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Numerical Methods of High-Order Accuracy for Nonlinear Boundary Value Problems

I. One Dimensional Problem*

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ERRATA for Numerical methods of high-order accuracy for nonlinear boundary value problems. I. One dimensional problem.

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p. 403 Replace the hypothesis of Theorem 4, "If $\bigcup_{i=1}^{\infty} S_{M_i}$ is dense in S in the norm $\|\cdot\|_\gamma$" by

"If $\lim_{i \to \infty} \{ \inf_{w \in S_{M_i}} \|w - g\|_\gamma \} = 0$ for all $g \in S$"

p. 421 In Theorem 17, delete the hypothesis that $\Lambda > 0$, and replace Λ by $\Lambda + \gamma$ in eq. (8.11).

§ 1. Introduction

We shall consider here the numerical approximation of the solution of the following real nonlinear two-point boundary value problem

$$L[u(x)] = f(x, u(x)), \qquad 0 < x < 1, \tag{1.1}$$

with boundary conditions

$$D^k u(0) = D^k u(1) = 0, \qquad D = \frac{d}{dx}, \qquad 0 \leqq k \leqq n-1, \tag{1.2}$$

where the linear differential operator L is defined by

$$L[u(x)] = \sum_{j=0}^{n} (-1)^{j+1} D^j [p_j(x) D^j u(x)], \qquad n \geqq 1. \tag{1.3}$$

We remark that nonhomogeneous boundary conditions: $D^k u(0) = \alpha_k$, $D^k u(1) = \beta_k$, $0 \leqq k \leqq n-1$, can always be reduced to those of (1.2) by means of a suitable change of the dependent variable. The coefficient functions $p_j(x)$ are assumed to be of class $C^j[0, 1]$, $j = 0, 1, \ldots, n$, although we will see in § 8 that these conditions can be weakened.

Let S denote the linear space of all real functions $w(x)$ satisfying the boundary conditions of (1.2), such that $w(x) \in C^{n-1}[0, 1]$ with $D^{n-1} w(x)$ absolutely continuous in $[0, 1]$ and $D^n w \in L^2[0, 1]$. We assume that there exist two real constants β and K such that

$$\|w\|_{L^\infty} \equiv \sup_{x \in [0,1]} |w(x)| \leqq K \left\{ \int_0^1 \left[\sum_{j=0}^{n} p_j(x) (D^j w(x))^2 + \beta (w(x))^2 \right] dx \right\}^{\frac{1}{2}} \tag{1.4}$$

for all $w \in S$. The assumption (1.4) is the weakest one that we will need, and in particular, it will be implied by either of the following:

$$\|D^l w\|_{L^\infty} \leqq K \left\{ \int_0^1 \left[\sum_{j=0}^{n} p_j(x) (D^j w(x))^2 + \beta (w(x))^2 \right] dx \right\}^{\frac{1}{2}}, \tag{1.5}$$

$$\|D^{l+1} w\|_{L^2} \leqq K \left\{ \int_0^1 \left[\sum_{j=0}^{n} p_j(x) (D^j w(x))^2 + \beta (w(x))^2 \right] dx \right\}^{\frac{1}{2}} \tag{1.5'}$$

for some l, $0 \leqq l \leqq n-1$, where $\|w\|_{L^2} \equiv \left\{ \int_0^1 (w(x))^2 dx \right\}^{\frac{1}{2}}$. We remark that (1.5′) is a weakened form of GÅRDING's inequality [*46*, p. 175]. As such, (1.5′) is thus valid for *all* $-1 \leqq l \leqq n-1$ for any *strongly elliptic* operator L, i.e., with the $p_j(x) \in C^0[0, 1]$, $0 \leqq j \leqq n$, the additional assumption that $p_n(x) \geqq \omega > 0$ for all $x \in [0, 1]$ is *sufficient* to insure the validity of (1.5′) for all $-1 \leqq l \leqq n-1$ for $w \in S$. Finally, we remark that the inequality (1.4) implies that the quantity in the right-hand side is a norm on the space S.

Next, we introduce the finite quantity (see Lemma 1)

$$\Lambda \equiv \inf_{\substack{w \in S \\ w \neq 0}} \frac{\int_0^1 \left\{ \sum_{j=0}^{n} p_j(x) \, [D^j w(x)]^2 \right\} dx}{\int_0^1 [w(x)]^2 dx}. \tag{1.6}$$

Although this fact is not used, Λ is a lower bound for the eigenvalues of the associated eigenvalue problem $L[u(x)] + \lambda u(x) = 0$, $0 < x < 1$, subject to the boundary conditions of (1.2).

Finally, we assume that the functions $f(x, u)$ and $\frac{\partial f(x, u)}{\partial u}$ are real and continuous in both variables, i.e., $f(x, u), \frac{\partial f(x, u)}{\partial u} \in C^0([0, 1] \times R)$, and there exists a constant γ such that

$$\frac{\partial f(x, u)}{\partial u} \equiv f_u(x, u) \geqq \gamma > -\Lambda \quad \text{for all } x \in [0, 1], \text{ and all real } u. \tag{1.7}$$

In the linear case $f(x, u) \equiv p(x)\, u + q(x)$, these conditions imply that $p(x)$ and $q(x)$ are continuous, with $p(x) \geqq \gamma$ which generalizes for example the assumption of [*42*]. We point out that still weaker conditions can be made on $f(x, u)$, and these will be described in § 8.

The techniques used here are such that it is necessary to make the essential hypothesis that a classical solution of (1.1)—(1.2) exists. Later, in [*11*], using the more general technique of monotone operators [*8*], we shall treat weak (or generalized) solutions of (1.1)—(1.2) and regularity conditions which insure that a weak solution is indeed a classical solution.

One of our main goals is to study the effects of applying the classical Rayleigh-Ritz procedure (cf. [*28*, p. 85]) to the variational formulation of (1.1)—(1.2) by minimizing over subspaces of polynomial functions, and piecewise-polynomial functions such as Hermite and spline functions. In so doing, we generalize the results of [*42*], [*9*], and [*10*], and we obtain new error estimates which considerably improve upon known results in the literature, both for discrete finite difference methods applied to (1.1)—(1.2) (cf. [*7*], [*12*, p. 141], [*18*, p. 228], [*19*], [*21*, p. 347], [*22*, p. 191], [*24*], [*27*], [*31*], and [*43*]), as well as continuous methods applied to (1.1)—(1.2) (cf. [*2*, p. 28], [*6*], [*12*, p. 207], [*15*], [*22*, p. 262], [*28*, p. 126], [*33*], [*34*], and [*39*]).

Another of our goals is to show that these techniques can, from a numerical point of view, be efficiently applied on modern high-speed digital computers. To illustrate these theoretical results, numerical results for particular examples of (1.1)—(1.2), such as the bending of a thin beam, will be examined in detail in §9.

§ 2. Variational Formulation

In this section, we determine certain properties of a functional associated with the variational formulation of (1.1)—(1.2). We begin with

Lemma 1. With the assumption of (1.4), then

$$\Lambda \equiv \inf_{\substack{w \in S \\ w \not\equiv 0}} \frac{\int_0^1 \left\{ \sum_{j=0}^{n} p_j(x) [D^j w(x)]^2 \right\} dx}{\int_0^1 [w(x)]^2 dx} > -\infty. \tag{2.1}$$

Proof. Since $\|w\|_{L^\infty} \geqq \|w\|_{L^2}$, then from the definition of Λ and the assumption (1.4), it follows that

$$\Lambda \geqq \frac{1}{K^2} - \beta. \quad \text{Q.E.D.}$$

Corollary 1. If $\beta < 1/K^2$, then Λ is positive.

As previously mentioned, we make the essential hypothesis that (1.1)—(1.2) has a classical solution $\varphi(x)$.

Theorem 1. With the assumptions of (1.4) and (1.7), let $\varphi(x)$ be a classical solution of (1.1)—(1.2). Then $\varphi(x)$ *strictly* minimizes the following functional[1]

$$F[w] \equiv \int_0^1 \left\{ \frac{1}{2} \sum_{j=0}^{n} p_j(x) (D^j w(x))^2 + \int_0^{w(x)} f(x, \eta) \, d\eta \right\} dx \tag{2.2}$$

over the space S, and $\varphi(x)$ is thus the unique solution of (1.1)—(1.2).

