}$$

As a result, as $h \downarrow 0$ for fixed q, t, the difference between f and g is extremely small. Furthermore, as t increases the high-frequency (large q) components become negligible very quickly, so that excellent accuracy is obtained. Finally, we note that even better results can be obtained by using higher-order central differences; see §5.

We now generalize some of the ideas suggested by the preceding example, to the case that $P(i\mathbf{q})$ has n eigenvalues $\lambda_l(\mathbf{q})$.

DEFINITION. The system (1) will be said to be of *smoothing type** when

$$\text{Lim}_{|q|\to\infty} \operatorname{Re}\{\lambda_l(\mathbf{q})\} = -\infty, \qquad l = 1, \cdots, n. \tag{21}$$

Evidently, any system (1) which is of smoothing type is regular, since the $\lambda_l(\mathbf{q})$ are continuous functions and hence bounded on compact sets. A similar remark applies trivially to *any* semi-discrete approximation. We have

LEMMA 3. Let $\Pi = \Pi(\Delta, \mathbf{h})$ be any *fixed* semi-discrete approximation to a Cauchy problem (1). Then the initial value problem $\partial\mathbf{u}/\partial t = \Pi(\Delta, \mathbf{h})\mathbf{u}$ is regular —i.e., $\Lambda(\Pi(\Delta, \mathbf{h})) < +\infty$.

Proof. By a change of basis, we can assume that the vector $\mathbf{h}$ describing the spatial mesh in X has unit components. Then, for any vector $\mathbf{n} = (n_1, \cdots, n_r)$ with integral entries n_i,

$$\Pi[\mathbf{b}e^{i(\mathbf{q}+2\pi\mathbf{n})\cdot\mathbf{x}}] = \Pi[\mathbf{b}e^{i\mathbf{q}\cdot\mathbf{x}}] = A\mathbf{b}e^{i\mathbf{q}\cdot\mathbf{x}}.$$

Since $A(\mathbf{q}, \Pi)$ is continuous in $\mathbf{q}$, so are its eigenvalues $\lambda_l(\mathbf{q}, \Pi)$. Therefore

$$\Lambda(\Pi) = \sup_{l,|q_j|\leqq\pi} \operatorname{Re}[\lambda_l(\mathbf{q}, \Pi)]$$

is the least upper bound of a continuous function on the *compact set* $|q_j| \leqq \pi$, whence $\Lambda(\Pi) < +\infty$. Q. E. D.

The same argument shows that the spectral radius of Π is bounded:

$$R(\mathbf{h}) = \sup_{l,|q_j|\leqq\pi} |\lambda_l(\mathbf{q}, \Pi)| < +\infty, \qquad j = 1, \cdots, r, \tag{22}$$

which will be useful later (in §7).

Note that the preceding results hold whether or not the initial value problem is well-set for (1)—i.e., whether or not $\Lambda(P) < +\infty$. Thus (for any *fixed* scheme Δ and mesh H) they hold for $t < 0$ as well as for $t > 0$.

5. Order of consistency. We now consider the asymptotic behavior of semi-discrete difference approximations $\Pi(\Delta, \mathbf{h})$ based on a fixed scheme Δ, as the mesh is indefinitely refined—i.e., as $\mathbf{h} \to \mathbf{0}$. For this purpose, we define (algebraic) consistency† in a way which is independent of the norm ("norm-free").

* Such systems are necessarily *parabolic*, but the converse is not true, as the example $u_t = v$, $v_t = -u_{xxxx}$ (vibrating rod) shows.

† This definition is a semi-discrete analog of the earlier notion of "compatibility" of Fritz John [8, p. 160], and of the notion of consistency for fully discretized problems of Lax and Richtmyer ([11], [13]).

DEFINITION. Let Δ be any semi-discrete difference scheme (7) for approximating spatial derivatives by difference quotients, and let its associated semi-discrete operators for (1) be $\Pi(\Delta, \mathbf{h})$ and $A(\mathbf{q}, \Pi)$. Then Δ is *consistent* with $P(i\mathbf{q})$ when

$$\lim_{\mathbf{h}\to 0} A(\mathbf{q}, \Pi) = P(i\mathbf{q}) \qquad \text{for any real } \mathbf{q}. \tag{23}$$

It is understood that the limit in (23) refers to each entry of the matrices $A(\mathbf{q}, \Pi)$ and $P(i\mathbf{q})$. The matrix $A(\mathbf{q}, \Pi)$ of Example 1 is a consistent approximation of $P(i\mathbf{q})$.

LEMMA 4. If the semi-discrete difference scheme Δ is consistent, then the error matrix $A(\mathbf{q}, \Pi) - P(i\mathbf{q}) = E(\mathbf{q}, \mathbf{h})$ is an entire function of the $2r$ variables $q_1, \cdots, q_r, h_1, \cdots, h_r$.

Proof. Evidently, $P(i\mathbf{q})$ is a polynomial function, while by (7) and (9′) $A(\mathbf{q}, \Pi)$ is defined (when all $h_j \neq 0$) as a power series in the q_i and h_j, everywhere convergent since exponential polynomials are. This series has a finite number of pole-like terms with power-products of the h_i in the denominator. If Δ is consistent, these terms must vanish (since $P(i\mathbf{q})$ is bounded on bounded sets); moreover the terms involving the q_i alone (without any factor h_j) must have $P(i\mathbf{q})$ as sum since the difference would otherwise be $0(1)$.

It follows that, if all terms with zero numerator are dropped, the error matrix $E(\mathbf{q}, \mathbf{h})$ gives a power series which is *everywhere* convergent since it vanishes identically on the hyperplanes $h_j = 0$. Such a series is *uniformly* convergent on any bounded set, and represents an entire function*. Q. E. D.

In one space dimension (if $r = 1$ in (1)), we can therefore write $A(q, h) = P(iq) + h^{\nu} P_{\nu}(q) + 0(h^{\nu+1})$ for some greatest positive integer ν, the *order of consistency* of the scheme whose amplification matrix is $A(q, h)$. When $r > 1$, the situation is more complicated, since the set of vectors $\mathbf{v}$ such that $E(\mathbf{q}, \mathbf{h})$ has a non-zero component of the form $h_1^{l(1)} \cdots h_r^{l(r)} P_l(i\mathbf{q})$ need not be characterized by a single "order of consistency".

