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MINIMAL GERSCHGORIN SETS

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If $A = (a_{i,j})$ is a fixed $n \times n$ complex matrix, then it is well known that the Gerschgorin disks G_i in the complex plane, defined by

$$(1) \qquad G_i = \left\{z : |z - a_{i,i}| \leqq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}|\right\}, 1 \leqq i \leqq n ,$$

are such that each eigenvalue of A lies in at least one disk, and, consequently, the union of these disks,

$$(2) \qquad G = \bigcup_{i=1}^{n} G_i ,$$

which we call the Gerschgorin set, contains all the eigenvalues of A. It is however clear from (1) that the radii of these Gerschgorin disks depend only on the moduli of the off-diagonal entries of A. Thus, if

$$(3) \qquad \Omega_A = \{B = (b_{i,j}) : b_{i,i} = a_{i,i}, 1 \leqq i \leqq n, \text{ and } |b_{i,j}| = |a_{i,j}|, 1 \leqq i, j \leqq n\} ,$$

then it is clear that the Gerschgorin set G contains all the eigenvalues of each $n \times n$ matrix B in Ω_A. It is natural to ask how far-reaching this elementary theory is in bounding the eigenvalues of Ω_A.

To extend the above results slightly, let $x > 0$ be any vector with positive components, and let $X(x) \equiv \text{diag}(x_1, x_2, \cdots, x_n)$. Applying the above results to $X^{-1}(x)AX(x)$ shows that if

$$(1') \qquad G_i(x) \equiv \left\{z : |z - a_{i,i}| \leqq \frac{1}{x_i} \sum_{j \neq i} |a_{i,j}| x_j \equiv \Lambda_i(x)\right\}, \quad 1 \leqq i \leqq n ,$$

then the associated Gerschgorin set

$$(2') \qquad G(x) = \bigcup_{i=1}^{n} G_i(x)$$

again contains all the eigenvalues of each $B \in \Omega_A$ for every $x > 0$. Thus, the closed bounded set

$$(4) \qquad G(\Omega_A) \equiv \bigcap_{x > 0} G(x) ,$$

which we call the minimal Gerschgorin set, also contains all the eigenvalues of each $B \in \Omega_A$.

One of the major results in this paper is that each boundary point of $G(\Omega_A)$ is an eigenvalue of some matrix B in Ω_A.

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Thus, the minimal Gerschgorin set $G(\Omega_A)$ can be thought of as being *optimal*.[1] In the irreducible case, it is shown moreover that each boundary point of $G(\Omega_A)$ is geometrically the intersection of n Gerschgorin circles, which is closely related to a result of Olga Taussky [7]. It is also shown (Corollary 1) in the irreducible case that the minimal Gerschgorin set $G(\Omega_A)$ contains n disks with positive radii. Finally, an analogue of a result of Gerschgorin [5] is obtained for disconnected minimal Gerschgorin sets.

It is worth pointing out that there are several other methods [6, 8] for determining n nonnegative numbers ρ_i which, like the radii $\Lambda_i(x)$ in (1′), have the property that each eigenvalue λ of any $B \in \Omega_A$ satisfies $|\lambda - a_{i,i}| \leqq \rho_i$ for at least one i, and analogous minimal Gerschgorin sets could be defined relative to these different methods. However, a very interesting result of Ky Fan [3] tells us that if A is irreducible, then there exists a positive vector $y > 0$ such that

$$(5) \qquad \rho_i \geqq \Lambda_i(y) , \qquad 1 \leqq i \leqq n .$$

Hence, in the interest of developing the *smallest* minimal Gerschgorin sets for either the irreducible case (§ 2) or the reducible case (§ 3), it is sufficient to consider only the minimal Gerschgorin set $G(\Omega_A)$ defined by the diagonal similarity transformations of (1′), (2′), and (4).

The author wishes to express his appreciation to Drs. A. S. Householder, Olga Taussky, and Bernard Levinger for several stimulating discussions on this topic.

