NUMERICAL SOLUTION PARTIAL DIFFERENTIAL EQUATIONS

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Hermite Interpolation-Type Ritz Methods for Two-Point Boundary Value Problems

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1. Introduction

In the important work of Bramble and Hubbard [*1*], higher-order difference approximations of order $O(h^4)$ are used at interior mesh points for two-point self-adjoint second-order boundary value problems, while lower-order difference approximations are used near the boundary points. The resulting matrix equations are such that the associated matrices A are nonsymmetric, but possess the attractive property that $A^{-1} \geq 0$. Moreover, it is shown that the discretization error is also $O(h^4)$.

This lack of symmetry in the matrix A is somewhat a matter of concern, in that one intuitively feels that such self-adjoint problems should give symmetric matrix approximations. This lack of symmetry also makes it much more difficult (if not impossible [1, p. 124]) to rigorously apply standard direct and iterative methods for the solution of such matrix equations. This raises the question if comparable difference equations can be derived which are symmetric and of high accuracy.

In this work, we shall affirmatively answer this question with the use of higher-order Hermite interpolation in conjunction with the Ritz method for numerically approximating the solution $u(x)$ of the two-point boundary value problem.

$$-u''(x) + \sigma(x)\,u(x) = f(x), \qquad 0 < x < 1, \qquad \sigma(x) \geq 0, \tag{0}$$

where $u(0) = u(1) = 0$. The Ritz method is of course classic, and other authors (e.g., [*2*]) have applied similar ideas to the numerical solution of this problem. However, the error estimates of Theorem 1 and the connection with Fox's differential correction method [*4*] seem new.

2. The Spaces $H_N^{(m)}$ and H

Let $\hat{H}$ be the space of all real-valued functions $w(x)$ on $0 \leq x \leq 1$ with $w(0) = w(1) = 0$ such that the following integral exists and is finite:

$$\int_0^1 [(w'(t))^2 + \sigma(t)\,(w(t))^2]\,dt < \infty. \tag{1}$$

Here, $\sigma(x)$ is a nonnegative smooth function on $0 \leq x \leq 1$. Defining an inner product on $\hat{H}$ as

$$\langle v, w \rangle = \int_0^1 [v'(t)\, w'(t) + \sigma(t)\, v(t)\, w(t)]\, dt, \qquad v, w \in \hat{H}, \tag{2}$$

then we denote the Dirichlet norm on $\hat{H}$ as

$$\|w\|_D{}^2 = \langle w, w \rangle = \int_0^1 [(w'(t))^2 + \sigma(t)\, (w(t))^2]\, dt. \tag{3}$$

Completing $\hat{H}$ in this norm gives us the Hilbert space H.

Let $H_N^{(m)}$ be the (finite dimensional) subspace of H of all piecewise polynomial functions $w(x; m) = w(x; d_1^{(0)},..., d_N^{(0)}; d_0^{(1)},..., d_{N+1}^{(1)}; ...; d_0^{(m)},..., d_{N+1}^{(m)})$ such that, for $x_i = ih$, $0 \leq i \leq N+1$, $h = 1/(N+1)$,

$$\begin{aligned} w(x_i; m) &= d_i^{(0)}, \qquad 1 \leq i \leq N, \\ w^{(k)}(x_i; m) &= d_i^{(k)}, \qquad 0 \leq i \leq N+1, \quad 1 \leq k \leq m, \end{aligned} \tag{4}$$

and, in each interval $ih \leq x \leq (i+1)h$, $0 \leq i \leq N$, $w(x; m)$ is a polynomial of degree $(2m+1)$. Clearly, $w \in H_N^{(m)}$ implies that $w \in C^{(m)}[0, 1]$. We remark that $w(x; m)$ is uniquely determined by the $(m+1)N + 2m$ parameters $d_i^{(0)}$, $d_j^{(k)}$, $1 \leq i \leq N$, $0 \leq j \leq N+1$, $1 \leq k \leq m$. In other words $w \in H_n^{(m)}$ can be viewed as the unique Hermite polynomial interpolation of these parameters.

