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Introduction. The classical inversion of

$$(*) \qquad F(t) = \int_0^\infty \phi(u) \cos tu \, du$$

is

$$\phi(t) = \frac{2}{\pi} \int_0^\infty F(u) \cos tu \, du.$$

In this paper we present a method of inverting (*) which uses no integration whatsoever. The method consists of an application of the Moebius inversion formula combined with a variation of the classical Poisson formula from Fourier analysis. The main result is contained in Theorem 3. (*Added in proof.* It has been called to our attention that a similar result was announced by R. J. Duffin in the Bulletin of the American Mathematical Society, vol. 47(1941), p. 383.)

Theorem 3. *If* 1. $\phi(u)$ *of bounded variation on* $(0 \leq u \leq R)$ *for every* $R > 0$,

$$2. \quad \int_1^\infty |\phi(u)| \log u \, du < \infty, \text{ and}$$

$$3. \quad F(t) = \int_0^\infty \phi(u) \cos tu \, du,$$

then A. $G(t) = \dfrac{1}{t}\left[\dfrac{F(0)}{2} + \sum_{k=1}^{\infty} (-1)^k F\left(\dfrac{k\pi}{t}\right)\right]$

is finite almost everywhere $(0 < t < \infty)$ *and*

$$\text{B.} \quad \phi(t) = \sum_{n=1}^{\infty} \mu_{2n-1} G[(2n-1)t]$$

almost everywhere $(0 < t < \infty)$.

Here the $\{\mu_n\}$ are the Moebius numbers, defined in Example 1 of Section II.

I. Two lemmas on sums.

Lemma 1. *If* 1. $\int_R^\infty |\phi(t)| \, dt < \infty$ *for every* $R > 0$,

then $\sum_{k=1}^{\infty} |\phi(kt)| < \infty$ *almost everywhere* $(0 < t < \infty)$.

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Proof. We first show

$$\int_1^\infty \frac{dt}{t} \sum_{k=1}^\infty |\phi(kt)| \leq \int_1^\infty |\phi(t)| \, dt. \tag{1}$$

We have

$$\begin{aligned}
\int_1^\infty \frac{dt}{t} \sum_{k=1}^\infty |\phi(kt)| &= \sum_{k=1}^\infty \int_1^\infty \frac{|\phi(kt)|}{t} \, dt \\
&= \sum_{k=1}^\infty \int_k^\infty \frac{|\phi(t)|}{t} \, dt = \sum_{k=1}^\infty \sum_{n=k}^\infty \int_n^{n+1} \frac{|\phi(t)|}{t} \, dt \\
&= \sum_{n=1}^\infty \int_n^{n+1} \frac{|\phi(t)|}{t} \, dt \sum_{k=1}^n 1 = \sum_{n=1}^\infty n \int_n^{n+1} \frac{|\phi(t)|}{t} \, dt \\
&\leq \sum_{n=1}^\infty \int_n^{n+1} |\phi(t)| \, dt = \int_1^\infty |\phi(t)| \, dt.
\end{aligned}$$

Using (1) we have for any $R > 0$

$$\int_R^\infty \frac{dt}{t} \sum_{k=1}^\infty |\phi(kt)| = \int_1^\infty \frac{dt}{t} \sum_{k=1}^\infty |\phi(kRt)| \leq \int_1^\infty |\phi(Rt)| \, dt = \frac{1}{R} \int_R^\infty |\phi(t)| \, dt.$$

The last term is finite by hypothesis so that

$$\int_R^\infty \frac{dt}{t} \sum_{k=1}^\infty |\phi(kt)| < \infty \quad \text{for any} \quad R > 0.$$

The conclusion follows immediately.