Proof. If $\varphi(x)$ is a classical solution of (1.1)—(1.2), then surely $\varphi(x) \in S$. Since S is a linear space, $w(x) - \varphi(x) \equiv \varepsilon(x)$ is in S, for any $w \in S$. After integration by parts, it then follows that

$$\begin{aligned} F[w] = F[\varphi] &+ \tfrac{1}{2} \int_0^1 \left\{ \sum_{j=0}^{n} p_j(x) (D^j \varepsilon(x))^2 \right\} dx + \\ &+ \int_0^1 \int_{\varphi(x)}^{\varphi(x)+\varepsilon(x)} [f(x, \eta) - f[x, \varphi(x))] \, d\eta \, dx. \end{aligned} \tag{2.3}$$

Using the hypothesis (1.7), a simple calculation gives that

$$\int_{\varphi(x)}^{\varphi(x)+\varepsilon(x)} [f(x, \eta) - f(x, \varphi(x)] \, d\eta \geqq \frac{\gamma}{2} \varepsilon^2(x),$$

so that

$$F[w] \geqq F[\varphi] + \frac{1}{2} \int_0^1 \left\{ \sum_{j=0}^{n} p_j(x) (D^j \varepsilon(x))^2 \right\} dx + \frac{\gamma}{2} \int_0^1 \varepsilon^2(x) \, dx. \tag{2.4}$$

Thus, from the definition of Λ in Lemma 1, it follows that

$$F[w] \geqq F[\varphi] + \left(\frac{\Lambda + \gamma}{2} \right) \int_0^1 [w(x) - \varphi(x)]^2 dx. \tag{2.5}$$

Thus, $F[w] > F[\varphi]$ for any $w \in S$ with $w \not\equiv \varphi$. Q.E.D.

[1] Such functionals for nonlinear boundary-value problems have been considered by LEVINSON [25] and MIKHLIN [29].

§ 3. Approximation Scheme

Consider now *any* finite dimensional subspace S_M of S of dimension M, and let $\{w_i(x)\}_{i=1}^M$ be M linearly independent functions from the subspace. The analogue of Theorem 1, concerning the minimization of the functional $F[w]$ over the subspace S_M, is given in

Theorem 2. With the assumptions of (1.4) and (1.7), there exists a unique function $\widehat{w}_M(x)$ in the subspace S_M which minimizes the functional $F[w]$ over S_M.

Proof. Suppose first that $\varphi(x)$ is in S_M. Then from Theorem 1, $\widehat{w}_M \equiv \varphi \in S_M$ uniquely minimizes $F[w]$ over S_M. If $\varphi(x) \notin S_M$, consider the $(M+1)$-dimensional subspace of S spanned by $\{w_i(x)\}_{i=1}^M$ and $\varphi(x)$. Any function in this subspace is expressible as $\sum_{i=1}^M u_i w_i(x) + \alpha\varphi(x)$ for suitable coefficients, and can thus be represented by the $(M+1)$-vector $\boldsymbol{v} = (u_1, u_2, \ldots, u_M, \alpha)$. Since the $w_i(x)$ and $\varphi(x)$ are linearly independent, then

$$\eta(\boldsymbol{v}) = \eta(u_1, u_2, \ldots, u_M, \alpha) \equiv \left\{\int_0^1 \left[\sum_{i=1}^M u_i w_i(x) + \alpha\,\varphi(x)\right]^2 dx\right\}^{\frac{1}{2}} \tag{3.1}$$

is a norm on this subspace. Thus, for any function $w_M(x) = \sum_{i=1}^M u_i w_i(x)$ in S_M, we can rewrite the inequality of (2.5) as

$$F\left[\sum_{i=1}^M u_i w_i(x)\right] \geqq F[\varphi] + \frac{(\Lambda+\gamma)}{2}\,\eta^2(u_1, u_2, \ldots, u_M, -1). \tag{3.2}$$

But, as all norms on this $(M+1)$-dimensional subspace are equivalent, there necessarily exists a positive constant C (which depends on S_M) such that

$$\begin{aligned} C\,\eta(u_1, u_2, \ldots, u_M, \alpha) &\geqq \{|u_1|^2 + \cdots + |u_M|^2 + |\alpha|^2\}^{\frac{1}{2}} \\ &\geqq \{|u_1|^2 + \cdots + |u_M|^2\}^{\frac{1}{2}}, \end{aligned} \tag{3.3}$$

the second inequality being obvious. Hence, for any $w(x) \in S_M$,

$$F\left[\sum_{i=1}^M u_i w_i(x)\right] \geqq F[\varphi] + \frac{(\Lambda+\gamma)}{2C^2}\{|u_1|^2 + \cdots + |u_M|^2\}. \tag{3.4}$$

Thus, if we view $F\left[\sum_{i=1}^M u_i w_i(x)\right]$ as a functional on R^M and write $F\left[\sum_{i=1}^M u_i w_i(x)\right] \equiv G(\boldsymbol{u}) = G(u_1, u_2, \ldots, u_M)$, then the equivalence of all norms on R^M coupled with (3.4) gives us that

$$\lim_{\|\boldsymbol{u}\| \to \infty} G(\boldsymbol{u}) = +\infty \tag{3.5}$$

for any norm $\|\cdot\|$ on R^M. Hence, as $G(\boldsymbol{u})$ is clearly a continuous function on R^M which is bounded below (by $F[\varphi]$) and satisfies (3.5), a standard compactness argument shows that there exists at least one vector $\widehat{\boldsymbol{u}} \in R^M$ for which $G(\boldsymbol{u}) \geqq G(\widehat{\boldsymbol{u}})$ for all $\boldsymbol{u} \in R^M$, or equivalently

$$F\left[w_M(x) = \sum_{i=1}^M u_i w_i(x)\right] \geqq F\left[\widehat{w}_M(x) = \sum_{i=1}^M \hat{u}_i w_i(x)\right] \quad \text{for all} \quad w_M \in S_M. \tag{3.6}$$

To show that $\hat{u} \in R^M$ is actually unique, we first observe that $G(u)$ is a twice differentiable function over R^M, and its derivatives explicity are

$$\frac{\partial G(u)}{\partial u_i} = \int_0^1 \left\{ \sum_{j=0}^{n} p_j(x) \left(\sum_{k=1}^{M} u_k D^j w_k(x) \right) D^j w_i(x) \right\} dx + \tag{3.7}$$
$$+ \int_0^1 f\left(x, \sum_{k=1}^{M} u_k w_k(x)\right) w_i(x)\, dx, \quad 1 \leq i \leq M,$$

and

$$\frac{\partial^2 G(u)}{\partial u_i \partial u_k} = \int_0^1 \left\{ \sum_{j=0}^{n} p_j(x) D^j w_i(x) D^j w_k(x) \right\} dx + \tag{3.8}$$
$$+ \int_0^1 \frac{\partial f}{\partial u}\left(x, \sum_{r=1}^{M} u_r w_r(x)\right) w_i(x) w_k(x)\, dx, \quad 1 \leq i, j \leq M.$$

Now, we define an $M \times M$ real matrix $B(u) = (b_{i,k}(u))$, where $b_{i,k}(u) = \frac{\partial^2 G(u)}{\partial u_i \partial u_k}$. From (3.8), it is clear that $B(u)$ is symmetric for all $u \in R^M$, and we now show that $B(u)$ is *uniformly* positive definite, i.e., for any (column) vectors u and y in R^M, there exists a positive constant c such that $y^T B(u) y \geq c y^T y$ for all $y \in R^M$ and all $u \in R^M$. By definition, it follows that

$$y^T B(u) y = \sum_{i,k=1}^{M} y_i b_{i,k}(u) y_k \tag{3.9}$$
$$= \int_0^1 \left\{ \sum_{j=0}^{n} p_j(x) \left(D^j Y(x)\right)^2 + f_u\left(x, \sum_{r=1}^{M} u_r w_r(x)\right) \left(Y(x)\right)^2 \right\} dx,$$

where $Y(x) \equiv \sum_{i=1}^{M} y_i w_i(x)$. Applying the definition of (1.6) and the inequality of (1.7) to the equation above yields

$$y^T B(u) y \geq (\Lambda + \gamma) \int_0^1 (Y(x))^2 dx. \tag{3.10}$$

But, since $\int_0^1 \left\{ \left(\sum_{i=1}^{M} y_i w_i(x) \right)^2 dx \right\}^{\frac{1}{2}} = \left\{ \int_0^1 \left(Y(x) \right)^2 dx \right\}^{\frac{1}{2}}$ is evidently a norm for the (column) vector $y = (y_1, y_2, \ldots, y_M)^T$ in R^M, the equivalence of all norms on R^M shows that exists a positive constant C' such that

$$\int_0^1 \left\{ \sum_{i=1}^{M} y_i w_i(x) \right\}^2 dx \geq C' y^T y \quad \text{for all} \quad y \in R^M. \tag{3.11}$$

Hence, combining (3.10) and (3.11), we have

$$y^T B(u) y \geq C'(\Lambda + \gamma) y^T y \quad \text{for all} \quad y \in R^M, \quad \text{all} \quad u \in R^M, \tag{3.12}$$

which establishes the uniform positive definite character of the $M \times M$ matrix $B(u)$.