2. The irreducible[2] case. In this section, we assume that the $n \times n$ matrix $A = (a_{i,j})$ is irreducible.[2] For any (finite) complex constant σ, consider the real $n \times n$ matrix $P(\sigma) = (p_{i,j})$ defined by

$$(6) \qquad \begin{cases} p_{i,j} = |a_{i,j}|, \ i \neq j , & 1 \leqq i, j \leqq n , \\ p_{i,i} = -|\sigma - a_{i,i}| & 1 \leqq i \leqq n . \end{cases}$$

Since the off-diagonal entries of $P(\sigma)$ are nonnegative, and $P(\sigma)$ is irreducible because A is, then $P(\sigma)$ is *essentially positive* [2; 9, p. 257]. Thus, $P(\sigma)$ possesses a real eigenvalue $\nu(\sigma)$ which is uniquely characterized by the property that if λ is any other eigenvalue of $P(\sigma)$, then

$$(7) \qquad Re\,\lambda < \nu(\sigma) .$$

[1] This was conjectured by Dr. A. S. Householder during the Summer Engineering Conference (1963) in Numerical Analysis at the University of Michigan.

[2] An $n \times n$ matrix A is *irreducible* if there exists no $n \times n$ permutation matrix P such that $PAP^T = \begin{bmatrix} C & D \\ 0 & E \end{bmatrix}$, where C and E are square nonvoid submatrices. Equivalently, the directed graph of A is *strongly connected*. See, for example, [9, p. 20].

Moreover, the eigenvector $\boldsymbol{y}$ corresponding to $\nu(\sigma)$ can be chosen to have positive components, and $\nu(\sigma)$ satisfies the following inclusion relationships [9, p. 261]

$$\min_{1 \leq i \leq n} \left\{ \left(\sum_{j=1}^{n} p_{i,j} x_j \right) \Big/ x_i \right\} \leqq \nu(\sigma) \leqq \max_{1 \leq i \leq n} \left\{ \left(\sum_{j=1}^{n} p_{i,j} x_j \right) \Big/ x_i \right\} \tag{8}$$

for any $\boldsymbol{x} > \boldsymbol{0}$, and

$$\sup_{x > 0} \left[\min_{1 \leq i \leq n} \left\{ \left(\sum_{j=1}^{n} p_{i,j} x_j \right) \Big/ x_i \right\} \right] = \nu(\sigma) = \inf_{x > 0} \left[\max_{1 \leq i \leq n} \left\{ \left(\sum_{j=1}^{n} p_{i,j} x_j \right) \Big/ x_j \right\} \right]. \tag{9}$$

From the definition of the matrix $P(\sigma)$ in (6), it follows for any $\boldsymbol{x} > \boldsymbol{0}$ that

$$\left(\sum_{j=1}^{n} p_{i,j} x_j \right) \Big/ x_i = \Lambda_i(\boldsymbol{x}) - |\sigma - a_{i,i}| , \tag{10}$$

which will be useful in conjunction with (8) and (9). Finally, we remark that $\nu(\sigma)$ is a *continuous* function of σ.

The reason for introducing the function $\nu(\sigma)$ is brought out by the following result.

THEOREM 1. *Let* $A = (a_{i,j})$ *be an irreducible* $n \times n$ *matrix. Then,* $\sigma \in G(\Omega_A)$ *if and only if* $\nu(\sigma) \geqq 0$.

Proof. If $\sigma \in G(\Omega_A)$, then, from (4), $\sigma \in G(\boldsymbol{x})$ for every $\boldsymbol{x} > \boldsymbol{0}$, so that for some j, $\Lambda_j(\boldsymbol{x}) - |\sigma - a_{j,j}| \geqq 0$. Coupled with (8), (9), and (10), we see that $\nu(\sigma) \geqq 0$. Conversely, if $\nu(\sigma) \geqq 0$, then for every $\boldsymbol{x} > \boldsymbol{0}$ there is a j such that

$$\Lambda_j(\boldsymbol{x}) - |\sigma - a_{j,j}| \geqq \nu(\sigma) \geqq 0 .$$

Thus, $\sigma \in G(\boldsymbol{x})$ for every $\boldsymbol{x} > \boldsymbol{0}$, and evidently $\sigma \in G(\Omega_A)$, which completes the proof.