It is easy to see that $H_N^{(m)}$ is a substance of H of dimension $(m+1)N+2m$. Specifically, we can construct $(m+1)N + 2m$ basis vectors in $H_N^{(m)}$ as follows. Let $s_{i,0}(x; m)$, $1 \leq i \leq N$, and $s_{i,k}(x; m)$, $0 \leq i \leq N+1$, $1 \leq k \leq m$, be defined as

$$s_{i,0}(x_j; m) = \delta_{i,j}; \qquad s_{i,0}^{(l)}(x_j; m) = 0, \qquad 1 \leq l \leq m, \tag{5}$$

$$s_{i,k}^{(k)}(x_j; m) = \delta_{i,j}; \qquad s_{i,k}^{(l)}(x_j; m) = 0, \qquad 0 \leq l \leq m, \quad l \neq k, \tag{6}$$

for all $0 \leq j \leq N+1$. For any $w \in H_N^{(m)}$, we can thus write

$$w(x; m) = \sum_{i=1}^{N} d_i^{(0)} s_{i,0}(x; m) + \sum_{k=1}^{m} \sum_{i=0}^{N+1} d_i^{(k)} s_{i,k}(x; m). \tag{7}$$

Since $H_N^{(m)}$ is a closed finite subspace of H, a standard result from Hilbert space theory [5] tells us that, given any $u \in H$, there exists a *unique* $w \in H_N^{(m)}$ of best approximation to u, called the *projection of* u on $H_N^{(m)}$, such that

$$\inf_{w \in H_N^{(m)}} \|u - w\|_D = \|u - \hat{w}\|_D. \tag{8}$$

To show the relationship of this minimization problem of Eq. (8) to variational problems, we define the functional F on H as

$$F[w] = \int_0^1 [(w'(t))^2 + \sigma(t)(w(t))^2 - 2w(t) f(t)]\, dt. \tag{9}$$

If $u(x)$ is the solution of the two-point boundary value problem

$$-u''(x) + \sigma(x) u(x) = f(x), \qquad 0 < x < 1, \tag{10}$$

subject to the boundary conditions

$$u(0) = u(1) = 0, \tag{11}$$

where $\sigma(x) \geq 0$ and $f(x)$ are given functions which will subsequently be assumed to be sufficiently smooth, then it is well known that

$$\|u - w\|_D{}^2 = F[w] - F[u] \geq 0, \qquad \forall w \in H, \tag{12}$$

and

$$\|u - \hat{w}\|_D{}^2 = \inf_{w \in H_N^{(m)}} \|u - w\|_D{}^2 = \inf_{w \in H_N^{(m)}} F[w] - F[u]. \tag{13}$$

Thus, the element $\hat{w}$ in $H_N^{(m)}$ of best approximation to u can be equivalently formulated as

$$F[\hat{w}] = \inf_{w \in H_N^{(m)}} F[w]. \tag{14}$$

3. Accuracy

In this section, we estimate the size of the norm $\|u - \hat{w}\|_D$ for $\hat{w} \in H_N^{(m)}$ as $h \to 0$. To begin, we first derive a norm inequality which will be useful. For any function $g(x) \in C^1[0, 1]$ for which $g(0) = g(1) = 0$, it is well known from the fundamental theorem of calculus and Schwarz's inequality that

$$2 \sup_{0 \leq x \leq 1} |g(x)| \leq \left(\int_0^1 (g'(t))^2 \, dt \right)^{1/2}. \tag{15}$$

Since $\sigma(x)$ is by assumption nonnegative in $0 \leq x \leq 1$, it then follows from Eq. (3) that

$$2\|w\|_\infty \leq \|w\|_D \tag{16}$$

for any differentiable function $w(x)$ in H, where $\|w\|_\infty = \sup_{0 \leq t \leq 1} |w(t)|$.

We assume now that the solution $u(x)$ of Eqs. (10)–(11) is of class $C^{2m+2}[0, 1]$. Evaluating $u(x)$ and its derivatives in the points $x_i = ih$, consider the vector $\tilde{w}(x; m) = \tilde{w}(x; u_1, \ldots, u_N;\ u_0^{(1)}, \ldots, u_{N+1}^{(1)};\ \ldots;\ u_0^{(m)}, \ldots, u_{N+1}^{(m)})$ in $H_N^{(m)}$, obtained by Hermite polynomial interpolation to $u(x)$ in each interval $ih \leq x \leq (i + 1)h$, $0 \leq i \leq N$; i.e., $u^{(l)}(ih) = \tilde{w}^{(l)}(ih; m)$ for $0 \leq l \leq m$, $0 \leq i \leq N + 1$. Then, estimating the interpolation error in a standard way (cf. [3]), we find for each $0 \leq i \leq N$ that