To avoid the resulting complications, we will therefore restrict attention when $r > 1$ to *one-parameter* families of semi-discretizations. By this we mean that Δ is fixed (i.e., that each term $\alpha_{jkl} D^l$ of (1′) is approximated by a fixed difference quotient (7)), *and* that the mesh-vector $\mathbf{h}$ is given explicitly by $\mathbf{h} = h\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is a fixed vector with all $\theta_j > 0$, so that spatial *mesh-ratios* are *fixed*. For simplicity, we will denote the operator

$$\Pi(\Delta, \mathbf{h}) = \Pi(\Delta, h\theta_1, h\theta_2, \cdots, h\theta_r) = \Pi_h(\Delta, \boldsymbol{\theta}) \tag{24}$$

of this one-parameter family of semi-discretizations by Π_h, and the matrix $A(\mathbf{q}, \Pi_h)$ by $A(\mathbf{q}, h)$.

For any fixed mesh-ratio vector $\boldsymbol{\theta}$ with all $\theta_j > 0$, collecting terms of the same total order $l = l(1) + \cdots + l(r)$ in h, we can therefore write the error matrix as

$$E(\mathbf{q}, \mathbf{h}) = \textstyle\sum_{l=\nu}^{\infty} h^l R_l(\mathbf{q}, \boldsymbol{\theta}), \qquad \text{for given } \Pi_h(\Delta, \boldsymbol{\theta}). \tag{24′}$$

* Loosely speaking, we may therefore say that $E(\mathbf{q}, \mathbf{h})$ has a *removable singular locus* on the hyperplanes $h_j = 0$.

Here $R_l(\mathbf{q}, \boldsymbol{\theta})$ is an entire function in $\mathbf{q}$ and $\mathbf{h}$, explicitly computable from the coefficients $\alpha_{j,k,l}$ in $(1')$, the weights $\mu_{\mathbf{m}}^{(l)}$, and the components $\theta_j \neq 0$ of $\boldsymbol{\theta}$.

This result motivates the following definition.

DEFINITION. A consistent one-parameter family $\Pi_h(\Delta, \boldsymbol{\theta})$ of semi-discretizations is *consistent to order* $\nu = \nu(\Delta, \boldsymbol{\theta})$ when

$$A(\mathbf{q}, h) - P(i\mathbf{q}) = 0(h^{\nu}) \quad \text{for each} \quad \mathbf{q}, \quad \text{as} \quad h \downarrow 0. \tag{25}$$

We say that Δ has *exact* order of consistency ν (for given $\boldsymbol{\theta}$) if there is no larger ν for which (25) is valid.

An examination of the one-parameter semi-discrete schemes of Examples 1–4 shows that $\nu = 1$ in Examples 1 and 2, while $\nu = 2$ in Examples 3 and 4. It is not difficult to see that semi-discrete schemes based on central difference approximations (14) have orders of consistency *two*, while the forward and backward difference approximations analogous to (14) have orders of consistency *one*.

By using the operational calculus, it is easy to construct $k + 1$ point central difference schemes having *arbitrarily high order of consistency*. Thus, if $C[u] = [u(x + h) - u(x - h)]/2h$, then, since e^{hD} is by Taylor's Theorem the shift operator $f(x) \to f(x + h)$ on any entire function, $hC = \sinh hD$. Hence,

$$D = (1/h) \sinh^{-1} hC = C - h^2C^3/6 + 3h^4C^5/40 - \cdots . \tag{26}$$

By truncating this series after k terms and, denoting the result $\sigma_k(hC)/h$, i.e.,

$$D \doteqdot \sigma_k(hC)/h, \tag{26'}$$

we get a finite difference scheme Δ whose order of consistency is $\nu = 2k$.

Remark. For future reference, we now bound

$$-i\sigma_k(i \sin \theta) = \sin \theta + (\sin^3 \theta)/6 + 3(\sin^5 \theta)/40 + \cdots .$$

As θ varies, $|\sin \theta| \leqq 1$; hence the sum of the preceding series of positive terms is bounded by $\sinh^{-1} i = \sin^{-1} 1 = \pi/2$, proving

$$|\sigma_k(i \sin \theta)| < \pi/2 \qquad \text{for all} \quad k = 1, 2, 3, \cdots, \quad \text{and real} \quad \theta. \tag{26''}$$

This procedure can be applied to each coordinate x_i in turn. Moreover, one can use repeated substitution to get from these a difference approximation to each $D^{(l)} = D_1^{l_1} \cdots D_r^{l_r}$ having an arbitrarily high order of accuracy. The governing principle is the rule that if each $D^{(l)}$ is approximated to order ν by some difference quotient $E^{(l)}$, then $\sum a_{jkl}D^{(l)}$ is approximated to order ν by $\sum a_{jkl}E^{(l)}$. Moreover, if each D_i is approximated to order ν by some expression E_i in divided differences and p_{jk} is any polynomial, then $p_{jk}(\mathbf{D})$ is approximated to order ν by $p_{jk}(E)$.

We now define a semi-discretization scheme Δ to be *uniform* when each derivative operator D_i is approximated in the same way in all its occurrences in (7). For example, using central differences one can replace each D_i by $\sigma_2(hC_i)/h = C_i - h^2C_i^3/6$ from $(26')$. Dropping subscripts, this gives

$$[C - h^2C^3/6]e^{iqx} = i\hat{q}e^{iqx}, \tag{27}$$

where

$$(27') \qquad \hat{q} = [(\sin qh + \tfrac{1}{6}\sin^3 qh)/qh]q = -i\sigma_2(i \sin qh)/h.$$

In this case, one verifies that $|\hat{q}| \leqq |q|$. In general, we have

THEOREM 1. Let $\Pi_h(\Delta, \boldsymbol{\theta})$ be any uniform one-parameter family of central semi-discretizations, where $D_i \doteqdot \sigma_k(h_i C_i)/h$ in (26′) and $k \geqq 1$. Then $\Pi_h(\Delta, \boldsymbol{\theta})$ is consistent to order $2k$, and $A(\mathbf{q}, h) = P(i\hat{\mathbf{q}})$, where $\hat{q}_j = -i\sigma_k(i \sin q_j h_j)/h_j$ and $|\hat{q}_j| \leqq |q_j|$ for all $j = 1, 2, \cdots, r$.

Proof. The previous discussion shows that the one-parameter semi-discretization is consistent to order $2k$. It remains to show in general that $|\hat{q}_j| \leqq |q_j|$. Setting $z = qh$, the identity $1 \equiv (1/iz) \sinh^{-1}(i \sin z)$ gives us that

$$1 = \frac{1}{z}\left\{\sin z + \frac{1}{6}\sin^3 z + \frac{3}{40}\sin^5 z + \cdots\right\}.$$

For $0 \leqq z \leqq \pi$, truncation of this series after n terms gives a function $\psi_n(z)$ which satisfies $0 \leqq \psi_n(z) \leqq 1$, since all terms are positive or zero. Since the function is even, the numerator periodic of period 2π, and the denominator increasing, it follows that $|\psi_n(z)| \leqq 1$ for all real z. But, since $i\hat{q}_j = \sigma_k(i \sin q_j h_j)/h_j = i\psi_k(q_j h_j) \cdot q_j$, then $|\hat{q}_j| \leqq |q_j|$, which completes the proof.