Several remarks are now in order. First, since $G(\Omega_A)$ is a closed bounded set, its complement $G'(\Omega_A)$ is open, and $G'(\Omega_A)$ is simply the set of all complex numbers σ such that $\nu(\sigma) < 0$. Denoting the boundary of $G(\Omega_A)$ by $\partial G(\Omega_A)$, then $\partial G(\Omega_A)$ is defined as usual by

$$\partial G(\Omega_A) = \{ \sigma : \sigma \in \overline{G(\Omega_A)} \cap \overline{G'(\Omega_A)} \} , \tag{11}$$

where $\overline{G'(\Omega_A)}$ is the closure of the complement of $G(\Omega_A)$. From Theorem 1, we see that $\partial G(\Omega_A)$ can also be described as the set of all complex numbers σ such that $\nu(\sigma) = 0$, and such that there exists a sequence of complex numbers $\{z_j\}_{j=1}^{\infty}$ with $\lim_{j \to \infty} z_j = \sigma$ and $\nu(z_j) < 0$. It would of course be simpler if one could describe $\partial G(\Omega_A)$ solely by $\nu(\sigma) = 0$, but this is not in general true.[2]

The above discussion allows one to deduce a rather interesting geometrical property of the boundary $\partial G(\Omega_A)$ of the minimal Gerschgorin set. If $\sigma \in \partial G(\Omega_A)$, then $\nu(\sigma) = 0$. Hence, there exists a vector $\boldsymbol{y} > \boldsymbol{0}$ such that $P(\sigma)\boldsymbol{y} = \nu(\sigma)\boldsymbol{y} = \boldsymbol{0}$, so that from (10),

$$|\sigma - a_{i,i}| = \Lambda_i(\boldsymbol{y}) \qquad \text{for all } 1 \leqq i \leqq n \,. \tag{12}$$

In other words, σ is the intersection of the n Gerschgorin circles $|z - a_{i,i}| = \Lambda_i(\boldsymbol{y})$.

With $A = (a_{i,j})$ an irreducible $n \times n$ matrix, we now show as a corollary to Theorem 1 that the minimal Gerschgorin set $G(\Omega_A)$ contains at least n disks with centers $a_{i,i}$ and positive radii ρ_i.

COROLLARY 1. *Let $A = (a_{i,j})$ be an irreducible $n \times n$ matrix. Then, there exists a vector $\boldsymbol{\rho} > \boldsymbol{0}$ such that all complex numbers z satisfying*

$$|z - a_{i,i}| \leqq \rho_i \tag{13}$$

for some i are contained in the minimal Gerschgorin set $G(\Omega_A)$.

Proof. From Theorem 1 and the discussion following it, any complex number σ with $\nu(\sigma) > 0$ is necessarily an interior point of $G(\Omega_A)$. Thus, it is sufficient to show that $\nu(a_{i,i}) > 0$ for each $1 \leqq i \leqq n$. For any σ, there is a $\boldsymbol{y} > \boldsymbol{0}$ such that $P(\sigma)\boldsymbol{y} = \nu(\sigma)\boldsymbol{y}$. Next, from the definition of the matrix $P(\sigma)$ in (6), it is clear that the diagonal entry $p_{i,i}$ of the particular matrix $P(a_{i,i})$ is zero, and all other entries in that row of $P(a_{i,i})$ are nonnegative. The irreducibility of A, implying the irreducibility of $P(a_{i,i})$, shows us that the ith component of $p(a_{i,i})\boldsymbol{y} = \nu(a_{i,i})\boldsymbol{y}$ is positive, and as $\boldsymbol{y} > \boldsymbol{0}$, we conclude that $\nu(a_{i,i}) > 0$. Consequently, each $a_{i,i}$ is an interior point of $G(\Omega_A)$, and there necessarily exists a vector $\boldsymbol{\rho} > \boldsymbol{0}$ such that $|z - a_{i,i}| \leqq \rho_i$ for some i implies that $z \in G(\Omega_A)$, completing the proof. To be more explicit, one can directly verify that choosing

$$\hat{\rho}_i = \nu(a_{i,i}) \,, \qquad 1 \leqq i \leqq n \,, \tag{14}$$

gives such radii, and it is easy to construct examples where the radii of (14) are best (i.e., largest) possible.

When A is an irreducible $n \times n$ matrix, a result of Olga Taussky [7] states that if λ is an eigenvalue of A, and λ is a boundary point of $G(\boldsymbol{x})$ for some $\boldsymbol{x} > \boldsymbol{0}$, then all the Gerschgorin circles pass through λ:

$$|\lambda - a_{i,i}| = \Lambda_i(\boldsymbol{x}) \,, \qquad 1 \leqq i \leqq n \,. \tag{15}$$

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[2] The author is indebted to Dr. J. H. Wilkinson for having constructed a simple counterexample.