Since $G(u)$ is a twice differentiable function of $u \in R^M$, we can write its Taylor series expansion as

$$G(u) = G(\hat{u}) + (u - \hat{u})^T \{grad\, G(\hat{u})\} + (u - \hat{u})^T B(w) (u - \hat{u}), \tag{3.13}$$

where $\boldsymbol{w}=\vartheta \boldsymbol{u}+(1-\vartheta)\hat{\boldsymbol{u}}$ for some ϑ with $0<\vartheta<1$. Then, the uniqueness of $\hat{\boldsymbol{u}}\in R^M$ is apparent, for if $G(\boldsymbol{u})\geqq G(\hat{\boldsymbol{u}})$ for all $\boldsymbol{u}\in R^M$, then evidently $\mathit{grad}\, G(\hat{\boldsymbol{u}})=\mathbf{0}$, and the positive definite character of the matrix B further gives us that

$$G(\boldsymbol{u})>G(\hat{\boldsymbol{u}}) \quad \text{for any} \quad \boldsymbol{u}\neq\hat{\boldsymbol{u}} \text{ in } R^M. \tag{3.14}$$

Thus, $\hat{\boldsymbol{u}}$ is the unique vector in R^M which minimizes $G(\boldsymbol{u})$. Q.E.D.

To find this unique element $\hat{w}_M(x)=\sum_{i=1}^{M}\hat{u}_i w_i(x)$ in S_M which minimizes $F[w]$ over S_M, we must solve the M nonlinear equations

$$\int_0^1\left\{\sum_{j=0}^{n} p_j(x)\left(\sum_{k=1}^{M} u_k D^j w_k(x)\right) D^j w_i(x) + f\left(x, w_M(x)\right) w_i(x)\right\} dx = 0, \quad 1\leqq i\leqq M, \tag{3.15}$$

for the M unknowns $u_1, u_2, \ldots, u_M$ which arise from $\mathit{grad}\, G(\boldsymbol{u})=\mathbf{0}$, or equivalently

$$\frac{\partial F\left[\sum_{i=1}^{M} u_i w_i(x)\right]}{\partial u_i}=0, \qquad 1\leqq i\leqq M.$$

It is convenient to write these Eqs. (3.15) in matrix form as

$$A\boldsymbol{u}+\boldsymbol{g}(\boldsymbol{u})=\mathbf{0}, \tag{3.16}$$

where $A=(a_{i,k})$ is an $M\times M$ real matrix, and $\boldsymbol{g}(\boldsymbol{u})=(g_1(\boldsymbol{u}),\ldots,g_M(\boldsymbol{u}))^T$ is a column vector, both being determined by

$$a_{i,k}=\int_0^1\left\{\sum_{j=0}^{n} p_j(x)\, D^j w_i(x)\, D^j w_k(x)\right\} dx, \qquad 1\leqq i,\ k\leqq M, \tag{3.17}$$

and

$$g_k(\boldsymbol{u})=\int_0^1 f\left(x, \sum_{i=1}^{M} u_i w_i(x)\right) w_k(x)\, dx, \qquad 1\leqq k\leqq M. \tag{3.18}$$

While we know that (3.16) have a unique solution vector in R^M, it is of practical importance to know that several iterative methods for the numerical solution of (3.16) have been shown to be convergent [*30, 35*]. To illustrate this, consider the Gauss-Seidel iterative method applied to (3.16):

$$\sum_{j\leqq i} a_{i,j} u_j^{(r+1)} + \sum_{j>i} a_{i,j} u_j^{(r)} + g_i\left(u_1^{(r+1)},\ldots,u_i^{(r+1)}, u_{i+1}^{(r)},\ldots,u_M^{(r)}\right)=0, \quad 1\leqq i\leqq M. \tag{3.19}$$

For each fixed i, $1\leqq i\leqq M$, this equation, a nonlinear equation in the single unknown $u_i^{(r+1)}$, has a *unique* solution, and the cyclic determination of the $u_i^{(r+1)}$ is convergent [*35*]. The practical implications of this will be considered in greater detail in § 9. It is, however, clear that the speed of convergence of such iterative methods depends in part on the sparseness of the matrix A.

§ 4. Convergence

We have seen that, given any finite-dimensional subspace S_M of S, we can find a unique element $\hat{w}_M(x)$ in S_M which is the best approximation to the solution $\varphi(x)$ of (1.1)—(1.2) in the sense of minimizing $F[w]$ over S_M. It is quite natural to expect that the difference $(\hat{w}_{M_i}(x)-\varphi(x))$ might converge to zero, i.e., in some topology, if we have a sequence of subspaces $\{S_{M_i}\}_{i=1}^{\infty}$ satisfying the necessary condition $\lim_{i\to\infty} \dim S_{M_i}=+\infty$, as well as appropriate asymptotic properties.

If $h(x)$ is a continuous function on $[0, 1]$, define

$$(4.1) \qquad \|w\|_h=\left\{\int_0^1\left[\sum_{j=0}^{n} p_j(x)\,(D^j w(x))^2+h(x)\,(w(x))^2\right]dx\right\}^{\frac{1}{2}} \quad \text{for all } w\in S.$$

If $h(x)\equiv\alpha$ on $[0, 1]$, we write $\|w\|_\alpha$ for $\|w\|_h$. Then, recalling the constant γ of (1.7), we have

Lemma 2. If $h(x)\geqq\gamma'>-\Lambda$ is a continuous function on $[0, 1]$, then $\|w\|_h$ and $\|w\|_{\gamma'}$ are both norms on S, and moreover they are equivalent.

Proof. With the notation of (4.1), then $h(x)\geqq\gamma'>-\Lambda$ coupled with Lemma 1 gives that $\|w\|_h^2\geqq(\Lambda+\gamma')\,\|w\|_{L^2}^2$ for all $w(x)\in S$. Similarly, $\|w\|_{\gamma'}^2\geqq(\Lambda+\gamma)\,\|w\|_{L^2}^2$ for all $w(x)\in S$, so that both $\|w\|_h$ and $\|w\|_{\gamma'}$ are norms on S. To prove the equivalence of these norms, we merely state that the following inequalities

$$(4.2) \qquad c_1\|w\|_h^2\leqq\|w\|_{\gamma'}^2\leqq c_2\|w\|_h^2 \quad \text{for all } w\in S,$$

where

$$(4.2') \qquad c_1=\left(1+\frac{\max(\Gamma-\gamma;\,0)}{\Lambda+\gamma}\right)^{-1}; \qquad c_2=\left(1+\frac{\max(\gamma-\gamma';\,0)}{\Lambda+\gamma'}\right); \qquad \Gamma\equiv\max_{x\in[0,1]} h(x),$$

follow routinely from the hypothesis $h(x)\geqq\gamma'>-\Lambda$ and the definition of Λ in (2.1). Q.E.D.

Corollary 2. If assumption (1.4) is satisfied for some real number β, then it is also satisfied for every γ' with $\gamma'>-\Lambda$.

Proof. Clearly, the number $\beta'=\max\{\beta,-\Lambda+1\}$ satisfies (1.4) since $\beta'\geqq\beta$ Now, take any $\gamma'>-\Lambda$. Since $\beta'\geqq-\Lambda+1>-\Lambda$, we can apply Lemma 2 to $\|w\|_{\beta'}$ and $\|w\|_{\gamma'}$. Thus, there necessarily exists a constant $c>0$ such that $\|w\|_{\beta'}\leqq c\|w\|_{\gamma'}$. Thus, $\|w\|_{L^\infty}\leqq K\|w\|_{\beta'}\leqq Kc\|w\|_{\gamma'}$ for all $w\in S$. Q.E.D.

As a consequence of Corollary 2, we can write for the particular constant γ of (1.7) that

$$(4.3) \qquad \|w\|_{L^\infty}\leqq K\|w\|_\gamma \quad \text{for all } w\in S,$$

where $\|w\|_\gamma$ is defined in (4.1). We shall regard K and γ in what is to follow as *fixed* constants satisfying (1.7).

One of the important consequences of Lemma 2 is that, for any continuous function $h(x)$ on $[0, 1]$ with $h(x)\geqq\gamma'>-\Lambda$, the norm $\|w\|_h$ can be induced by the following *inner product* on S:

$$(4.4) \quad \langle w, v\rangle_h\equiv\int_0^1\left[\sum_{j=0}^{n} p_j(x)\,D^j w(x)\,D^j v(x)+h(x)\,w(x)\,v(x)\right]dx, \qquad w, v\in S.$$

This will be useful in establishing Theorem 3.