LEMMA 2. *If*

1. $\int_R^\infty |\phi(t)| \, dt < \infty$ *for every* $R > 0$ *and*

2. $\int_1^\infty |\phi(t)| \log t \, dt < \infty,$

then

$$\sum_{n=1}^\infty \sum_{k=1}^\infty |\phi(knt)| < \infty \quad \textit{almost everywhere} \quad (0 < t < \infty).$$

Proof. From (1) we have for any $n = 1, 2, \cdots$

$$\int_1^\infty \frac{dt}{t} \sum_{k=1}^\infty |\phi(knt)| \leq \int_1^\infty |\phi(nt)| \, dt$$

so that

$$\int_1^\infty \frac{dt}{t} \sum_{n=1}^\infty \sum_{k=1}^\infty | \phi(knt) | = \sum_{n=1}^\infty \int_1^\infty \frac{dt}{t} \sum_{k=1}^\infty | \phi(knt) | \leq \sum_{n=1}^\infty \int_1^\infty | \phi(nt) | \, dt$$

$$= \sum_{n=1}^\infty \frac{1}{n} \int_n^\infty | \phi(t) | \, dt = \sum_{n=1}^\infty \frac{1}{n} \sum_{k=n}^\infty \int_k^{k+1} | \phi(t) | \, dt$$

$$= \sum_{k=1}^\infty \int_k^{k+1} | \phi(t) | \, dt \sum_{n=1}^k \frac{1}{n} \leq \sum_{k=1}^\infty (\log k + \gamma) \int_k^{k+1} | \phi(t) | \, dt$$

$$\leq \sum_{k=1}^\infty \int_k^{k+1} | \phi(t) | (\log t + \gamma) \, dt = \int_1^\infty | \phi(t) | (\log t + \gamma) \, dt.$$

Here γ is Euler's constant.

Hence, for any $R > 0$,

$$\int_R^\infty \frac{dt}{t} \sum_{n=1}^\infty \sum_{k=1}^\infty | \phi(knt) | = \int_1^\infty \frac{dt}{t} \sum_{n=1}^\infty \sum_{k=1}^\infty | \phi(knRt) |$$

$$\leq \int_1^\infty | \phi(Rt) | (\log t + \gamma) \, dt = \frac{1}{R} \int_R^\infty | \phi(t) | \left(\log \frac{t}{R} + \gamma\right) dt$$

$$\leq \frac{1}{R} \int_R^\infty | \phi(t) \log t | \, dt + \frac{1}{R} (| \log R | + \gamma) \int_R^\infty | \phi(t) | \, dt.$$

Our hypotheses show that the last two integrals are finite. Thus

$$\int_R^\infty \frac{dt}{t} \sum_{n=1}^\infty \sum_{k=1}^\infty | \phi(knt) | < \infty \quad \text{for any} \quad R > 0$$

from which the conclusion is evident.

(We are indebted to the referee for the simple proofs of Lemmas 1 and 2. For more general lemmas of the above type, as well as a discussion of Moebius inversions of other integral transforms, see [1].)

II. The Moebius Inversion Formula. Let $\{a_k\}_{k=1}^\infty$ be any sequence of numbers with $a_1 \neq 0$, and let $\{b_n\}_{n=1}^\infty$ be the (unique) sequence such that

$$\sum_{d | m} a_d b_{m/d} = \begin{matrix} 1 & m = 1 \\ 0 & m = 2, 3, \cdots \end{matrix} \tag{2}$$

the sum running over all divisors d of the positive integer m. If for some function $\phi(t)$ we have

$$\sum_{n=1}^\infty \sum_{k=1}^\infty | a_k b_n \phi(knt) | < \infty \qquad \text{(some fixed } t\text{)}, \tag{3}$$

then

$$\sum_{n=1}^\infty \sum_{k=1}^\infty a_k b_n \phi(knt) = \phi(t). \tag{4}$$

For if (3) holds, we may rearrange the double series to obtain

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_k b_n \phi(knt) = \sum_{m=1}^{\infty} \phi(mt) \sum_{d \mid m} a_d b_{m/d} = \phi(t).$$