$$u(x) - \tilde{w}(x; m) = \frac{u^{(2m+2)}(\xi_i)}{(2m+2)!}[(x - ih)((i+1)h - x)]^{m+1}, \; x \in [ih, (i+1)h] \tag{17}$$

where $ih < \xi_i < (i+1)h$. Thus, if $|u^{(2m+2)}(x)| \leq M_{2m+2}$ for all $0 \leq x \leq 1$, then

$$|u(x) - \tilde{w}(x; m)| \leq \frac{M_{2m+2}}{(2m+2)!}(h/2)^{2m+2}, \qquad 0 \leq x \leq 1. \tag{18}$$

In a similar way, we find that

$$|u'(x) - \tilde{w}'(x; m)| \leq \frac{M_{2m+2}}{2(2m+2)!}(h/2)^{2m+1}, \qquad 0 \leq x \leq 1. \tag{19}$$

From the definition of the Dirichlet norm in Eq. (3), the inequalities of Eqs. (18)–(19) then show us that there exists a constant Q_m, independent of h, such that

$$\|u(x) - \tilde{w}(x; m)\|_D \leq Q_m h^{2m+1}. \tag{20}$$

But as

$$\|u(x) - \hat{w}(x; m)\|_D = \inf_{w \in H_N^{(m)}} \|u - w\|_D \leq \|u - \tilde{w}\|_D,$$

we have proved

Theorem 1. Assume that the solution $u(x)$ of Eqs. (10)–(11) is of class $C^{2m+2}[0, 1.]$ Then, the element $\hat{w}(x; m)$ of best approximation to $u(x)$ in $H_N^{(m)}$ satisfies

$$2\|u(x) - \hat{w}(x; m)\|_\infty \leq \|u(x) - \hat{w}(x; m)\|_D \leq Q_m h^{2m+1}, \tag{21}$$

where Q_m is independent of h.

We remark that $\|u(x) - \tilde{w}(x; m)\|_\infty$ is $O(h^{2m+2})$ from Eq. (18), and it is probably also true that $\|u(x) - \hat{w}(x; m)\|_\infty$ is $O(h^{2m+2})$. Nevertheless, the result of Theorem 1, even for the special case $m = 2$, represents an improvement over the recent results of Bramble and Hubbard [1]. Specifically, assuming $u(x)$ is of class $C^6[0, 1]$, Bramble and Hubbard [1, Theorem 2.6], using five-diagonal matrices, obtained discrete approximations w_i to $u(ih)$ such that $|u(ih) - w_i| \leq Mh^4$ for $0 \leq i \leq N + 1$. With the same hypothesis, namely, $u \in C^6[0, 1]$, Theorem 1 gives us a *continuous* approximation $\hat{w}(x; 2)$ such that $|u(x) - \hat{w}(x; 2)| \leq Q_2 h^4$ for $0 \leq x \leq 1$. The results of [1] are however more closely connected in form with the case $m = 1$, since we shall see in the next section that the matrix equation whose solution determined $\hat{w}(x; 1)$ is such that the associated matrix is essentially five-diagonal.

4. Determination of the Best Approximation in $H_N^{(m)}$

We have seen that finding the unique element of best approximation in $H_N^{(m)}$ to the solution $u(x)$ of Eqs. (10)–(11) is equivalent to minimizing the functional $F[w]$ of Eq. (9) over $H_N^{(m)}$. As each element in $H_N^{(m)}$ depends on $P \equiv (m + 1)N + 2m$ parameters, we now derive a system of linear equations whose solution uniquely determines the P associated parameters for $\hat{w}(x; m)$ in $H_N^{(m)}$.

It is first convenient to order the basis vectors of $H_N^{(m)}$ as follows. Let $v_1(x; m) = s_{0,1}(x; m)$, $v_2(x; m) = s_{0,2}(x; m), \ldots, v_m(x; m) = s_{0,m}(x; m)$; $v_{m+1}(x; m) = s_{1,0}(x; m), \ldots, v_{2m+1}(x; m) = s_{1,m}(x; m), \ldots, v_P(x; m) = s_{N+1,m}(x; m)$. This corresponds to numbering the basis vectors of $H_N^{(m)}$ consecutively for each mesh point. For any $w \in H_N^{(m)}$, we can write