Lemma 3. With the assumption of (1.7), let $\mathcal{M} \equiv \max_{x\in[0,1]} |f(x, 0)|$. Then, for any real λ, any $\varepsilon > 0$, and any $0 \leqq x \leqq 1$,

$$\int_0^\lambda f(x, \eta)\, d\eta \geqq (\gamma - \varepsilon) \frac{\lambda^2}{2} - \frac{\mathcal{M}^2}{2\varepsilon}. \tag{4.5}$$

Proof. If $\lambda \geqq 0$, then as $0 \leqq \eta \leqq \lambda$, we have that

$$f(x, \eta) = f(x, 0) + f_u(x, \vartheta\eta) \cdot \eta \geqq f(x, 0) + \gamma\eta.$$

Hence,

$$\int_0^\lambda f(x, \eta)\, d\eta \geqq f(x, 0) \cdot \lambda + \frac{\lambda^2 \gamma}{2} \geqq -\mathcal{M}\lambda + \frac{\lambda^2\gamma}{2}.$$

Similarly for $\lambda \leqq 0$, we obtain

$$\int_0^\lambda f(x, \eta)\, d\eta \geqq \mathcal{M}\lambda + \frac{\lambda^2\gamma}{2}.$$

Finally, for any $\varepsilon > 0$,

$$\frac{\lambda^2\gamma}{2} \pm \mathcal{M}\lambda \geqq (\gamma - \varepsilon)\frac{\lambda^2}{2} - \frac{1}{2\varepsilon}\mathcal{M}^2,$$

since

$$\frac{1}{2}\left\{\sqrt{\varepsilon}\,\lambda \pm \frac{\mathcal{M}}{\sqrt{\varepsilon}}\right\}^2 \geqq 0. \qquad \text{Q.E.D.}$$

Finally, we have the following *a priori* bounds for both the solution $\varphi(x)$ and the best approximation $\hat{w}_M(x)$ in S_M.

Lemma 4. Let $w(x)$ be any function in S with $F[w] \leqq 0$. Then, the following *a priori* bound is valid:

$$\|w\|_\gamma \leqq \frac{2\mathcal{M}}{\sqrt{\Lambda + \gamma}}, \qquad \mathcal{M} \equiv \max_{x\in[0,1]} |f(x, 0)|, \tag{4.6}$$

and thus,

$$\|w\|_{L^\infty} \leqq \frac{2K\mathcal{M}}{\sqrt{\Lambda+\gamma}}. \tag{4.7}$$

Proof. From the definition of $F[w]$ in (2.2), the hypothesis $F[w] \leqq 0$ combined with the result of Lemma 3 gives us that

$$\int_0^1 \left\{\frac{1}{2}\sum_{j=0}^n p_j(x)\,(D^j w(x))^2 + (\gamma - \varepsilon)\frac{(w(x))^2}{2} - \frac{\mathcal{M}^2}{2\varepsilon}\right\} dx \leqq F[w] \leqq 0$$

for any $\varepsilon > 0$. Thus, if $\gamma - \varepsilon > -\Lambda$ in addition, then the above inequality can be written as $\|w\|^2_{\gamma-\varepsilon} \leqq \mathcal{M}^2/\varepsilon$. To obtain an analogous inequality for $\|w\|^2_\gamma$, we use the second inequality of (4.2):

$$\|w\|^2_\gamma \leqq \left(1 + \frac{\varepsilon}{\Lambda + \gamma - \varepsilon}\right)\|w\|^2_{\gamma-\varepsilon} \leqq \left(\frac{1}{\varepsilon} + \frac{1}{\Lambda+\gamma-\varepsilon}\right)\mathcal{M}^2,$$

$0 < \varepsilon < \Lambda + \gamma$. Then, minimizing the right-hand side of the above inequality as a function of ε directly gives (4.6). Then, (4.7) follows from (4.3). Q.E.D.

Since $w(x)\equiv 0$ is an element of both spaces S and S_M, it follows from the definitions of $\varphi(x)$ and $\hat{w}_M(x)$ that $F[\varphi]\leqq F[\hat{w}_M]\leqq F[0]=0$. Thus, both $\varphi(x)$ and $\hat{w}_M(x)$ satisfy the a priori bounds of (4.6) and (4.7).

We come now to a *key fact*. In [42], it was shown that the element of best approximation $\hat{w}_M(x)$ in S_M could be viewed as a *projection* of the solution $\varphi(x)$ on S_M with respect to a *fixed* inner product associated with a norm of the form (4.1). Basically, this was possible because the problem considered in [42] was linear. However, we can still view [9] $\hat{w}_M(x)$ as a projection of $\varphi(x)$ on S_M with respect to an inner product which now *varies* with the subspace S_M. To show this, let $\hat{w}_M(x)$ be the element of best approximation to $\varphi(x)$ on S_M. Then, $\hat{w}_M(x)=\sum_{i=1}^{M}\hat{u}_i w_i(x)$ satisfies (3.15), i.e.,

$$\text{(4.8)}\qquad \int_0^1\left\{\sum_{j=0}^{n}p_j(x)\,D^j\hat{w}_M(x)\,D^j w_i(x)+f\big(x,\hat{w}_M(x)\big)\,w_i(x)\right\}dx=0,\qquad 1\leqq i\leqq M.$$

Similarly, it is readily seen, after integration by parts, that we analogously have

$$\text{(4.9)}\qquad \int_0^1\left\{\sum_{j=1}^{n}p_j(x)\,D^j\varphi(x)\,D^j w_i(x)+f\big(x,\varphi(x)\big)\,w_i(x)\right\}=0,\qquad 1\leqq i\leqq M.$$

By simply subtracting (4.9) from (4.8), we obtain

$$\text{(4.10)}\qquad \int_0^1\left\{\sum_{j=0}^{n}p_j(x)\,D^j\big(\hat{w}_M(x)-\varphi(x)\big)\,D^j w_i(x)+g_M(x)\,\big(\hat{w}_M(x)-\varphi(x)\big)\,w_i(x)\right\}dx=0,\qquad 1\leqq i\leqq M,$$

where $g_M(x)\equiv f_u\big(x,\vartheta(x)\,\varphi(x)+(1-\vartheta(x))\,\hat{w}_M(x)\big)$ with $0<\vartheta(x)<1$. By hypothesis, we know that $g_M(x)$ is a continuous function on $[0,1]$, and moreover, from (1.7) and Lemma 4, we have the bounds

$$\text{(4.11)}\qquad -\Lambda<\gamma\leqq g_M(x)\leqq\Gamma_0,\quad \text{for all}\quad x\in[0,1],$$

where

$$\text{(4.12)}\qquad \Gamma_0\equiv\max\frac{\partial f}{\partial u}(x,u),\qquad 0\leqq x\leqq 1,\qquad |u|\leqq\frac{2K\mathcal{M}}{\sqrt{\Lambda+\gamma}}.$$

Note that these bounds for $g_M(x)$ are valid for *any* subspace S_M of S, a fact which we shall use to advantage later. Thus, the equations of (4.10) rewritten in terms of the inner product of (4.4) are just

$$\text{(4.13)}\qquad \langle\hat{w}_M-\varphi,w_i\rangle_{g_M}=0,\qquad 1\leqq i\leqq M.$$

Consequently, $\hat{w}_M$ is the projection of φ in S_M with respect to this inner product, and thus, using a well known property of inner product spaces concerning the projection of an element on a closed subspace, we have

$$\text{(4.14)}\qquad \|\hat{w}_M-\varphi\|_{g_M}=\inf_{w\in S_M}\|w-\varphi\|_{g_M}.$$

Now, from (4.2), (4.3), (4.11), and (4.14), we obtain the following crucial chain of inequalities:

$$\text{(4.15)}\qquad \begin{aligned}\|\hat{w}_M-\varphi\|_{L^\infty}&\leqq K\|\hat{w}_M-\varphi\|_\gamma\leqq K\|\hat{w}_M-\varphi\|_{g_M}=K\inf_{w\in S_M}\|w-\varphi\|_{g_M}\\&\leqq K\left(1+\frac{\max(\Gamma_0-\gamma;\,0)}{\Lambda+\gamma}\right)^{\frac12}\inf_{w\in S_M}\|w-\varphi\|_\gamma,\end{aligned}$$

which proves the fundamental result of

Theorem 3. Let $\varphi(x)$ be the solution of (1.1)—(1.2), subject to the conditions of (1.4) and (1.7), let S_M be any finite dimensional subspace of S and let $\hat{w}_M(x)$ be the unique function which minimizes $F[w]$ over S_M. Then, with the constant $C \equiv K\left(1+\frac{\max(\Gamma_0-\gamma;\,0)}{\Lambda+\gamma}\right)^{\frac{1}{2}}$ which can be explicitly determined a priori, the following error bound is valid:

$$\|\hat{w}_M-\varphi\|_{L^\infty} \leqq K\|\hat{w}_M-\varphi\|_\gamma \leqq C \inf_{w\in S_M}\|w-\varphi\|_\gamma. \tag{4.16}$$

Moreover, if in addition either (1.5) or (1.5′) is valid, then

$$\|D^k(\hat{w}_M-\varphi)\|_{L^\infty} \leqq \frac{C}{2^{l-k}} \inf_{w\in S_M}\|w-\varphi\|_\gamma \quad \text{for all } 0\leqq k\leqq l. \tag{4.17}$$

Next, as an immediate consequence of Theorem 3, we have

Theorem 4. Let $\varphi(x)$ be the solution of (1.1)—(1.2), subject to the conditions of (1.4) and (1.7), let $\{S_{M_i}\}_{i=1}^\infty$ be any sequence of finite dimensional subspaces of S, and let $\{\hat{w}_{M_i}(x)\}_{i=1}^\infty$ be the sequence of functions obtained by minimizing $F[w]$ respectively, over the subspaces S_{M_i}. If $\bigcup_{i=1}^\infty S_{M_i}$ is dense in S in the norm $\|\cdot\|_\gamma$, then $\{\hat{w}_{M_i}(x)\}_{i=1}^\infty$ converges *uniformly* to $\varphi(x)$. Moreover, if in addition either (1.5) or (1.5′) is valid, then $\{\hat{w}_{M_i}^{(k)}(x)\}_{i=1}^\infty$ converges uniformly to $\varphi^{(k)}(x)$ for all $0\leqq k\leqq l$.