If we set

$$G(t) = \sum_{k=1}^{\infty} a_k \phi(kt), \tag{5}$$

then if (3) holds, we have from (4)

$$\phi(t) = \sum_{n=1}^{\infty} b_n G(nt). \tag{6}$$

Thus if we obtain G from ϕ using $\{a_k\}$, we invert (obtain ϕ from G) using $\{b_n\}$. For this reason (4), or equivalently (5) and (6), is called the (Moebius) inversion formula. Here are two examples of pairs of sequences satisfying (2):

EXAMPLE 1. If $a_k = 1$, $k = 1, 2, \cdots$, it is well known (see [2]) that $b_n = \mu_n$ where $\{\mu_n\}_{n=1}^{\infty}$ are the Moebius numbers defined as $\mu_1 = 1$, $\mu_n = (-1)^s$ if n is the product of s distinct primes, $\mu_n = 0$ if n is divisible by a square. For this example, (2) reads

$$\sum_{d \mid m} \mu_{m/d} = \begin{cases} 1 & m = 1 \\ 0 & m = 2, 3, \cdots . \end{cases} \tag{7}$$

EXAMPLE 2. If

$$a_{2k-1} = 1 \quad k = 1, 2, \cdots; \qquad a_{2k} = 0 \quad k = 1, 2, \cdots,$$

then

$$b_{2n-1} = \mu_{2n-1} \quad n = 1, 2, \cdots; \qquad b_{2n} = 0 \quad n = 1, 2, \cdots .$$

For if m is even, then each term in (2) is zero while if m is odd, then each divisor d of m is also odd and (2) follows from (7). Hence (2) holds for this pair of sequences.

We can now prove the following theorem.

THEOREM 1. *If* 1. $\int_R^{\infty} |\phi(t)|\, dt < \infty$ *for every* $R > 0$,

2. $\int_1^{\infty} |\phi(t)| \log t\, dt < \infty$, *and*

3. $G(t) = \sum_{k=1}^{\infty} \phi[(2k-1)t]$

(which converges almost everywhere $(0 < t < \infty)$ *by Lemma* 1)

then

$$\phi(t) = \sum_{n=1}^{\infty} \mu_{2n-1} G[(2n-1)t]$$

almost everywhere $(0 < t < \infty)$.

Proof. In view of preceding remarks we need only prove

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} | a_k b_n \phi(knt) | < \infty$$

almost everywhere $(0 < t < \infty)$ where $\{a_k\}$ and $\{b_n\}$ are defined in Example 2. Since $| a_k | \leq 1$, $| b_n | \leq 1$, it is sufficient to prove

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} | \phi(knt) | < \infty$$

almost everywhere $(0 < t < \infty)$. But this follows from Lemma 2, and the theorem is proved.

III. **The Poisson Formula.** This is the formula

$$\sqrt{\beta}\left[\frac{F(0)}{2} + \sum_{n=1}^{\infty} F(n\beta)\right] = \sqrt{\alpha}\left[\frac{f(0)}{2} + \sum_{n=1}^{\infty} f(n\alpha)\right]$$

where $\alpha > 0$, $\alpha\beta = 2\pi$, and

$$F(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \cos tu \, du.$$

In [3] the formula (8) is established for functions $f(u)$ which are of bounded total variation on $(0 \leq u < \infty)$ and which vanish at infinity. We will establish a formula similar to (8) under conditions which will make Lemma 1 applicable.

THEOREM 2. *If* 1. *$\phi(u)$ is of bounded variation on $(0 \leq u \leq R)$ for every $R > 0$,*

2. $\int_R^{\infty} | \phi(u) | \, du < \infty$ *for some* $R > 0$,

3. $F(t) = \int_0^{\infty} \phi(u) \cos tu \, du$, *and*

4. $G_N(t) = \frac{1}{t}\left[\frac{F(0)}{2} + \sum_{k=1}^{N} (-1)^k F\left(\frac{k\pi}{t}\right)\right] \qquad N = 1, 2, \cdots,$

then A. $G(t) = \lim_{N\to\infty} G_N(t)$

exists almost everywhere $(0 < t < \infty)$ and

B. $G(t) = \sum_{k=1}^{\infty} \phi[(2k-1)t]$

almost everywhere $(0 < t < \infty)$.