In particular, we have proved

COROLLARY 1. Any system (1)–(1′) can be approximated to an arbitrarily high order of consistency $\nu = 2k$ by a suitable one-parameter family of uniform central semi-discretizations.

Note that, although the definition of consistency assumes fixed mesh-ratios θ_j, the order of consistency is independent of the mesh-ratio vector $\boldsymbol{\theta}$.

A much sharper result concerns the stability index of semi-discretizations, as defined in §3. By analogy with ordinary DE's, we call the semi-discretization $(\Delta, \mathbf{h})$ *strictly stable* when $\Lambda(\Delta, \mathbf{h}) < 0$, and we define the *asymptotic stability index* of a one-parameter family of semi-discretizations as

$$(28) \qquad \Lambda(\Pi) = \lim_{h \to 0} \sup \Lambda(\Delta, h\boldsymbol{\theta}).$$

Theorem 1 has the following additional consequence.

COROLLARY 2. If Δ is any consistent one-parameter family of *uniform* central semi-discretizations, then its asymptotic stability index is at least that in R^r of the system (1)–(1′) which it approximates.

Proof. For any fixed $\mathbf{h}$, $\Lambda(\Delta, \mathbf{h}) \leqq \Lambda(P)$ since the eigenvalues of $A(\mathbf{q}, \mathbf{h})$ are a subset of the eigenvalues of $P(i\hat{\mathbf{q}})$. Moreover, since $\hat{q}_j(q_j, \mathbf{h}) \to q_j$ as $\mathbf{h} \to \mathbf{0}$, and the eigenvalues λ_l of $P(i\mathbf{q})$ are continuous functions of $\mathbf{q}$, for any fixed P, strict inequality is impossible. Q.E.D.

We now show that, for any one-parameter family of semi-discretizations Π_h, the exact order of consistency $\nu = \nu(\Delta, \boldsymbol{\theta})$ is a positive integer in non-trivial cases.

Any such family Π_h is defined as in (7) by a specific divided difference approximation scheme Δ to each separate term of (1′). This approximation is of the form (7), where the coefficients $\mu_{\mathbf{m}}^{(l)}$ are independent of h. From (7) and (9′), it follows that the entry $a_{jk}(\mathbf{q}, \mathbf{h})$ of $A(\mathbf{q}, \mathbf{h})$, the infinitesimal amplification matrix, is given by a finite sum of the form

$$a_{jk}(\mathbf{q}, \mathbf{h}) = \left(\prod_k (\theta_k h)^{-l_k}\right) \sum_{\mathbf{m}} \mu_{\mathbf{m}}^{(l)} e^{i\mathbf{q}\cdot\mathbf{mh}},$$

where $\mathbf{mh} = (m_1\theta_1 h, \cdots, m_r\theta_r h)$. The corresponding term $p_{jk}(i\mathbf{q})$ of $P(i\mathbf{q})$ is

$$p_{jk}(i\mathbf{q}) = (iq_1)^{l_1} \cdots (iq_r)^{l_r}.$$

If $A(\mathbf{q}, h)$ is a consistent approximation of $P(i\mathbf{q})$, then $a_{jk}(\mathbf{q}, h) \to p_{jk}(i\mathbf{q})$ as $h \to 0$, and their difference can be expanded as a power series in h for each fixed $\mathbf{q}$, due to the fact that each term $e^{i\mathbf{q}\cdot\mathbf{mh}}$ is an entire function of h. This shows that, for *each* $\mathbf{q}$, there is a largest positive integer $\nu = \nu(j, k, \mathbf{q})$ such that

$$a_{jk}(\mathbf{q}, h) = p_{jk}(i\mathbf{q}) + 0(h^\nu), \qquad h \to 0, \tag{29}$$

unless $a_{jk} = p_{jk}$ for all h, in which case we put $\nu(j, k, \mathbf{q}) = +\infty$.

Now let $j, k, \mathbf{q}$ vary; it is obvious then that

$$\nu(\Pi_h(\Delta, \boldsymbol{\theta}) = \inf_{j,k,\mathbf{q}} \nu(j, k, \mathbf{q}). \tag{29′}$$

Moreover, since the *trivial case* that $P(\mathbf{D})$ is a matrix of constants has been ruled out, then there is some entry of $P(i\mathbf{q})$ such that $p_{jk}(i\mathbf{q})$ is a non-constant polynomial in $\mathbf{q}$, and $a_{jk}(\mathbf{q}, h)$ is similarly a non-constant polynomial in $e^{i\mathbf{q}\cdot\mathbf{mh}}$. Their difference cannot vanish identically for all $\mathbf{q}$, so that inf $\nu(j, k, \mathbf{q}) < +\infty$, proving that $\nu(\Delta, \boldsymbol{\theta}) < +\infty$. This proves

LEMMA 5. For any consistent one-parameter family of semi-discretizations' there exists a greatest positive integer $\nu = \nu(\Delta, \boldsymbol{\theta})$ $(1 \leqq \nu < \infty)$, such that for any $\mathbf{q}$,

$$A(\mathbf{q}, h) = P(i\mathbf{q}) + 0(h^\nu) \quad \text{as} \quad h \to 0. \tag{30}$$

Let $A(\mathbf{q}, h)$ be consistent with $P(i\mathbf{q})$, and let $\nu = \nu(\Delta, \boldsymbol{\theta})$ be the order of consistency of $A(\mathbf{q}, h)$ to $P(i\mathbf{q})$. If $N(\mathbf{f})$ is any norm* on the space of n-vectors $\mathbf{f}$ (cf. (3)), we define as usual $\| B \|_N = \sup_{\mathbf{f}\neq 0} [N(B\mathbf{f})/N(\mathbf{f})]$ as the associated matrix norm of the matrix B. It then follows, for given N, that for all $h \leqq 1$,

$$\| \exp [tA(\mathbf{q}, h)] - \exp [tP(i\mathbf{q})] \|_N \leqq M_N(\mathbf{q}, t)h^\nu, \tag{31}$$

where $M_N(\mathbf{q}, t)$ is independent of h. This implies that

$$\lim_{h\to 0} \exp [tA(\mathbf{q}, h)] = \exp [tP(i\mathbf{q})] \tag{31′}$$

for any fixed $\mathbf{q}$ and t. Moreover, for $t = \Delta t$ sufficiently small, $M_N(\mathbf{q}, \Delta t) \leqq \tilde{M}_N(\mathbf{q})\Delta t$, so that

$$\| \exp [\Delta t A(\mathbf{q}, h)] - \exp [\Delta t P(i\mathbf{q})] \|_N \leqq \tilde{M}_N(\mathbf{q})h^\nu \Delta t. \tag{31″}$$

* See A. S. Householder, "The Theory of Matrices in Numerical Analysis," Bla isdell 1964, p. 37.