§ 5. Polynomial Subspaces

To give an example of subspaces S_M satisfying the sufficient conditions of Theorem 4, let N be any integer with $N\geqq 2n$, where n is the number of boundary conditions in (1.2) imposed at each boundary point. Then, the polynomial space $P^{(N)}$ is the collection of all real polynomials of degree N. We observe that the total number of parameters associated with any element of $P^{(N)}$ is $N+1$.

Those elements of $P^{(N)}$ which satisfy the boundary conditions of (1.2) form an $N+1-2n$ dimensional subspace of S, denoted by $P_0^{(N)}$. It is clear that the elements of $P_0^{(N)}$ are polynomials of the form

$$x^n(1-x)^n\{a_0+a_1x+\cdots,+a_{N-2n}x^{N-2n}\}.$$

We now discuss the error of best approximation by elements of $P_0^{(N)}$.

Theorem 5. If $u(x)\in C^t[0,1]$, $t\geqq n$, and $u(x)$ satisfies the boundary conditions of (1.2), then there exists a sequence of polynomials $\{\tilde{p}_N(x)\}_{N=r}^\infty$ with $r\equiv\max(t, 2n-1)$ such that $\tilde{p}_N(x)\in P_0^{(N)}$ and

$$\|D^k(u-\tilde{p}_N)\|_{L^\infty} \leqq \frac{K}{(N-n)^{t-n}}\,\omega\left(D^t u;\frac{1}{N-n}\right) \tag{5.1}$$

for all $0\leqq k\leqq n$, where K is a constant which depends only on t and n, and ω is the modulus of continuity.

Proof. Fixing $N\geqq r=\max(t, 2n-1)$, the proof is accomplished by recursively generating a finite sequence of polynomials $\{q_{j,N}(x)\}_{j=0}^n$ for each N, such that $q_{j,N}(x)$ is a polynomial of degree $N-n+j$ which approximates $D^{n-j}u(x)$, and satisfies the boundary conditions

$$D^k q_{j,N}(0)=D^k q_{j,N}(1)=0, \qquad 0\leqq k\leqq j-1, \quad 1\leqq j\leqq n.$$

We then will choose $\tilde{p}_N(x)=q_{n,N}(x)$. We now construct the required sequence.

By a classical theorem [26, Theorem 2, p. 66] of approximation theory, there exists a polynomial $q_{0,N}(x)$ of degree $N-n$ and a constant K depending only on t and n such that

$$\|D^n u - q_{0,N}\|_{L^\infty} \leqq \frac{K}{(N-n)^{t-n}} \, \omega\left(D^t u; \frac{1}{N-n}\right), \quad N \geqq r, \tag{5.2}$$

where the modulus of continuity $\omega(f; \delta)$ is defined as usual by

$$\omega(f; \delta) = \max_{\substack{x, y \in [0,1] \\ |x-y| \leqq \delta}} |f(x) - f(y)|.$$

Define recursively

$$q_{j,N}(x) = \int_0^x q_{j-1,N}(x)\,dx - t_j(x) \int_0^1 q_{j-1,N}(x)\,dx, \quad 1 \leqq j \leqq n, \tag{5.3}$$

where $t_j(x)$ is a polynomial of degree $2j-1$. As such, $q_{j,N}(x)$ is then a polynomial of $N-n+j$. In order that $q_{j,N}(x)$ satisfy the boundary conditions $D^l q_{j,N}(0) = D^l q_{j,N}(1) = 0$ for $0 \leqq l \leqq j-1$, we take $t_j(x)$ to be the (unique) interpolation polynomial of degree $2j-1$ such that

$$D^l t_j(0) = 0, \quad 0 \leqq l \leqq j-1; \qquad D^l t_j(1) = \delta_{0,l}, \quad 0 \leqq l \leqq j-1.$$

Thus, for example, $t_1(x) = x$. By virtue of our definitions, we now establish

$$\|D^k(D^{n-j} u - q_{j,N})\|_{L^\infty} \leqq \frac{K'}{(N-n)^{t-n}} \, \omega\left(D^t u; \frac{1}{N-n}\right), \quad N \geqq r, \tag{5.4}$$

for $0 \leqq k \leqq j$ and $0 \leqq j \leqq n$, where K' is a constant which depends only on n and t.

We first prove (5.4) for the case $k=0$, using induction. For $j=0$, (5.4) is nothing more than the inequality of (5.2). Assuming that (5.4) is valid for $k=0$, and $j-1$ we write

$$D^{n-j} u(x) - q_{j,N}(x) = \int_0^x (D^{n-j+1} u(x) - q_{j-1,N}(x))\,dx + \\ + t_j(x) \int_0^1 q_{j-1,N}(x)\,dx.$$

By subtracting the term $t_j(x) \int_0^1 D^{n-j+1} u(x)\,dx$ which is necessarily zero because of the boundary conditions, we obtain

$$|D^{n-j} u(x) - q_{j,N}(x)| \leqq (1 + |t_j(x)|) \int_0^1 |D^{n-j+1} u(x) - q_{j-1,N}(x)|\,dx,$$

from which the case $k=0$ of (5.4) follows. Because we can express $D^k q_{j,N}(x)$ from (5.3) as

$$D^k q_{j,N}(x) = q_{j-k,N}(x) - \sum_{l=0}^{k-1} D^{k-l} t_{j-l}(x) \int_0^1 q_{j-l-1,N}(x)\,dx,$$

the same method of proof can be used to establish the general case of (5.4). Then, setting $q_{n,N}(x) \equiv \tilde{p}_N(x)$ gives the desired result of (5.1). Q.E.D.

Since $\omega\left(D^t u; \frac{1}{N-n}\right)$ by definition is bounded above by $2\|D^t u\|_{L^\infty}$, we have the

Corollary 3. $\|D^k(u - \tilde{p}_N)\|_{L^\infty} \leqq \frac{2K}{(N-n)^{t-n}} \|D^t u\|_{L^\infty}$ for all $0 \leqq k \leqq n$, where $N \geqq \max(t, 2n-1)$.

We remark that the result of Theorem 5 gives no estimates such as those of (5.1) for polynomials $p_N(x)$ of low degree, i.e., for $N<\max(t, 2n-1)$.

As an application of the result of Theorem 5, we have

Theorem 6. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4) and (1.7), be of class $C^t[0, 1]$ with $t\geqq 2n$, and let $\hat{p}_N(x)$ be the unique function which minimizes $F[w]$ over $P_0^{(N)}$, where $N\geqq t$. Then, there exists a constant M depending on t, n, and γ such that

$$\|\hat{p}_N-\varphi\|_{L^\infty}\leqq K\|\hat{p}_N-\varphi\|_\gamma\leqq\frac{M}{(N-n)^{t-n}}\|D^t\varphi\|_{L^\infty} \tag{5.5}$$

for all $N\geqq t$. Moreover, if in addition, either (1.5) or (1.5′) is valid, then

$$\|D^k(\hat{p}_N-\varphi)\|_{L^\infty}\leqq\frac{M}{2^{l-k}(N-n)^{t-n}}\|D^t\varphi\|_{L^\infty}\quad\text{for all}\quad 0\leqq k\leqq l. \tag{5.6}$$

Proof. With Corollary 3, it follows from the definition of the norm $\|\cdot\|_\gamma$ in (4.1) that there exists a constant M' such that

$$\|\tilde{p}_N-\varphi\|_\gamma\leqq\frac{M'}{(N-n)^{t-n}}\|D^t\varphi\|_{L^\infty}.$$

The rest then follows from (4.16) and (4.17) of Theorem 3. Q.E.D.