As a converse to this, we have

THEOREM 2. *Let $A = (a_{i,j})$ be an arbitrary $n \times n$ matrix. If, for some $x > 0$, σ is a complex number with*

$$|\sigma - a_{k,k}| = \Lambda_k(x), \qquad 1 \leqq k \leqq n, \tag{16}$$

then σ is an eigenvalue of some $B \in \Omega_A$, and hence $\sigma \in G(\Omega_A)$.

Proof. Writing $(\sigma - a_{k,k}) = |\sigma - a_{k,k}| \exp(i\phi_k)$, let the matrix $B = (b_{k,j})$ be defined by

$$b_{k,k} = a_{k,k}, 1 \leqq k \leqq n; b_{k,j} = |a_{k,j}| \exp(i\phi_k), k \neq j, 1 \leqq k, j \leqq n. \tag{17}$$

Thus $B \in \Omega_A$, and (16) can be written equivalently as

$$\sum_{j=1}^{n} b_{k,j} x_j = \sigma x_k, \qquad 1 \leqq k \leqq n. \tag{18}$$

As $x > 0$ then σ is an eigenvalue of B, and thus $\sigma \in G(\Omega_A)$, which completes the proof.

The importance of this theorem lies in its application in the following

COROLLARY 2. *If $A = (a_{i,j})$ is an irreducible $n \times n$ matrix and $\nu(\sigma) = 0$, then σ is an eigenvalue of some matrix $B \in \Omega_A$. Thus, every boundary point $\sigma \in \partial G(\Omega_A)$ of the minimal Gerschgorin set is an eigenvalue of some matrix $B \in \Omega_A$.*

Proof. If A is irreducible, and σ is a complex number such that $\nu(\sigma) = 0$, then there exists a vector $y > 0$ such that $P(\sigma)y = \nu(\sigma)y = 0$. From (10), it follows that

$$\Lambda_i(y) = |\sigma - a_{i,i}| \qquad \text{for all } 1 \leqq i \leqq n.$$

Thus, applying Theorem 2, σ is an eigenvalue of some matrix $B \in \Omega_A$. From the discussion following Theorem 1, we know that $\nu(\sigma) = 0$ is a necessary condition that $\sigma \in \partial G(\Omega_A)$. Thus, we conclude that each boundary point of minimal Gerschgorin set is an eigenvalue of some matrix $B \in \Omega_A$, which completes the proof.

In terms of finding inclusion regions for eigenvalues of matrices B in Ω_A, Corollary 2 tells us that the minimal Gerschgorin set $G(\Omega_A)$ is optimal.

In analogy to the discussion following Theorem 1, the boundary $\partial G(x)$ of the Gerschgorin set $G(x)$ of (2′) can be described as the set of all complex numbers σ for which there exists an integer j, $1 \leqq j \leqq n$, such that $|\sigma - a_{j,j}| = \Lambda_j(x)$, and there exists a sequence of complex numbers $\{z_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} z_j = \sigma$ for which $|z_j - a_{i,i}| > \Lambda_i(x)$ for all $1 \leqq i \leqq n$. With this, we now give sufficient conditions for a complex number σ to be a boundary point of the minimal Gerschgorin set.

THEOREM 3. *Let $A = (a_{i,j})$ be an irreducible $n \times n$ matrix. If $\sigma \in \partial G(x)$, $x > 0$, and $|\sigma - a_{i,i}| = \Lambda_i(x)$ for all $1 \leqq i \leqq n$, then $\sigma \in \partial G(\Omega_A)$.*

Proof. Since $x > 0$, it follows from $|\sigma - a_{i,i}| = \Lambda_i(x)$, $1 \leqq i \leqq n$, that $\nu(\sigma) = 0$. Next, as $\sigma \in \partial G(x)$, there exists a sequence of complex numbers $\{z_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} z_j = \sigma$ for which $|z_j - a_{i,i}| > \Lambda_i(x)$ for all $1 \leqq i \leqq n$. Hence,

$$0 > \max_{1\leqq i\leqq n} \{\Lambda_i(x) - |z_j - a_{i,i}|\} .$$

But as $x > 0$, we deduce from (8) and (10) that

$$0 > \max_{1\leqq i\leqq n} \{\Lambda_i(x) - |z_j - a_{i,i}|\} \geqq \nu(z_j)$$

for each $j > 1$. Thus, $\sigma \in \partial G(\Omega_A)$, which completes the proof.