6. Full discretization. On digital computers, one must discretize in space and in time. We will consider below only the case of uniform *rectangular* meshes M in space-time, with meshpoints $(\mathbf{mh}, \Delta t) = (m_1h_1, m_2h_2, \cdots, m_rh_r, n\Delta t)$, $h_i = h\theta_i$. Moreover, we will usually make $\Delta t = rh^\alpha$ for some fixed exponent α and factor r, considering Δt (or h) or a variable *parameter* which is (theoretically) allowed to tend to zero.

Many such discretizations can be constructed from semi-discretization schemes Π_h, by applying familiar methods for the approximate numerical integration systems of first-order *ordinary* DE's:

$$d\mathbf{u}/dt = \Pi_h[\mathbf{u}]. \tag{32}$$

For example, the Cauchy polygon method yields from any such Π_h the *forward* time-difference scheme

$$\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \Delta t\Pi_h[\mathbf{v}(t)] = (I + \Delta t\Pi_h)[\mathbf{v}(t)]. \tag{33}$$

Other discretization schemes which can be constructed from any Π_h are the (implicit) *backward* time-difference scheme

$$(I - \Delta t\Pi_h)[\mathbf{v}(t + \Delta t)] = \mathbf{v}(t), \tag{34}$$

and the *trapezoidal* (Crank-Nicolson) discretization

$$\mathbf{v}(t + \Delta t) = \mathbf{v}(t) + \Delta t\{\Pi_h[\mathbf{v}(t + \Delta t) + \mathbf{v}(t)]\}/2, \tag{35}$$

which can also be written in operator notation as

$$(I - (\Delta t\Pi_h)/2)[\mathbf{v}(t + \Delta t)] = (I + (\Delta t\Pi_h)/2)[\mathbf{v}(t)].$$

For instance, if applied to Example 1 with $\Pi_h[\mathrm{u}] = [u(x - h) - u(x)]/h$ and $\Delta t = rh$, the forward difference scheme (33) gives for $u_t + u_x = 0$ the discretization

$$v(x, t + \Delta t) = rv(x - h, t) + (1 - r)v(x, t). \tag{36}$$

The backward difference scheme (34), applied in the same way, gives the implicit discretization

$$(1 + r)v(x, t + \Delta t) = v(x, t) + rv(x - h, t + \Delta t). \tag{36$'$}$$

Ten other discretizations can be constructed from Examples 1–4 by use of (33)–(35). In each case, it is important to know the order of *consistency and stability index* of the scheme. We now define these conepts precisely, for rectangular discretizations of any system of the form (1).

DEFINITION. For each wave-vector $\mathbf{q}$, let $B(\mathbf{q}, \Delta, \mathrm{h}) = B(\mathbf{q}, \Delta, h, r, \alpha)$ be the matrix expressing the effect of letting the one-parameter discretization scheme Δ operate on $\mathbf{b}e^{i\mathbf{q}\cdot\mathbf{x}}$ with $\Delta t = rh^\alpha$, where r and α are positive constants. Then the scheme will be said to have *order of consistency* ν as $h \downarrow 0$ if entry-wise

$$B(\mathbf{q}, \Delta, h) - \exp[\Delta tP(i\mathbf{q})] = 0(h^\nu\Delta t), \qquad \Delta t = rh^\alpha \tag{37}$$

for all $\mathbf{q} \in Q$. It will be said to be *consistent* if (37) holds for some $\nu > 0$.

For discretization schemes Δ constructed from one-parameter families of semi-discretizations, it is particularly simple to compute $B(\mathfrak{q}, \Delta, h)$ from the matrix $A(\mathfrak{q}, h)$ for fixed Δt (cf. §5). Thus, if (33) is used, we have for $\Delta t = rh^\alpha$:

$$B_1(\mathfrak{q}, \Delta, h) = I + \Delta t A(\mathfrak{q}, h) = I + rh^\alpha A(\mathfrak{q}, h). \tag{33'}$$

Likewise, if (34) is used,

$$B_2(\mathfrak{q}, \Delta, h) = (I - \Delta t A(\mathfrak{q}, h))^{-1}, \tag{34'}$$

while the Crank-Nicolson (trapezoidal) formula (35) gives

$$B_3(\mathfrak{q}, \Delta, h) = (2I - \Delta t A(\mathfrak{q}, h))^{-1}(2I + \Delta t A(\mathfrak{q}, h)). \tag{35'}$$

As one would expect, discretizations defined from *consistent* one-parameter families of semi-discretizations by any of formulas (33)–(35) are themselves consistent. More precisely, we have

LEMMA 6. Let Δ be a consistent one-parameter family of semi-discretizations which is consistent to order ν. Then for $\Delta t = rh^\alpha$, the $B_i(\mathfrak{q}, \Delta, h)$ of (33′)–(34′) are both consistent to order min $[\nu, \alpha]$ for $i = 1, 2$ and $B_3(\mathfrak{q}, \Delta, h)$ of (35′) is consistent to order min $[\nu, 2\alpha]$.

Since this result will be proven in greater generality later (Theorem 4), we omit the proof.

EXAMPLE 5. From the DE $u_t + u_x = 0$ and the semi-discretization $v_t = [v(x - h, t) - v(x, t)]/h$ of Example 1, the forward difference method (27) yields for $\Delta t = rh$, the discretization

$$v(x, t + \Delta t) = (1 - r)v(x, t) + rv(x - h, t). \tag{38}$$

By Lemma 6, this is consistent to order $\nu = 1$, for any r. For $r = 1$, it is *exact*. This shows that the order of consistency computed from Lemma 6 may be merely a *lower bound* to the exact order of consistency; the latter may be greater through cancellation of errors.

Order of efficiency. The order of consistency of a one-parameter family of discretizations of a system (1)–(1′) is not the best measure of its computational efficiency. This is because the total computing time for a given difference scheme on a given domain is proportional to the number of mesh-points per unit volume: the mesh-point *density*. For $\Delta t = Rh^\alpha$ in r space dimensions, this density is $0(h^{-(r+\alpha)})$. Hence, if the order of consistency of a given one-parameter family is ν, the mesh-point density required to reduce the discretization error to ϵ is $0(\epsilon^{-(r+\alpha)/\nu})$, and the asymptotic computing time is also $0(\epsilon^{-(r+\alpha)/\nu})$, as $\epsilon \downarrow 0$. For given r, $\nu/(r + \alpha)$ is therefore a better measure of computational efficiency than ν. Allowing for the fact that, with $\alpha = 1$ (typical of most schemes for hyperbolic systems), the computation time would normally be expected to have a factor $r + 1$ in the exponent, we define $(r + 1)\nu/(r + \alpha)$ as the *asymptotic order of efficiency** of a given one-parameter family with $\Delta t = Rh^\alpha$, in r space dimensions.