The importance of the results of (5.5) and (5.6) can be illustrated with the following examples. If the solution $\varphi(x)$ of the linear problem: $D^2u(x)=f(x)$, $0<x<1$, with $u(0)=u(1)=0$, is *only* of class $C^2[0, 1]$, the results of (5.5) with $t=2$ and $n=1$ give us

$$\|\hat{p}_N-\varphi\|_{L^\infty}\leqq\frac{M}{(N-1)}\|D^2\varphi\|_{L^\infty},\quad N\geqq 2,$$

i.e., the sequence of polynomials $\{\hat{p}_N(x)\}_{N=2}^\infty$ converges at least *linearly* (in $h=1/N$) and uniformly to $\varphi(x)$, as $N\to\infty$. Similarly, if the solution $\varphi(x)$ of the linear thin beam problem:

$$D^4u(x)=f(x),\quad 0<x<1,\quad D^k u(0)=D^k u(1)=0,\quad 0\leqq k\leqq 1,$$

is only of class $C^4[0, 1]$, the result of (5.5) with $t=4$ and $n=2$ gives us

$$\|\hat{p}_N-\varphi\|_{L^\infty}\leqq\frac{M}{(N-2)^2}\|D^4\varphi\|_{L^\infty},\quad N\geqq 4,$$

i.e., the sequence $\{\hat{p}_N(x)\}_{N=4}^\infty$ converges at least *quadratically* (in $h=1/N$) and uniformly to $\varphi(x)$, as $N\to\infty$. Moreover, as this last boundary value problem is derived from a strongly elliptic operator, then (1.5′) is valid for $l=1$, in which case from (5.6) we have

$$\|\hat{p}_N^{(1)}-\varphi^{(1)}\|_{L^\infty}\leqq\frac{M}{(N-2)^2}\|D^4\varphi\|_{L^\infty},\quad N\geqq 4.$$

Thus, even the sequence of derivatives $\{D\hat{p}_N(x)\}_{N=4}^\infty$ converges quadratically and uniformly to $D\varphi(x)$ as $N\to\infty$. Later, we shall derive similar results for other particular choices of subspaces of S. It is worth mentioning that such results, to the best of our knowledge, are not obtainable from conventional Taylor series and Gerschgorin-type convergence arguments for discrete methods (cf. [*16*, p. 283], [*41*, p. 165]).

If $u(x)$ is actually analytic in some open set of the complex plane containing the interval $[0, 1]$, we can of course apply the result of Theorem 5 for any $t \geqq n$. But, an even stronger result, indicating exponential convergence, is possible.

Theorem 7. Assume that $u(x)$ is analytic in some open set containing the interval $[0, 1]$. The, there exists a constant μ with $0 \leqq \mu < 1$, and a sequence of polynomials $\{\tilde{p}_N(x)\}_{N=2n-1}^{\infty}$ with $\tilde{p}_N \in P_0^{(N)}$ such that

$$\overline{\lim_{N\to\infty}} (\|D^k(u-\tilde{p}_N)\|_{L^\infty})^{1/N} \leqq \mu \quad \text{for all} \quad 0 \leqq k \leqq n. \tag{5.7}$$

Proof. By a classical result of Bernstein (cf. [26, p. 76]), there exists a constant μ with $0 \leqq \mu < 1$ and a sequence of polynomials $\{q_{0,N}(x)\}_{N=n}^{\infty}$ with

$$\overline{\lim_{N\to\infty}} (\|D^n u - q_{0,N}\|_{L^\infty})^{1/N-n} = \mu, \tag{5.8}$$

where $q_{0,N}(x)$ is a polynomial of degree $N-n$. By repeating the construction of (5.3), we form a new sequence of polynomials $\{p_N(x)\}_{N=2n-1}^{\infty}$ with $p_N(x) \in P_0^{(N)}$, such that

$$\|D^k(u-p_N)\|_{L^\infty} \leqq K(n) \|D^n u - q_{0,N}\|_{L^\infty} \quad \text{for all} \quad 0 \leqq k \leqq n,$$

where $K(n)$ is a constant. Taking N-th roots in this expression and using (5.8) then gives the desired result. Q.E.D.

Applying this result now to the solution of the differential Eq. (1.1)—(1.2), we have

Theorem 8. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the condition of (1.4) and (1.7), be analytic in some open set of the complex plane containing the interval $[0, 1]$, and let $\hat{p}_N(x)$ be the unique function which minimizes $F[w]$ over $P_0^{(N)}$, where $N \geqq 2n-1$. Then, there exists a constant μ with $0 \leqq \mu < 1$ such that

$$\overline{\lim_{N\to\infty}} (\|\hat{p}_N - \varphi\|_{\nu})^{1/N} = \mu, \tag{5.9}$$

and consequently from (4.3)

$$\overline{\lim_{N\to\infty}} (\|\hat{p}_N - \varphi\|_{L^\infty})^{1/N} \leqq \mu. \tag{5.9'}$$

Moreover, if in addition either (1.5) or (1.5') is valid, then

$$\overline{\lim_{N\to\infty}} (\|D^k(\hat{p}_N - \varphi)\|_{L^\infty})^{1/N} \leqq \mu \quad \text{for all} \quad 0 \leqq k \leqq l. \tag{5.10}$$

For any basis $\{w_i(x)\}_{i=1}^{N-2n+1}$ of $P_0^{(N)}$, the element of best approximation $\hat{p}_N(x) = \sum_{i=1}^{N-2n+1} u_i w_i(x)$ in $P_0^{(N)}$ is determined (cf. (4.8)) from the solution of the nonlinear matrix equation

$$A\boldsymbol{u} + \boldsymbol{g}(\boldsymbol{u}) = \boldsymbol{0}, \tag{5.11}$$

where the $(N-2n+1) \times (N-2n+1)$ matrix $A = (a_{i,j})$ and the column vector $g(\boldsymbol{u})$ are given by

$$a_{i,j} = \int_0^1 \left\{ \sum_{k=0} p_k(x) D^k w_i(x) D^k w_j(x) \right\} dx = \langle w_i, w_j \rangle_0, \tag{5.12}$$

and

$$g_i(\boldsymbol{u}) = \int_0^1 f\left(x, \sum_{i=1}^{N-2n+1} u_i w_i(x)\right) w_i(x)\, dx. \tag{5.13}$$

It is, of course, always possible to select an *orthonormal* basis for $P_0^{(N)}$, i.e., one for which $a_{i,j} = \delta_{i,j}$, and such a choice makes the numerical solution of (5.11) comparatively easy. But what is important is that in certain special, but nevertheless interesting cases, it is easy to deduce *explicitly* such an orthonormal basis. For example, for $n=1$, $p_1(x) \equiv 1$, $p_0(x) \equiv 0$, the polynomials

$$w_k(x) = \sqrt{\frac{2k+1}{2}} \int_0^x L_k(x)\, dx, \qquad k = 1, 2, \ldots, \tag{5.14}$$

satisfy

$$w_k(0) = w_k(1) = 0; \qquad \int_0^1 D w_k(x)\, D w_m(x)\, dx = \delta_{k,m}, \tag{5.15}$$

where $L_k(x)$ is the k-th Legendre polynomial of degree k for the interval $[0, 1]$. This example will be further discussed in § 9.

We remark that if one instead chooses the basis functions $\{x^n(1-x)^n x^i\}_{i=0}^{N-2n}$ for $P_0^{(N)}$, the resulting matrix A of (5.11) closely resembles a segment of the Hilbert matrix. As such, it signals the fact that the numerical solution of (5.11) based on direct matrix inversion may be numerically unstable with regard to the growth of rounding errors when N is large.

In the next two sections, we shall consider certain finite dimensional subspaces of S of piecewise-polynomial functions. The study of such subspaces is important in that the minimization of the functional $F[w]$ over these subspaces gives nonlinear matrix equations of the form (5.11) which closely resemble the corresponding nonlinear matrix equations arising from discrete techniques applied to (1.1)—(1.2), (cf. [*17, 19, 24, 32*]). Moreover, such Rayleigh-Ritz techniques for piecewise-polynomial subspaces of S can be efficiently solved on high-speed computers. Further, as we shall see in § 8, these piecewise-polynomial subspaces of S are especially useful in solving two-point boundary value problems which admit internal physical interfaces.