$$w(x; m) = \sum_{j=1}^{P} c_j v_j(x; m). \tag{22}$$

Thus, the functional $F[w]$ is

$$F[w] = \int_0^1 \left[\left\{ \sum_{j=1}^{P} c_j v_j' \right\}^2 + \sigma(t) \left\{ \sum_{j=1}^{P} c_j v_j \right\}^2 - 2f(t) \left\{ \sum_{j=1}^{P} c_j v_j \right\} \right] dt, \tag{23}$$

which can be expressed as the quadratic form

$$F[w] = \mathbf{x}^T A \mathbf{x} - 2\mathbf{x}^T \mathbf{k}, \tag{24}$$

where $A = (a_{i,j})$ is a real $P \times P$ matrix with

$$a_{i,j} = \int_0^1 (v_i' v_j' + \sigma(t) v_i v_j)\, dt = \langle v_i, v_j \rangle, \tag{25}$$

$\mathbf{k}$ is a column vector whose entries are $\int_0^1 f v_i \, dt$, $1 \le i \le P$, and $\mathbf{x} = (c_1, c_2, \ldots, c_P)^T$ is the column vector of coefficients which determine $w(x; m)$ in Eq. (22). Minimizing $F[w]$ by setting $\partial F[w]/\partial c_j = 0$, $1 \le j \le P$, gives us the matrix equation

$$A\mathbf{x} = \mathbf{k}. \tag{26}$$

Because the vectors $v_i(x; m)$ are linearly independent, the matrix A is of course positive definite.

Returning to the definition of the vectors $s_{i,k}(x; m)$ in Eqs. (5)–(6), it is clear that $s_{i,k}(x; m)$ is zero for any x not in $(i - 1)h \le t \le (i + 1)h$. We can make use of this observation as follows. From Eq. (25), $a_{i,j} = 0$ for $|j - 1| > 2(m + 1)$. Thus, the matrix A of Eq. (26) is a band matrix of semiwidth $2m + 1$, which is independent of h. A more precise statement, however, can be made. We can partition the matrix A of Eq. (26) so that

$$A = \begin{bmatrix} A_{0,0} & A_{0,1} & & & \bigcirc \\ A_{0,1}^T & A_{1,1} & A_{1,2} & & \\ & \ddots & A_{N,N} & A_{N,N+1} & \\ \bigcirc & & A_{N,N+1}^T & A_{N+1,N+1} & \end{bmatrix}; \tag{27}$$

i.e., A is a block tridiagonal matrix. Here, the order of the submatrix $A_{j,j}$ is the number of parameters associated with the jth mesh point in the interval $0 \leq x \leq 1$. In particular, the submatrices $A_{1,1},\ldots, A_{N,N}$ are all of order $m + 1$, while $A_{0,0}$ and $A_{N+1,N+1}$ are of order m.

These derived properties, viz., that the matrix A of Eq. (26) is real symmetric and positive definite, a band matrix with semiwidth independent of h, and block tridiagonal, are of considerable numerical usefulness when one actually wishes to solve this matrix equation. Because A is a positive definite symmetric band matrix, the direct solution of Eq. (26) by Gaussian elimination without pivoting is efficient and stable, relative to the growth of rounding errors [9]. Similarly, since A is block tridiagonal, the direct solution of Eq. (26) can also be obtained by applying Gaussian elimination by blocks to this matrix equation (cf. [8, p. 196]). Iterative methods can also be applied to solve Eq. (26). Since A is real symmetric and positive definite, the point successive overrelaxation is convergent for any ω with $0 < \omega < 2$ (cf. [8, p. 177]); and as A is block tridiagonal and thus consistently ordered, block successive overrelaxation is also convergent for any ω with $0 < \omega < 2$ [8, p. 111]. These statements are true, *independent of* h. Thus, divergent iterative techniques or iterative techniques which are convergent only for h sufficiently small, as in [1], are completely avoided.

Finally, as another computational point, each $s_{i,k}(x; m)$ for any $1 \leq i \leq m$ has an expansion

$$s_{i,k}(x; m) = \sum_{l=0}^{2m+1} c_l(k; m)(x - x_i)^l, \quad 1 \leq i \leq m, \quad x \in [(i-1)h, (i+1)h], \tag{28}$$

where the coefficients $c_l(k; m)$ do not depend on i. Thus, it is sufficient to determine just $s_{1,k}(x; m)$, $0 \leq k \leq m$. Similarly, by symmetry, the determination of $s_{0,k}(x; m)$, $1 \leq k \leq m$, gives also the vectors $s_{N+1,k}(x; m)$, $1 \leq k \leq m$.