* A related concept was described in [5a, p. 30] for parabolic problems with $r = 1$.

7. Von Neumann Condition. It is well known that many plausible explicit difference schemes "blow up" if too large time-steps (for a given space-step) are taken. Thus, it was observed by Courant, Friedrichs and Lewy [4] that this happens with the wave equation $u_{tt} = u_{xx}$ for $\Delta t = r(\Delta x)^2$, if $r > \frac{1}{2}$. (This also happens with the convection equation $u_t + u_x = 0$ for any Δt, if one uses the "bad" semi-discretization $du_j/dt = (u_j - u_{j+1})/h$ in place of that of Example 1, §3. In general, explicit schemes derived from semi-discretizations whose asymptotic stability index (28) is ∞ never converge.)

A systematic general criterion for the bound on Δt is provided by the *von Neumann condition* ([12], [13, Ch. IV, §8]). This is the condition that the eigenvalues of the set of discretizations under consideration should have *uniformly bounded eigenvalues* on any *finite interval* $0 \leqq t \leqq T$. As is observed in [13], this is a *necessary* condition for convergence in any Banach space. We now correlate the von Neumann condition with the concept of stability index.

DEFINITION. The *stability index* of a fully discrete approximation, with given $\mathbf{h}$ and Δt, is defined in terms of the eigenvalues $\mu_l(\mathbf{q}, \mathbf{h})$ of the matrices $B(\mathbf{q}, \Delta, \mathbf{h}, \Delta t)$ of (37), as the number

$$\Lambda(\Delta, \mathbf{h}, \Delta t) = \frac{1}{\Delta t} \ln \{\sup_{l,\mathbf{q}} | \mu_l(B(\mathbf{q}, \Delta, \mathbf{h}, \Delta t) | \}. \tag{39}$$

A one-parameter family of discretizations, defined by a difference scheme Δ with $\Delta x_j = h\theta_j$ and $\Delta t = rh^\alpha(\boldsymbol{\theta}, r, \alpha$ fixed) is said to satisfy the *von Neumann condition* when

$$\limsup_{h \downarrow 0} \Lambda(\Delta, h\boldsymbol{\theta}, rh^\alpha) < +\infty. \tag{40}$$

The preceding definition is easy to apply. Thus, in Example 5 of §6, there is only one space variable ($r = 1$) and $\alpha = 1$, so that $\mu_1 = \mu = B(q, \Delta, h, 1, 1)$. Hence

$$\mu(B) = 1 + r(e^{-iqh} - 1), \qquad \sup | \mu_1(B) | = \begin{cases} 1 & \text{if } 0 \leqq r \leqq 1, \\ | 2r - 1 | & \text{otherwise,} \end{cases}$$

and the von Neumann stability condition (40) is satisfied if and only if $0 \leqq r \leqq 1$.

Again consider the forward difference scheme based on the semi-discretization of Example 3 of §4, with $\Delta t = rh^\alpha$:

$$v(x, t + \Delta t) = v(x, t) + \frac{r}{2} [v(x - h, t) - v(x + h, t)]h^{\alpha-1}. \tag{41}$$

In this case, $\mu_l = 1 + \Delta t[e^{-iqh} - e^{iqh}]/2\text{h} = 1 - irh^{\alpha-1} \sin qh$, so that $\sup_q | u_l | = (1 + r^2h^{2(\alpha-1)})^{\frac{1}{2}}$. For $\alpha \geqq 1$, we find from (37) that

$$\Lambda(\Delta, h, rh^\alpha) \sim \begin{cases} \ln (1 + r^2)h^{-1}/2r, & \alpha = 1 \\ rh^{\alpha-2}/2, & \alpha > 1 \end{cases}, \ h \downarrow 0.$$

Thus, this particular discretization satisfies the von Neumann condition if and only if $\alpha \geqq 2$.

We now prove a very general theorem, which guarantees the *existence* of a consistent one-parameter family of difference schemes satisfying the von Neumann condition (40), for *any* well-set ("regular") Cauchy problem (1).

THEOREM 2. For any well-set Cauchy problem (1) in R^r, any fixed spatial mesh-ratios θ_j, and any consistent one-parameter family of uniform central semi-discretizations Π_h, there exist constants R and α such that the combination of the Π_h with forward time-differences using $\Delta t = Rh^\alpha$ yields a consistent one-parameter family of *explicit* discretizations satisfying the von Neumann condition.*

Proof. By Lemma 6, the resulting one-parameter family of full discretizations is consistent. Moreover since uniform central space differences are used, we can write (from §5):

$$\mu_l = 1 + \Delta t \lambda_l(\mathbf{q}, h) = 1 + \Delta t \gamma_l(\hat{\mathbf{q}}), \qquad \hat{q}_j = -i\sigma_k(i \sin q_j h_j)/h_j,$$

where $h_j = h\theta_j$ and the $\gamma_l(\hat{\mathbf{q}})$ are the eigenvalues of $P(i\hat{\mathbf{q}})$. Thus

$$|\mu_l|^2 = 1 + \Delta t\{2Re\gamma_l(\mathbf{q}) + \Delta t|\gamma_l(\mathbf{q})|^2\}.$$

In order to find an asymptotic bound to this expression as $h \downarrow 0$, we first note that, by (26″):

$$|\hat{q}_j| \leqq \pi/2h\theta_j, \qquad j = 1, 2, \cdots, r.$$

By a theorem of Gelfand and Shilov [6a, p. 67], there exists a constant $C > 0$ such that $\max_l |\gamma_l(\hat{\mathbf{q}})| \leqq C(1 + \|\hat{\mathbf{q}}\|)^\beta$, where $\|\hat{\mathbf{q}}\|^2 = \sum_{j=1}^r |\hat{q}_j|^2$ and $\beta = \max_k (d_k/k)$, d_k being the *degree* (as a polynomial in the q_k) of the coefficient of γ^{n-k} in the characteristic polynomial of $P(i\mathbf{q})$. Setting $\eta_j = \theta_j^{-1}$ and $K = \pi\|\mathbf{n}\|/2$, $\mathbf{n} = (\eta_1, \cdots, \eta_r)$, these inequalities together give

$$|\mu_l|^2 \leqq 1 + \Delta t\{2\Lambda(P) + \Delta t C^2(1 + (K/h))^{2\beta}\}.$$

For $\Delta t = Rh^{2\beta}$, we thus have

$$(42) \qquad |\mu_l|^2 \leqq 1 + \Delta t\{2\Lambda(P) + RC^2(h + K)^{2\beta}\}.$$

In (39)–(40), we therefore obtain (since $\ln(1 + x) \leqq x$):

$$(42') \qquad \Lambda(\Delta, h\boldsymbol{\theta}, Rh^{2\beta}) \leqq \Lambda(P) + RC^2(h + K)^{2\beta}/2.$$

which shows that the von Neumann condition is satisfied. Q.E.D.