COROLLARY 3. *Let $A = (a_{i,j})$ be a nonnegative irreducible $n \times n$ matrix. Then, its spectral radius $\rho(A)$ is a boundary point of the minimal Gerschgorin set $G(\Omega_A)$.*

Proof. By the Perron-Frobenius theory of nonnegative matrices (see [8] or [9]), there exists a vector $x > 0$ such that $Ax = \rho(A)x$, and $\rho(A) > a_{i,i}$ for all $1 \leqq i \leqq n$. It follows that $-(\rho(A) - a_{i,i}) + \Lambda_i(x) = 0$ for $1 \leqq i \leqq n$, and we conclude from (6) and (10) that $P(\rho(A))x = 0$, whence $\nu(\rho(A)) = 0$. Next, it is obvious that for any $\delta > 0$,

$$|\rho(A) + \delta - a_{i,i}| = \rho(A) + \delta - a_{i,i} > \Lambda_i(x) \qquad \text{for all } 1 \leqq i \leqq n .$$

Thus, we see that $\rho(A) \in \partial G(x)$. Applying Theorem 3, we conclude that $\rho(A) \in \partial G(\Omega_A)$, which completes the proof.

3. The reducible case.

If the $n \times n$ matrix $A = (a_{i,j})$, first considered in §1, is reducible, we cannot geometrically characterize each boundary point of $G(\Omega_A)$ as the intersection of n Gerschgorin circles. Nevertheless, we can prove

THEOREM 4. *Let $A = (a_{i,j})$ be an arbitrary $n \times n$ matrix. Then, every boundary point of $G(\Omega_A)$ is an eigenvalue of some matrix $B \in \Omega_A$.*

Proof. From Corollary 2, we can assume that A is reducible. There exists an $n \times n$ permutation matrix P such that PAP^T is in its *normal reduced form* [4, 9]:

$$PAP^T = \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,N} \\ 0 & R_{2,2} & \cdots & R_{2,N} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & R_{N,N} \end{bmatrix}, \tag{19}$$

where each square submatrix $R_{j,j}$, $1 \leqq j \leqq n$, is either irreducible or a 1×1 null matrix. Since row sums are invariant under such permutation transformations, we see from (19) that the Gerschgorin radii $\Lambda_i(x)$ for any submatrix $R_{j,j}$, $1 \leqq j \leqq N$, are not increased by diminishing the components x_j associated with submatrices $R_{k,k}$, $k > j$. Thus, the minimal Gerschgorin set $G(\Omega_A)$ is just the union of the minimal Gerschgorin sets $G(\Omega_{R_{j,j}})$ determined from the matrices $R_{j,j}$:

$$G(\Omega_A) = \bigcup_{j=1}^{N} G(\Omega_{R_{j,j}}) . \tag{20}$$

If $R_{j,j}$ is a 1×1 null matrix, then $G(\Omega_{R_{j,j}})$ consists of the sole point $z = 0$, and clearly zero is then an eigenvalue of $A \in \Omega_A$. If $R_{j,j}$ is irreducible, then $\partial G(\Omega_{R_{j,j}})$ is characterized by the result of Corollary 2. From this, it follows that every point of $G(\Omega_A)$ is an eigenvalue of some $B \in \Omega_A$, which completes the proof.

4. **Disconnected minimal Gerschgorin sets.** Gerschgorin [5] showed that if $n_1(<n)$ disks of the Gerschgorin set $G(x)$, obtained from the $n \times n$ matrix A, are disjoint from the remaining $n - n_1$ disks of $G(x)$, then these n_1 disks contain *exactly* n_1 eigenvalues of A. The proof of this result (see [8, p. 287]), basically a continuity argument, extends easily to the case where the minimal Gerschgorin set $G(\Omega_A)$ is disconnected. First, let $G_j(\Omega_A)$ denote the disjoint closed connected components of $G(\Omega_A)$:

$$G(\Omega_A) = \bigcup_{j=1}^{m} G_j(\Omega_A) , \qquad 1 \leqq m \leqq n . \tag{21}$$

Further, let the *order* r_j of each $G_j(\Omega_A)$ be defined as the number of diagonal entries $a_{i,i}$ of A(or any $B \in \Omega_A$) in $G_j(\Omega_A)$. By replacing the off-diagonal entries $a_{i,j}$ by $\alpha a_{i,j}$ for all $i \neq j$, where $0 \leqq \alpha \leqq 1$, and letting α increase to unity, it is readily seen that $1 \leqq r_j \leqq n$. With this notation, we give the following result, whose proof is omitted.