§ 6. Piecewise-Polynomial Subspaces

To give other examples of subspaces of S which have connections with practical computation, we begin with the notion of the *smooth Hermite space* $H^{(m)}(\pi)$. Here, $\pi\colon 0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ denotes a partition of the unit interval with *joints* x_i, and m is a positive integer. Then, $H^{(m)}(\pi)$ is the collection of all real piecewise-polynomial functions $w(x)$ defined on $[0, 1]$ such that $w(x) \in C^{m-1}[0, 1]$, and such that on each subinterval $[x_i, x_{i+1}]$ defined by π, $w(x)$ is a polynomial of degree $2m-1$. An equivalent way of describing an arbitrary element of $H^{(m)}(\pi)$ is as follows. At each joint x_i, consider m interpolation parameters $d_i^{(k)}$, $0 \leq k \leq m-1$, $0 \leq i \leq N+1$. In each subinterval $[x_i, x_{i+1}]$, there is a unique interpolating polynomial $v_i(x)$ of degree $2m-1$ such that

$$D^k v_i(x_i) = d_i^{(k)}; \qquad D^k v_i(x_{i+1}) = d_{i+1}^{(k)}, \qquad 0 \leq k \leq m-1. \tag{6.1}$$

This type of interpolation is actually a special case of Hermite interpolation (cf. [44, p. 1]), and accounts for the naming of $H^{(m)}(\pi)$. The associated function $v(x)$, defined on $[0, 1]$ by the $v_i(x)$ on each subinterval of π, is of class $C^{m-1}[0, 1]$, and is thus an element of $H^{(m)}(\pi)$. As the number of parameters $d_i^{(k)}$ associated with any element of $H^{(m)}(\pi)$ is $m(N+2)$, then $H^{(m)}(\pi)$ is a linear space of dimension $m(N+2)$. As in [42], a convenient basis for $H^{(m)}(\pi)$ is $\{s_{i,k}(x; m; \pi)\}_{i=0,k=0}^{N+1,m-1}$ where the element $s_{i,k}(x; m; \pi)$ is defined by

$$(6.2)\qquad D^l s_{i,k}(x_j; m; \pi) = \delta_{i,j}\delta_{l,k}, \qquad 0 \leqq l \leqq m-1, \qquad 0 \leqq j \leqq N+1,$$

so that the support of $s_{i,k}(x; m; \pi)$ is contained in $[x_{i-1}, x_{i+1}]$. More explicitly, with $h_i = x_{i+1} - x_i$, the basis function $s_{i,0}(x; 1; \pi)$ for the special case $m = 1$ is the piecewise linear function

$$s_{i,0}(x; 1; \pi) = \begin{cases} (x - x_{i-1})/h_{i-1}, & x_{i-1} \leqq x \leqq x_i, \\ (x_{i+1} - x)/h_i, & x_i \leqq x \leqq x_{i+1}, \\ 0, & x \in \{[0, 1] - [x_{i-1}, x_{i+1}]\}, \ 1 \leqq i \leqq N, \end{cases}$$

$$s_{0,0}(x; 1; \pi) = \begin{cases} (x_1 - x)/h_0, & 0 \leqq x \leqq x_1, \\ 0, & x_1 \leqq x \leqq 1, \end{cases}$$

$$s_{N+1,0}(x; 1; \pi) = \begin{cases} 0, & 0 \leqq x \leqq x_N, \\ (x - x_N)/h_N, & x_N \leqq x \leqq 1, \end{cases}$$

sometimes called the "chapeau" or "roof" function, which is drawn in Fig. 1A. Similarly, the piecewise-cubic basis functions $s_{i,0}(x; 2; \pi)$ and $s_{i,1}(x; 2; \pi)$ are shown respectively in Figs. 1B and 1C.

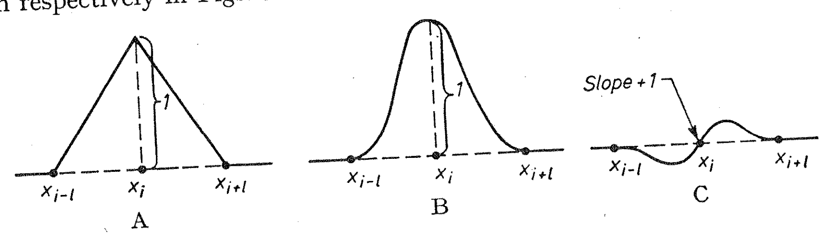

Fig. 1

Those elements of $H^{(m)}(\pi)$ which satisfy the boundary conditions of (1.2) form an $m(N+2) - 2n$ dimensional subspace of S, denoted by $H_0^{(m)}(\pi)$. This amounts to the restriction that $m \geqq n$ and that $d_0^{(k)} = d_{N+1}^{(k)} = 0$ for all $0 \leqq k \leqq n-1$.

We now discuss the interpolation error in such piecewise-Hermite interpolation. For notation, let $\bar{h}(\pi) = \max_{0 \leqq i \leqq N} h_i$ where $h_i = x_{i+1} - x_i$.

Theorem 9. Let $r(x) \in C^t[0, 1]$ with $t \geqq 2m$, let π be any partition of $[0, 1]$, and let $\tilde{w}(x)$ be the unique interpolation of $r(x)$ in $H^{(m)}(\pi)$, i.e., $D^l r(x_i) = D^l \tilde{w}(x_i)$ for all $0 \leqq i \leqq N+1$, $0 \leqq l \leqq m-1$. If $x \in [x_i, x_{i+1}]$, then

$$(6.3)\qquad |D^k(\tilde{w}(x) - r(x))| \leqq \frac{\|D^{2m} r\|_{L^\infty} h_i^k}{k!\,(2m-2k)!} [(x - x_i)(x_{i+1} - x)]^{m-k}$$

for all k with $0 \leq k \leq m$. Hence,

$$\|D^k(\widetilde{w} - r)\|_{L^\infty} \leq \frac{\|D^{2m} r\|_{L^\infty} (\bar{h}(\pi))^{2m-k}}{2^{2m-2k} k! (2m-2k)!}, \qquad 0 \leq k \leq m. \tag{6.4}$$

Proof. Although the expression of (6.3) is an easy extension of a known result ([*14, 17, 44*]) corresponding to the case $k=0$, we have not found this more general result in the literature. Because of its specific use in results to follow, we sketch its proof for completeness. Let σ be any point of (x_i, x_{i+1}) and consider

$$\psi(x) \equiv D^k(\widetilde{w}(x) - r(x)) - \{D^k(\widetilde{w}(\sigma) - r(\sigma))\} \left[\frac{(x-x_i)(x-x_{i+1})}{(\sigma-x_i)(\sigma-x_{i+1})}\right]^{m-k} \tag{6.5}$$

for all $x \in [x_i, x_{i+1}]$. Then, $D^l\psi(x_i) = D^l\psi(x_{i+1}) = 0$ for $0 \leq l \leq m-k-1$, and in addition $\psi(\sigma) = 0$. Hence, repeated applications of ROLLE's Theorem shows that a point ξ in (x_i, x_{i+1}) exists such that $D^{2m-2k}\psi(\xi) = 0$. Thus,

$$0 = D^{2m-2k}\psi(\xi)$$
$$= D^{2m-k}(\widetilde{w}(\xi) - r(\xi)) - ((2m-2k)!)\{D^k(\widetilde{w}(\sigma) - r(\sigma))\}/[(\sigma - x_i)(\sigma - x_{i+1})]^{m-k},$$

and substituting the resulting expression for $D^k(\widetilde{w}(\sigma) - r(\sigma))$ in (6.5) gives

$$\psi(x) = D^k(\widetilde{w}(x) - r(x)) - \frac{[D^{2m-k}(\widetilde{w}(\xi) - r(\xi))]}{(2m-2k)!} [(x - x_i)(x - x_{i+1})]^{m-k}.$$

Since $\psi(\sigma) = 0$ and σ is an arbitrary point in the open interval (x_i, x_{i+1}), then using continuity gives us that

$$D^k(\widetilde{w}(x) - r(x)) = \frac{[D^{2m-k}(\widetilde{w}(\xi) - r(\xi))]}{(2m-2k)!} [(x - x_i)(x - x_{i+1})]^{m-k}. \tag{6.6}$$

for *all* $x \in [x_i, x_{i+1}]$. It then remains to get an upper bound for $|D^{2m-k}(\widetilde{w}(\xi) - r(\xi))|$ in terms of $\|D^{2m} r\|_{L^\infty}$. If $\Omega(x) \equiv \widetilde{w}(x) - r(x)$, then $0 = D^l\Omega(x_i) = D^l\Omega(x_{i+1})$ for $0 \leq l \leq m-1$, and ROLLE's Theorem again gives us that there exist points η_r in $[x_i, x_{i+1}]$ such that $D^{2m-1-r}\Omega(\eta_r) = 0$, $0 \leq r \leq 2m-1$. But as $\widetilde{w}(x)$ is a polynomial of degree $2m-1$, then

$$D^{2m-1}\Omega(x) = \int_{\eta_0}^{x} D^{2m}\Omega(\varrho_1)\, d\varrho_1 = \int_{\eta_0}^{x} D^{2m} r(\varrho_1)\, d\varrho_1,$$

and in general

$$D^{2m-k}\Omega(\xi) = \int_{\eta_{k-1}}^{\xi} d\varrho_k \int_{\eta_{k-2}}^{\varrho_k} d\varrho_{k-1} \ldots \int_{\eta_0}^{\varrho_2} D^{2m} r(\varrho_1)\, d\varrho_1.$$

Thus,

$$|D^{2m-k}\Omega(\xi)| = |D^{2m-k}(\widetilde{w}(\xi) - r(\xi))| \leq \|D^{2m} r\|_{L^\infty} h_i^k/k!.$$

This proves (6.3); the inequality of (6.4) is then an obvious consequence of (6.3). Q.E.D.