COROLLARY. For any well-set Cauchy problem of the form (1), and any fixed spatial mesh-ratios $\boldsymbol{\theta}$, combination of the uniform central space-difference schemes (14) or (26′) with the forward time-difference schemes (33) yields, for suitable R, α, a consistent *explicit* difference scheme which satisfies the von Neumann condition, for any $\Delta t(h) \leqq Rh^\alpha$.

Note that, from (42′), the asymptotic stability index is

$$(42'') \qquad \limsup_{h\to 0} \Lambda(\Delta, h\boldsymbol{\theta}, Rh^{2\beta}) \leqq \Lambda(P) + RC^2K^{2\beta}/2.$$

* We owe to Mr. Martin Schultz the precise form of Theorem 2, as well as other helpful comments. For the case $\nu = 1$, see Aronson [1]. Our results for $\nu = 1$ were obtained in 1960; see Abstract 567–8, Notices Am. Math. Soc. 7 (1960), p. 888.

Much stronger results can be proved for the *implicit* schemes defined by [34]–[35] from Π_h, by backwards and trapezoidal time-differences. We have

THEOREM 3. Let Δ be any of the uniform central difference schemes defined by (26′) from a given stable Cauchy problem (1). Then, for *any* fixed $\boldsymbol{\theta}, r > 0$ and $\alpha > 0$, the (implicit) discretizations defined from $\Pi(\Delta, h\boldsymbol{\theta})$ by (34)–(35) and $\Delta t = rh^{\alpha}$ are consistent and satisfy the von Neumann condition.

For, the conformal transformations $z \rightarrow 1/(1 - z)$ and $z \rightarrow (2 + z)/(2 - z)$ map the left half-plane $Re\{z\} \leqq 0$ into the unit disc $|z| \leqq 1$. Actually, the scheme defined by (35) has *second-order* consistency.

8. Padé approximation. In §6, we studied difference schemes (33′)–(35′) obtained from any semi-discretization $\mathbf{u}_t = \Pi_h[\mathbf{u}]$ by three standard formulas (33)–(35) for numerically integrating systems of first-order ordinary DE's. Letting $z = \Delta t \Pi_h$, these formulas correspond to three approximations

$$(43) \qquad e^z \doteqdot 1 + z, \qquad e^z \doteqdot (1 - z)^{-1}, \qquad e^z \doteqdot (2 + z)(2 - z),$$

to the operator $\exp(\Delta t \Pi_h)$ which gives the *exact* solution to the semi-discretization $\mathbf{u}_t = \Pi_h[\mathbf{u}]$. We now show how one can obtain *arbitrarily high order of consistency* ν in discretizations, by combining highly consistent semi-discretizations with Padé approximations to e^z—an idea first suggested in [14] (cf. also [10]).

The three approximations (43) are just the first three non-trivial upper left entries in the doubly infinite *Padé table* [16, Ch. XX] for the best rational approximation to e^z in the neighborhood of $z = 0$, of the form

$$(44) \qquad e^z \doteqdot r_{i,j}(z) = n_{i,j}(z)/d_{i,j}(z),$$

where $n_{i,j}(z)$ and $d_{i,j}(z)$ are polynomials of fixed degrees j and i, respectively. Since e^z is analytic, one has [16]

$$(44') \qquad e^z = r_{i,j}(z) + 0(|z|^{i+j+1}), \qquad z \rightarrow 0.$$

It is interesting to note that other entries in the Padé table for e^z also correspond to familiar time-discretizations of $\mathbf{u}_t = \Pi_n[\mathbf{u}]$. Thus, the entry $r_{0,2}(z) = 1 + z + z^2/2$ corresponds to the modified Euler method, and $r_{0,4}(z) = 1 + z + z^2/2 + z^3/6 + z^4/24$ corresponds to the fourth-order Runge-Kutta method (cf. [3, pp. 179, 208]).

Given $r_{i,j}(z)$ and a semi-discretization $\mathbf{u}_t = \Pi_h[\mathbf{u}]$ of (1), one can obtain a full discretization of (1) by writing

$$(45) \qquad d_{i,j}(\Delta t \Pi_h)[\mathbf{v}] = n_{i,j}(\Delta t \Pi_h)[\mathbf{v}],$$

as an implicit (for $i > 0$) difference scheme*. To obtain higher-order consistency, we now appeal to the following extension of Lemma 6.

THEOREM 4. For any system (1), ν, and α, let $\Delta t = rh^{\alpha}$ and $(i + j)\alpha \geqq \nu$.

* The use of $i > 1$ has the inconvenience in practice of requiring the inversion of matrices which become progressively fuller as i increases.

Then the combination of any semi-discretization consistent to order ν with (45) gives a discretization which is consistent to order ν.

Proof. Since $e^0 = 1$, $d_{i,j}(0) = 1$ and so (by continuity) $d_{i,j}(z) \neq 0$ for all sufficiently small z. Hence, for all Δt sufficiently small, $d_{ij}(\Delta t A(\mathbf{q}, h))$ is nonsingular and $r_{ij}(\Delta t A)$ is well-defined. Writing

$$R = \exp((\Delta t P(i\mathbf{q})) - r_{ij}(\Delta t A(\mathbf{q}, h)),$$

we have for each $\mathbf{q}$ by (31″) and (44′):

$$R = [\exp(\Delta t P(i\mathbf{q})) - \exp(\Delta t A(\mathbf{q}, h))] + [\exp(\Delta t A(\mathbf{q}, h)) - r_{ij}(\Delta t A(\mathbf{q}, h))]$$
$$= 0(\Delta t h^\nu) + 0((\Delta t)^{i+j+1}).$$

Since $(\Delta t)^{i+j+1} = (rh^\alpha)^{i+j+1} = 0(h^\nu \Delta t)$, we have the desired result.

Diagonal Padé approximations. We next show that the von Neumann stability condition can also be satisfied for *all* Δt, by choosing $i = j$. The proof depends on some special properties of the diagonal entries $r_{i,i}(z)$ in the Padé table for e^z, which we now describe.