5. Connection with Other Methods

In the previous section, we deduced the positive definite symmetric nature of the matrix A of Eq. (26). Now, we reorder the parameters $d_k^{(l)}$

which from Eq. (4) uniquely determine $\hat{w}(x)$. Let R be the $P \times P$ permutation matrix, $P = (m + 1)N + 2m$, such that

$$Rx = (d_1^{(0)}, \ldots, d_N^{(0)};\ d_0^{(1)}, \ldots, d_{N+1}^{(1)};\ \ldots;\ d_0^{(m)}, \ldots, d_{N+1}^{(m)})^T = \mathbf{y}; \qquad (29)$$

i.e., we group the parameters corresponding to the same derivative at the mesh points. With $RAR^T \equiv \tilde{A}$, we can partition the matrix $\tilde{A}$ and the vector $\mathbf{y}$ corresponding to the above natural grouping of the parameters so that the matrix equation $A\mathbf{x} = \mathbf{k}$ becomes

$$\begin{bmatrix} \tilde{A}_{0,0} & \tilde{A}_{0,1} & \cdots & \tilde{A}_{0,m} \\ \tilde{A}_{0,1}^T & \tilde{A}_{1,1} & \cdots & \tilde{A}_{1,m} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \tilde{A}_{0,m}^T & \tilde{A}_{1,m}^T & \cdots & \tilde{A}_{m,m} \end{bmatrix} \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{y}_m \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{k}}_0 \\ \tilde{\mathbf{k}}_1 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{\mathbf{k}}_m \end{bmatrix}, \qquad (30)$$

where $\tilde{\mathbf{k}} = R\mathbf{k}$. Because A is real symmetric and positive definite, so is $\tilde{A}$. Thus, the block Gauss–Seidel iterative procedure

$$\left\{ \sum_{j \le i} \tilde{A}_{j,i} y_j^{(m+1)} + \sum_{j > i} \tilde{A}_{i,j} y_j^{(m)} = \tilde{k}_i \right\}_{i=0}^{m} \qquad (31)$$

is convergent (8, p. 77) for any initial $\mathbf{y}^{(0)}$.

Interestingly enough, this can be viewed as a variant of the *Fox differential correction method* [4], which we shall show is necessarily rigorously convergent. To see this, assume that $u(x) \in C^{2m+2}[0, 1]$, and assume that we have found the element $\hat{w}(x; m - 1)$ of best approximation to $u(x)$ in $H_N^{(m-1)}$. This uniquely determines the $P - (N + 2)$ parameters $\hat{w}_1^{(0)}$, $\hat{w}_2^{(0)}, \ldots, \hat{w}_N^{(0)};\ \hat{w}_0^{(1)}, \ldots, \hat{w}_{N+1}^{(1)};\ \ldots;\ \hat{w}_0^{(m-1)}, \ldots, \hat{w}_{N+1}^{(m-1)}$. With these parameters, define the initial vector $\mathbf{y}^{(0)}$ of Eq. (31) so that $\mathbf{y}_0^{(0)} = (\hat{w}_1^{(0)}, \ldots, \hat{w}_N^{(0)})^T, \ldots,$ $\mathbf{y}_{m-1}^{(0)} = (\hat{w}_0^{(m-1)}, \ldots, \hat{w}_{N+1}^{(m-1)})^T$, and $\mathbf{y}_m^{(0)} = (0, \ldots, 0)^T$. These parameters in turn determine a vector $y^{(0)}(x; m)$ in $H_N^{(m)}$. Since the vector $\hat{w}(x; m - 1)$ of $H_N^{(m-1)}$ is itself a Hermite interpolation of $y^{(0)}(x)$, then

$$\|\hat{w}(x; m - 1) - y^{(0)}(x; m)\|_\infty \le Mh^{2m-1}. \qquad (32)$$

But as $\|u(x) - \hat{w}(x; m - 1)\|_\infty$ is itself $O(h^{2m-1})$, then clearly

$$\|y^{(0)}(x; m) - u(x)\|_\infty \le Mh^{2m-1}. \qquad (33)$$

On the other hand, with this initial vector $y^{(0)}(x; m - 1)$ the iteration in Eq. (31) is necessarily convergent to the unique element $\hat{w}(x; m)$ of best approximation to y in $H_N^{(m)}$, so that

$$\|y^{(\infty)}(x; m) - u(x)\|_\infty \leq Mh^{2m+1}. \tag{34}$$

Thus, the iteration of Eq. (31) improves the order of the approximation to $u(x)$ by two in h. In other words, iteration in Eq. (31) determines a sequence of vectors $\{y^{(l)}(x; m)\}^\infty$ in $H_N^{(m)}$ which necessarily converges to the element of best approximation $\hat{w}(x; m)$ in $H_N^{(m)}$. It may also be true, as in Professor Lees's work in this volume, that just one or two iterates of Eq. (31) are necessary to produce this higher-order accuracy.