THEOREM 5. *For the set of $n \times n$ matrices Ω_A, let the $G_j(\Omega_A)$, $1 \leqq j \leqq m$, be the disjoint closed connected components of the minimal Gerschgorin set $G(\Omega_A)$. Then, each $G_j(\Omega_A)$ (of order r_j) contains exactly r_j eigenvalues of the matrix B for any $B \in \Omega_A$.*

5. The extended set $\hat{\Omega}_A$. If $S(\Omega_A)$ is the set of all eigenvalues of all $B \in \Omega_A$, then the results of Corollary 2 and Theorem 4 show us that

$$\partial G(\Omega_A) \subseteqq S(\Omega_A) \subseteqq G(\Omega_A) . \tag{22}$$

In the next section, we shall give examples where $\partial G(\Omega_A) = S(\Omega_A) \subset G(\Omega_A)$, so that $S(\Omega_A)$ need not be the entire set $G(\Omega_A)$.

Let us expand the set Ω_A as follows. Letting $\hat{\Omega}_A$ denote the set of all $n \times n$ matrices $B = (b_{i,j})$ such that

$$b_{i,i} = a_{i,i}, 1 \leqq i \leqq n; |b_{i,j}| \leqq |a_{i,j}|, i \neq j, 1 \leqq i, j \leqq n , \tag{23}$$

then it is obvious that $\Omega_A \subseteqq \hat{\Omega}_A$. If $S(\hat{\Omega}_A)$ analogously denotes the eigenvalues of all $B \in \hat{\Omega}_A$, then we prove

THEOREM 6. *Let A be an arbitrary $n \times n$ matrix. Then, $S(\hat{\Omega}_A) = G(\Omega_A)$, i.e., every $z \in G(\Omega_A)$ is an eigenvalue of some matrix $B \in \hat{\Omega}_A$.*

Proof. The expression (20) in the proof of Theorem 4 shows us that we may assume, without loss of generality, that A is irreducible. If $z \in G(\Omega_A)$, then $\nu(z) \geqq 0$ by Theorem 1, and there exists a vector $\boldsymbol{x} > \boldsymbol{0}$ such that

$$\Lambda_i(\boldsymbol{x}) - |z - a_{i,i}| = \nu(z) , \qquad 1 \leqq i \leqq n . \tag{23'}$$

Let the $n \times n$ matrix $B = (b_{i,j})$ be defined by

$$b_{i,i} = a_{i,i}, 1 \leqq i \leqq n; b_{i,j} = \mu_i a_{i,j}, i \neq j, 1 \leqq i, j \leqq n , \tag{24}$$

where

$$\mu_i = \{\Lambda_i(\boldsymbol{x}) - \nu(z)\}/\Lambda_i(\boldsymbol{x}) , \qquad 1 \leqq i \leqq n . \tag{25}$$

Then, $0 \leqq \mu_i \leqq 1, 1 \leqq i \leqq n$, and as $|b_{i,j}| \leqq |a_{i,j}|$ for all $i \neq j$, then $B \in \hat{\Omega}_A$.

Utilizing the expressions of (23'), (24), and (25), it follows that

$$|z - b_{i,i}| = \Big(\sum_{j \neq i} |b_{i,j}| x_j\Big)\Big/ x_i, 1 \leqq i \leqq n . \tag{26}$$

From Theorem 2, z is evidently an eigenvalue of some matrix $C \in \Omega_B$, which is surely contained in $\hat{\Omega}_A$, completing the proof.

6. An example. To illustrate the results of § 2, consider the $n \times n$ irreducible matrix A_n given by

(27) $$A_n = \begin{bmatrix} a_{1,1} & a_{1,2} & & 0 \\ & a_{2,2} & a_{2,3} & \\ & 0 & \ddots & a_{n-1,n} \\ a_{n,1} & & & a_{n,n} \end{bmatrix},$$

where

(27′) $$|a_{1,2}\, a_{2,3} \cdots a_{n,1}| = 1 .$$

By direct computation, we find that $\Lambda_i(\boldsymbol{x}) = |a_{i,i+1}|\, x_{i+1}/x_i$, $1 \leqq i \leqq n-1$, and $\Lambda_n(\boldsymbol{x}) = |a_{n,1}|\, x_1/x_n$. Thus,