It can be shown by an application of Darboux's two-point expansion*, that $d_{j,j}(z) = n_{j,j}(-z)$:

$$e^z = n_{j,j}(z)/n_{j,j}(-z) + 0(|z|^{2j+1}), \qquad |z| \to 0, \tag{46}$$

where

$$n_{j,j}(z) = \sum_{k=0}^{j} \frac{(2j-k)!j!}{(2j)!k!(j-k)!} z^k, \qquad j \geqq 0, \tag{47}$$

is a polynomial of degree j in z.

This last expression for $n_{j,j}(z)$ shows that it converges *uniformly* to $e^{z/2}$ in any bounded domain, as $j \to \infty$. Thus, it follows that

$$r_{j,j}(z) = n_{j,j}(z)/n_{j,j}(-z) \to e^z \quad \text{as} \quad j \to \infty, \tag{48}$$

uniformly in any bounded domain.

The property (48) of the $r_{j,j}(z)$ is of course shared by the partial sums $r_{0,j}(z)$ of e^z. However, the next lemma establishes a property which these partial sums do not share.

LEMMA 7. Let $z_j^{(l)}$, $1 \leqq l \leqq j$ be the zeros of the polynomial $d_{j,j}(z) = n_{j,j}(-z)$. Then

$$\operatorname{Re} z_j^{(l)} > 0, \qquad 1 \leqq l \leqq j, \qquad j \geqq 1. \tag{49}$$

Proof. This is a well known result from the theory of passive networks†, so we merely sketch the proof. Decompose $n_{j,j}(z)$ into two polynomials $e_j(z)$

* J. de Math. 2 (1876), p. 271. See also P. M. Hummel and C. L. Seebeck, Am. Math. Monthly 56 (1949), 243–7.

† Cf. Dov Hazony, *Elements of Network Synthesis*, Reinhold, New York, 1963, p. 206.

and $f_j(z)$, which respectively contain only the even and odd terms of $n_{j,j}(z)$. Then, with $g_j(z) \equiv e_j(z)/f_j(z)$, it follows that

$$e^z = \frac{n_{j,j}(z)}{n_{j,j}(-z)} + 0(|z|^{2j+1})$$

$$= \frac{e_j(z) + f_j(z)}{e_j(z) - f_j(z)} + 0(|z|^{2j+1}) = \frac{g_j(z) + 1}{g_j(z) - 1} + 0(|z|^{2j+1}).$$

Multiplying through and collecting terms, we obtain

$$g_m(z) = \frac{e^z + 1}{e^z - 1} + 0(|z|^{2m}) = \coth(z/2) + 0(|z|^{2m}).$$

The function coth z has a continued fractions expansion given by

$$\coth z = \frac{1}{z} + \cfrac{1}{3z + \cfrac{1}{5z + \cfrac{1}{7z + \cdots}}} \tag{50}$$

and it can be verified that $g_j(2z)$ is the j-th continued fraction approximant of coth z. Since the coefficients of this continued fraction expansion of coth z are positive real numbers, it follows from a result of Wall [16, p. 178] that $n_{j,j}(z)$ has all its zeros in the open *left* half plane. But since $n_{j,j}(z) = d_{j,j}(-z)$, we have the desired result of (49).

A final property of the diagonal Padé approximations is given by

LEMMA 8. For all Re $z \leqq 0$, $|r_{j,j}(z)| \leqq 1$.

Proof. Since $r_{j,j}(z) = n_{j,j}(z)/n_{j,j}(-z)$, it follows by symmetry that $|r_{j,j}(i\sigma)| = 1$ for all real σ. Next, it is obvious that $|r_{j,j}(z)| \to 1$ as $|z| \to \infty$. By Lemma 7, $r_{j,j}(z)$ is analytic in the left half-plane Re $z \leqq 0$, and the result is then a consequence of the maximum principle.

Now, let (1)–(1′) define a regular Cauchy problem; we can suppose it strictly stable after a trivial substitution, as shown in §2. Using the preceding lemmas, we now prove finally

THEOREM 5. Let (1)–(1′) define a strictly stable Cauchy problem, and let Π_h be any uniform semi-discretization obtained by a systematic use of central difference approximations having order of consistency $\nu = 2k$. Then for $\Delta t = rh^\alpha$, the Padé discretization (45) with $i = j$ has order of consistency ν for any $\alpha \geqq k/j$, *and* satisfies the von Neumann condition (40) with index $\Lambda(\Delta, h\boldsymbol{\theta}, rh^\alpha) \leqq 0$ for all α and r.

Proof. The consistency is covered by Theorem 4. By Theorem 1, we know also that $\Lambda(\Pi_h) \leqq 0$, or equivalently, that Re $\lambda_l(\mathbf{q}, h\boldsymbol{\theta}) \leqq 0$ for all eigenvalues of $A(\mathbf{q}, h\boldsymbol{\theta})$. Since the eigenvalues μ_l for the discretization (45) are given by

$$\mu_l(\mathbf{q}, h, \Delta t) = n_{i,j}(\Delta t \lambda_l(\mathbf{q}, h\boldsymbol{\theta}))/d_{i,j}(\Delta t \lambda_l(\mathbf{q}, h\boldsymbol{\theta})),$$

it follows from Lemma 8 that $|\mu_l| \leqq 1$ for *all* h. Thus, by (39) and (40), the stability index (39) is nonpositive for all h, which completes the proof.

Theorem 5 generalizes the well known result [5a] that the coupling of three-point central space-differences with trapezoidal time discretization is *unconditionally stable* for the heat equation.

COROLLARY. Any regular Cauchy problem of the form (1)–(1′) has a discretization which satisfies the von Neumann condition *and* has arbitrarily high order of consistency.

9. Characteristic methods. The preceding methods by no means yield all good partial difference schemes. For hyperbolic systems, it is often best to use the method of characteristics. Thus, in Example 2 of §3, using the variables $U = u + v$, $V = u - v$, one gets the system

$$U_t = U_x, \qquad V_t = -V_x. \tag{51}$$

This can be solved exactly by setting

$$U(x, t + h) = U(x + h, t), \qquad V(x, t + h) = V(x - h, t), \tag{51′}$$

a scheme motivated by the method of characteristics.

In the preceding example, one can also obtain (51′) (artificially) by combining the semi-discretization

$$U_t = [U(x + h, t) - U(x, t)]/h, \qquad V_t = [V(x - h, t) - V(x, t)]/h, \tag{52}$$

with the forward time difference Padé approximation (33) with $\Delta t = h$. However, we do not believe that such a treatment is possible for the following, slightly more complicated example.