6. A Posteriori Error Bounds and Splines

First, we consider the case when $m \geq 2$. By definition, every element of $H_N^{(m)}$ is of class $C^m[0, 1]$, and thus for $m \geq 2$ these elements are twice differentiable. Let $L(w) \equiv -w''(x) + \sigma(x)\, w(x)$, so that $L(u) = f(x)$, $0 < x < 1$. For any twice-differentiable function $w(x)$ in H, integrating by parts gives us the identity

$$\|w\|_D{}^2 = \int_0^1 \{(w'(t))^2 + \sigma(t)\,(w(t))^2\}\, dt = \int_0^1 w(t)\, L[w]\, dt. \tag{35}$$

Thus, if $L(w) = f(x) - \tau(x)$, then

$$\begin{aligned} \|u - w\|_D{}^2 &= \int_0^1 (u(t) - w(t))\, \tau(t)\, dt \\ &\leq \left(\int_0^1 (u(t) - w(t))^2\, dt\right)^{1/2} \left(\int_0^1 (\tau(t))^2\, dt\right)^{1/2} \end{aligned} \tag{36}$$

by Schwarz's inequality. But for any differentiable $s(t)$ with $s(0) = s(1) = (0)$, it is known [6] that

$$\begin{aligned} \int_0^1 (s(t))^2\, dt &\leq \frac{1}{\pi^2} \int_0^1 (s'(t))^2\, dt \leq \frac{1}{\pi^2} \int_0^1 \{(s'(t))^2 + \sigma(x)\,(s(t))^2\, dt \\ &= \frac{1}{\pi^2} \|s\|_D{}^2. \end{aligned}$$

Hence, we deduce that

$$2\|u - w\|_\infty \leq \|u - w\|_D \leq \frac{1}{\pi}\left(\int_0^1 (\tau(t))^2\, dt\right)^{1/2} = \frac{1}{\pi} \|\tau\|_2. \tag{37}$$

The point of this inequality is that for $m \geq 2$ any element in $H_N^{(m)}$ is twice differentiable, and consequently the quantity $\tau(x)$ can be *explicitly* determined. This means that the L_2 norm of $\tau(x)$ can be explicitly calculated.

For the case when $m = 1$, the elements in $H_N^{(1)}$ are not in general twice differentiable, but nonetheless given specific ordinate values $w_0 = 0$, $w_1^{(0)}, w_2^{(0)},\ldots, w_N^{(0)}, w_{N+1}^{(0)} = 0$, it is possible to *assign* derivative values $w_1^{(1)},\ldots, w_N^{(1)}$ at the interior mesh points so that the corresponding function $w(x)$ in $H_N^{(1)}$ is actually twice differentiable. Specifically, regarding p_0 and p_{N+1} as free parameters, let $p_1,\ldots, p_N$ be N parameters which satisfy the system of linear equations

$$4p_i + p_{i-1} + p_{i+1} = \frac{3}{h}(w_{1+i}^{(0)} - w_{i-1}^{(0)}), \qquad 1 \leq i \leq N. \tag{38}$$

Since the matrix of coefficients in Eq. (38) is strictly diagonally dominant, the parameters $p_1,\ldots, p_N$ are uniquely determined. Then, it is known [7] that $w(x) = w(x; w_1,\ldots, w_N, p_0, p_1,\ldots, p_N, p_{N+1}) \in H_N^{(1)}$ is twice differentiable, and $w(x)$ is called the *spline interpolation* of the parameters $w_0^{(0)},\ldots, w_{N+1}^{(0)}$, $w_0^{(1)}, w_{N+1}^{(1)}$. This means again that, after modifying the derivatives at the interior mesh points, the error function $\tau(x)$ can again be determined, and Eq. (37) is again valid.

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