(28) $$\prod_{i=1}^{n} \Lambda_i(\boldsymbol{x}) = |a_{1,2}a_{2,3} \cdots a_{n,1}| = 1$$

for all $\boldsymbol{x} > \boldsymbol{0}$. If $\sigma \in \partial G(\Omega_A)$, then $\nu(\sigma) = 0$, and there exists a vector $\boldsymbol{y} > \boldsymbol{0}$ such that

(29) $$|\sigma - a_{i,i}| = \Lambda_i(\boldsymbol{y}) , \qquad 1 \leqq i \leqq n .$$

Hence, taking the product over all i and using (28), we conclude that

(30) $$\prod_{i=1}^{n} |\sigma - a_{i,i}| = 1 .$$

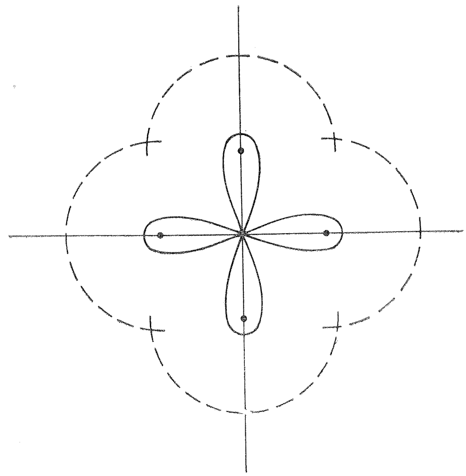
Conversely, it is readily shown that any σ for which (30) is valid is necessarily a boundary point of $G(\Omega_A)$. Thus, we conclude that all boundary points of $G(\Omega_A)$ lie on an algebraic curve of degree at most $2n$. To carry our example further, let us assume that the diagonal entries of A_n of (27) are given by

(31) $$a_{k,k} = \exp\left(\frac{2\pi i(k-1)}{n}\right), \qquad 1 \leqq k \leqq n .$$

Then for $\sigma = r \exp(i\theta)$, (30) reduces simply to

(32) $$r^n = 2 \cos n\theta ,$$

which is a higher order *lemniscate*. The minimal Gerschgorin set for the particular matrix A_4 of (27), (27′), and (31), is shown below. For comparison, the boundary of the "usual" Gerschgorin set for A_4, corresponding to the particular case $|a_{1,2}| = |a_{2,3}| = |a_{3,4}| = |a_{4,1}| = 1$ and the choice $x_i = 1$ in (2′), is indicated by dotted lines.



To illustrate the results of § 4, we again consider the matrix A_4 determined by (27) and (31), but now the off-diagonal terms of (27) are of modulus $0 < \alpha < 1$. Referring to the figure above, it is apparent that the minimal Gerschgorin set $G(\Omega_A)$ for A_4 is *disconnected*, with four disjoint connected components, each of order unity. Although $G(\Omega_A)$ is disconnected, it can be shown, for α sufficiently close to unity and all $n \geqq 4$, that every Gerschgorin set $G(\boldsymbol{x}), \boldsymbol{x} > \boldsymbol{0}$, is always connected. The point of this remark is that the minimal Gerschgorin set $G(\Omega_A)$ can sometimes isolate eigenvalues of $B \in \Omega_A$ which cannot be isolated by the Gerschgorin sets $G(\boldsymbol{x})$.

Let us again consider the matrix A_n of (27), subject to (27′). Since

$$\det(A_n - \lambda I) = \prod_{i=1}^{n} (a_{i,i} - \lambda) - (-1)^n a_{1,2} a_{2,3} \cdots a_{n,1}, \tag{33}$$

it follows that if λ is any eigenvalue of any $B \in \Omega_{A_n}$, then

$$\prod_{i=1}^{n} |a_{i,i} - \lambda| = 1 . \tag{34}$$

But from (30), we know that (34) precisely describes the boundary $\partial G(\Omega_{A_n})$ of the minimal Gerschgorin set for A_n. Thus, *no interior point* of the minimal Gerschgorin set can be an eigenvalue of any $B \in \Omega_{A_n}$ in this case. Using the notation of § 5, we have therefore shown for the matrices A_n of (27) that

$$\partial G(\Omega_{A_n}) = S(\Omega_{A_n}) . \tag{35}$$

Finally, it is interesting to point out that for the special case $n = 2$ of (27), which is the *general* 2×2 case, (30) and (34) reduce to the well known *oval of Cassini* considered by Brauer [1] and others. Thus, in the 2×2 case, the minimal Gerschgorin set $G(\Omega_A)$ is precisely the oval of Cassini.

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