EXAMPLE 6. Consider the (mildly dissipative) hyperbolic system

$$u_t - u_x + u = v, \qquad v_t + v_x + v = u. \tag{53}$$

Exact solutions of (53) satisfy the integral equations

$$\begin{aligned} u(x, t + h) &= u(x + h, t) + \int_0^h w(x + h - s, t + s)\, ds \\ v(x, t + h) &= v(x - h, t) - \int_0^h w(x - h + s, t + s)\, ds \end{aligned} \tag{54}$$

where $w = v - u$. In particular, this is true of all solutions of the form $u = f(t)e^{iqx}$, $v = g(t)e^{iqx}$.

Hence, by the standard trapezoidal formula for approximate quadrature, on any *square mesh* with $\Delta x = \Delta t = h$, solutions of class C^2 will satisfy

$$\begin{aligned} u_j^{n+1} &= u_{j+1}^n + \tfrac{1}{2}h[(v - u)_j^{n+1} + (v - u)_{j+1}^n] + 0(h^3) \\ v_j^{n+1} &= v_{j-1}^n + \tfrac{1}{2}h[(u - v)_j^{n+1} + (u - v)_{j-1}^n] + 0(h^3). \end{aligned} \tag{55}$$

The *order of consistency* of the scheme obtained by neglecting the $0(h^3)$ remainder is $\nu = 2$, by the following result.

LEMMA 9. If a discretization with $\Delta t = Rh^\alpha$ is obtained by neglecting terms $0(\Delta x^m \Delta t^n)$ with $m + n\alpha - \alpha \geqq \nu$, then it has order of accuracy at least ν.

Proof. We fix q, and refer to (37). The difference in question involves only terms of order $m + n\alpha - \alpha$ or more.

Collecting terms and eliminating, we get from the approximate equations (55) the *four-point, explicit,* (one-step) discretization

$$(56)\quad (4 + 4h)u_j^{n+1} = (4 - h^2)u_{j+1}^n + (2h + h^2)v_{j+1}^n + (2h - h^2)v_{j-1}^n + h^2u_{j-1}^n .$$

$$(56')\quad (4 + 4h)v_j^{n+1} = (4 - h^2)v_{j-1}^n + (2h - h^2)u_{j+1}^n + (2h + h^2)u_{j-1}^n + h^2v_{j+1}^n$$

It seems unlikely that the discretization (56)–(56′) can be obtained from a semi-discretization of (53) by approximating e^z, since the convection terms $u_j^{n+1} = u_{j+1}^n$, $v_j^{n+1} = v_{j-1}^n$ are formed from the semi-discretization of $u_t + u_x = v_t - v_x = 0$ by using the forward explicit approximation $e^z \doteq 1 + z$, while the trapezoidal quadrature terms $h(w_j^{n+1} + w_{j+1}^n)/2$ and $-h(w_j^{n+1} + w_{j-1}^n)$ are obtained by an implicit approximation.

The characteristic equation of the system (53) is $u_{tt} + 2u_t = u_{xx}$, which is a form of the *telegraph* equation and well known to be *stable* (dissipative). We now show that the same is true of the difference approximation (56)–(56′), whose amplification matrix is

$$(57)\qquad \frac{1}{2 + 2h}\begin{pmatrix} 2e^{iqh} - ih^2 \sin qh & 2h \cos qh + ih^2 \sin qh \\ 2h \cos qh - ih^2 \sin qh & 2e^{-iqh} + ih^2 \sin qh \end{pmatrix}.$$

The matrix of (59) has the form $\begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}$, and so its roots λ_1, λ_2 satisfy $\lambda_i = \text{Re}\,\{a\} \pm [|\,b\,|^2 - (\text{Im}\,\{a\})^2]^{\frac{1}{2}}$, by elementary algebra. In the present case, one finds that

$$\text{Re}\,\{a\} = (1 + h)^{-1} \cos qh,$$

$$|\,b\,|^2 - (\text{Im}\,\{a\})^2 = (1 + h)^{-2}[h^2 - \sin^2 qh]$$

$$\lambda_j = (1 + h)^{-1}[\cos qh \pm \sqrt{h^2 - \sin^2 qh}].$$

When $h \geqq |\sin qh|$, λ_i is real and the radical is at most h; hence $|\lambda_j| \leqq (1 + h)/(1 + h) = 1$. When $h \leqq |\sin qh|$, λ_j is complex and $|\lambda_j| \leqq 1/(1 + h) < 1$. Hence *the difference scheme* (56)–(56′) *is stable*; as a corollary, it satisfies the von Neumann condition.

It is interesting to consider also the analogous discretization of a related conservative (non-dissipative) hyperbolic system of partial DE's, whose characteristic equation is $u_{tt} + u = u_{xx}$, the *Klein-Gordon* equation.

EXAMPLE 7. Consider the system

$$(58)\qquad u_t - u_x = v, \qquad v_t + v_x = -u.$$

Much as in Example 6, solutions of class C^2 of (58) must satisfy

$$(59)\qquad \begin{aligned} u_j^{n+1} &= u_{j+1}^n + \frac{h}{2}[v_j^{n+1} + v_{j+1}^n] + 0(h^3) \\ v_j^{n+1} &= v_{j-1}^n - \frac{h}{2}[u_j^{n+1} + u_{j-1}^n] + 0(h^3). \end{aligned}$$

This yields the following analog of (56), whose order of consistency is again $\nu = 2$ by Lemma 9:

$$(60)\qquad \begin{aligned}(4 + h^2)u_j^{n+1} &= 4u_{j+1}^n + 2h(v_{j+1}^n + v_{j-1}^n) - h^2u_{j-1}^n\,,\\ (4 + h^2)v_j^{n+1} &= 4v_{j-1}^n - 2h(u_{j-1}^n + u_{j+1}^n) - h^2v_{j+1}^n\,.\end{aligned}$$

The amplification matrix associated with this discretization is (setting $\theta = h/2$):

$$(61)\qquad \frac{1}{1+\theta^2}\begin{pmatrix} e^{iqh} - \theta^2 e^{-iqh} & h\cos qh \\ -h\cos qh & e^{-iqh} - \theta^2 e^{iqh}\end{pmatrix}.$$

This matrix now has the form $\begin{pmatrix} a & b \\ -b & a^* \end{pmatrix}$, where b is real. Its characteristic equation $\lambda^2 - (a + a^*)\lambda + (aa^* + b^2) = 0$ has a negative discriminant. Since $aa^* + b^2 = 1$, as elementary calculations show, the complex conjugate roots have magnitude 1, whence the discretization (60) is also (neutrally) *stable*. Hence the von Neumann condition is satisfied.

Since $\nu = 2$ and the von Neumann condition is satisfied, the discretization (60) seems satisfactory.

We do not know of any comparably simple discretization for the telegraph or Klein-Gordon equations which has an order of consistency $\nu > 2$, and satisfies the von Neumann condition.

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