Proof. Since

$$F\left(\frac{k\pi}{t}\right) = \int_0^{\infty} \phi(u) \cos\frac{k\pi u}{t} \, du = t\int_0^{\infty} \phi(tu) \cos k\pi u \, du,$$

we have

$$(-1)^k F\left(\frac{k\pi}{t}\right) = t\int_0^\infty \phi(tu)\cos k\pi(u+1)\,du \qquad (k = 1, 2, \cdots; 0 < t < \infty).$$

Also

$$F(0) = t\int_0^\infty \phi(tu)\,du \qquad (0 < t < \infty).$$

Hence

$$G_N(t) = \int_0^\infty \phi(tu)D_N[\pi(u+1)] = \sum_{k=0}^\infty \int_{2k}^{2k+2} \phi(tu)D_N[\pi(u+1)]\,du \tag{9}$$

where

$$D_N(u) = \tfrac{1}{2} + \cos u + \cos 2u + \cdots + \cos Nu = \frac{\sin(N+\frac{1}{2})u}{2\sin u/2}.$$

For any $g(u)$ of bounded variation it is known from Fourier analysis that

$$\lim_{N\to\infty}\int_{2k-1}^{2k+1} g(u)D_N(\pi\mu)\,du = g(2k) \qquad k = 0, 1, 2, \cdots$$

and hence

$$\lim_{N\to\infty}\int_{2k}^{2k+2} \phi(tu)D_N[\pi(u+1)]\,du = \phi[(2k+1)t] \qquad k = 0, 1, 2, \cdots.$$

Thus, taking the limit under the summation sign in (9) we have

$$G(t) = \lim_{N\to\infty} G_N(t) = \sum_{k=0}^\infty \phi[(2k+1)t] = \sum_{k=1}^\infty \phi[(2k-1)t],$$

and the proof of the theorem will be complete if we can justify the limiting process for almost all t. To do this we note that by a mean value theorem

$$\int_{2k}^{2k+2} \phi(tu)D_N[\pi(u+1)]\,du = \phi(2kt)\int_{2k}^{\xi_k} D_N[\pi(u+1)]\,du + \phi[(2k+2)t]\int_{\xi_k}^{2k+2} D_N[\pi(u+1)]\,du$$

where $(2k \le \xi_k \le 2k+2)$. Hence

$$\sum_{k=0}^\infty \left|\int_{2k}^{2k+2} \phi(tu)D_N[\pi(u+1)]\,du\right| \le A\sum_{k=0}^\infty [|\phi(2kt)| + |\phi[(2k+2)t]|] \tag{10}$$

where $A > 0$ is a constant such that

$$\left|\int_a^b D_N(u)\,du\right| \le A \qquad (n = 1, 2, \cdots; 0\, a \le b < \infty).$$

(That A exists is shown in [3; 69].) Since the right side of (10) is independent of N and is, by Lemma 1, finite almost everywhere $(0 < t < \infty)$, the above limit process is valid almost everywhere $(0 < t < \infty)$, and the theorem is proved.

The inversion theorem (Theorem 3) is a combination of Theorem 1 and Theorem 2.

In closing we note that if the double series in Lemma 2 converges everywhere, then it follows readily that the conclusions A and B of Theorem 3 also hold everywhere. But the double series converges everywhere for a large class of $\phi(t)$, for example, any $\phi(t)$ such that

$$\phi(t) = O\left(\frac{1}{t^\alpha}\right) \qquad (\alpha > 1;\, t \to \infty).$$

This indicates that the above inversion may be used as a feasible numerical device. The authors have used it successfully in a number of cases.

References

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