As an application of this result, we have

Theorem 10. Let $\varphi(x)$, the solution of (1.1)—(1.2) subject to the conditions of (1.4) and (1.7), be of class $C^t[0, 1]$, with $t \geq 2m \geq 2n$, let π be any partition of $[0, 1]$, and let $\widehat{w}_m(x)$ be the unique function which minimizes $F[w]$ over $H_0^{(m)}(\pi)$. Then, there exists a constant M, which can be determined a priori and is independent of π, such that

$$\|\hat{w}_m - \varphi\|_{L^\infty} \leqq K \|\hat{w}_m - \varphi\|_\gamma \leqq KM \|D^{2m} \varphi\|_{L^\infty} (\bar{h}(\pi))^{2m-n}. \tag{6.7}$$

If the stronger hypothesis of (1.5) or (1.5′) is valid, then there exists a similar constant M' which can also be determined a priori and is independent of π, such that

$$\|D^k(\hat{w}_m - \varphi)\|_{L^\infty} \leqq KM' \|D^{2m} \varphi\|_{L^\infty} (\bar{h}(\pi))^{2m-n/2-k} \tag{6.8}$$

for all integers k with $0 \leqq k \leqq l$.

Proof. To establish the basic inequality of (6.7), let $S_M = H_0^{(m)}(\pi)$ with $m \geqq n$, and let $\tilde{w}(x)$ be the interpolation of $\varphi(x)$ in $H_0^{(m)}(\pi)$. Then, the inequality of (6.4) of Theorem 9 gives us the bounds

$$\|D^k(\tilde{w} - \varphi)\|_{L^\infty} \leqq \frac{\|D^{2m}\varphi\|_{L^\infty} \cdot [\bar{h}(\pi)]^{2m-k}}{2^{2m-2k}\, k!\, (2m-2k)!}, \qquad 0 \leqq k \leqq n. \tag{6.9}$$

Next, the coefficient functions $p_j(x)$ are, by hypothesis, at least continuous, and hence bounded in $[0, 1]$. Thus, from the definition of the norm $\|\cdot\|_\gamma$ in(4,3). it follows that

$$\|\tilde{w} - \varphi\|_\gamma \leqq Q \|D^{2m} \varphi\|_{L^\infty} (\bar{h}(\pi))^{2m-n},$$

where Q is a constant independent of π and can be determined a priori. The desired result of (6.7) then follows from the inequalities (cf. (4.15))

$$\|\hat{w}_m - \varphi\|_\gamma \leqq \|\hat{w}_m - \varphi\|_{g_m} \leqq \|\tilde{w} - \varphi\|_{g_m} \leqq \left(1 + \frac{\max(\Gamma_0 - \gamma; 0)}{\Lambda + \gamma}\right)^{\frac{1}{2}} \|\tilde{w} - \varphi\|_\gamma, \tag{6.10}$$

where Γ_0, γ, and Λ are all independent of π. Similarly, if the stronger hypothesis of (1.5) or (1.5′) is valid, then one makes use of (4.17) of Theorem 3 to establish (6.8). Q.E.D.

As an obvious consequence of the fact that the constant of (6.8) is independent of π, we have

Corollary 4. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4) and (1.7), be of class $C^t[0, 1]$ with fixed $t \geqq 2m \geqq 2n$, and let $\{\pi_i\}_{i=1}^\infty$ be any sequence of partitions of $[0, 1]$ such that $\lim_{i\to\infty} \bar{h}(\pi_i) = 0$. If the sequence of functions $\{\hat{w}_m(x; \pi_i)\}_{i=1}^\infty$ is obtained by minimizing $F[w]$ over $H_0^{(m)}(\pi_i)$, then $\{\hat{w}_m(x; \pi_i)\}_{i=1}^\infty$ converges uniformly to $\varphi(x)$.

As in § 5, the importance of the results of Theorem 10 and Corollary 4 can be illustrated with the following examples. If the solution $\varphi(x)$ of the linear problem: $D^2 u(x) = f(x)$, $0 < x < 1$, with $u(0) = u(1) = 0$, is *only* of class $C^2[0,1]$, then the results of (6.7) with $t=2$, $m=n=1$, give

$$\|\hat{w}_m - \varphi\|_{L^\infty} \leqq KM \|D^2 \varphi\|_{L^\infty} (\bar{h}(\pi)),$$

i.e., if $\{\pi_i\}_{i=1}^\infty$ is any sequence of partitions of $[0, 1]$ such that $\lim_{i\to\infty} \bar{h}(\pi_i) = 0$ then the sequence of functions $\{\hat{w}_m(x; \pi_i)\}_{i=1}^\infty$ converges at least *linearly* (in $\bar{h}(\pi_i)$) and uniformly to $\varphi(x)$, as $i \to \infty$. Similarly, if the solution $\varphi(x)$ of the linear thin beam problem: $D^4 u(x) = f(x)$, $0 < x < 1$, $D^k u(0) = D^k u(1) = 0$, $0 \leqq k \leqq 1$, is only of class $C^4[0, 1]$, the result of Theorem 10 with $t=4$, $m=n=2$, gives us

$$\|\hat{w}_m - \varphi\|_{L^\infty} \leqq KM \|D^4 \varphi\|_{L^\infty} (\bar{h}(\pi))^2,$$

i.e., the sequence of functions $\{\hat{w}_m(x;\pi_i)\}_{i=1}^{\infty}$ converges at least *quadratically* and uniformly to $\varphi(x)$. Moreover, $\|D(\hat{w}_m-\varphi)\|_{L^\infty}\leqq KM'\|D^1\varphi\|_{L^\infty}(h(\pi))^2$. Thus, the sequence $\{D\hat{w}_m(x;\pi_i)\}_{i=1}^{\infty}$ converges quadratically and uniformly to $D\varphi(x)$.

For the smooth Hermite spaces $H_0^{(m)}(\pi)$, one can vary the parameter m as well as the partition π, and convergence results, analogous to Corollary 4, can be proved for a sequence of smooth Hermite spaces $\{H_0^{(m_i)}(\pi)\}_{i=1}^{\infty}$ for a *fixed* partition π of $[0, 1]$. One such convergence result which can be proved is based on the interpolation result (6.4) of Theorem 9 and assumes that $\varphi(x)\in C^\infty[0, 1]$, with growth conditions

$$\lim_{i\to\infty}\left\{\frac{\|D^{2m_i}\varphi\|_{L^\infty}(\bar{h}(\pi))^{2m_i}}{2^{2m_i}(2m_i-2n)!}\right\}^{1/(2m_i)}=\alpha<1. \tag{6.11}$$

But, recognizing that $P_0^{(N)}$ is a subset of $H_0^{(m)}(\pi)$ for *any* partition of $[0, 1]$ if $2m-1\geqq N$, then the following stronger convergence result follows as a consequence of Theorem 6.

Corollary 5. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4) and (1.7), be of class $C^t[0, 1]$ where $t\geqq 2n$, and let $\{m_i\}_{i=1}^{\infty}$ be any sequence of positive integers with $m_i\geqq t$ and $\lim_{i\to\infty} m_i=+\infty$. If the sequence of functions $\{\hat{w}_i(x)\}_{i=1}^{\infty}$ is obtained by minimizing $F[w]$ over $H_0^{(m_i)}(\pi)$, where π is a fixed partition of $[0, 1]$, then $\{\hat{w}_i(x)\}_{i=1}^{\infty}$ converges uniformly to $\varphi(x)$.

The basic result of (6.7), based on a relatively crude interpolation theory, gives us that the exponent of $\bar{h}(\pi)$ is not less than $2m-n$, and one might suspect that this exponent is not sharp. Therefore, it is of interest now to show that this result in the norm $\|\cdot\|_\nu$ is actually *best possible* in the following sense. If π: $0=x_0<x_1<\cdots<x_{N+1}=1$ is a partition of $[0, 1]$, define $\underline{h}(\pi)=\min_{0\leqq i\leqq N} h_i$ where $h_i=x_{i+1}-x_i$.

Lemma 5. Let $\{\pi_i\}_{i=1}^{\infty}$ be any sequence of partitions of $[0, 1]$. Then, for $\varphi(x)=x^n(1-x)^{2m-n}$, $m\geqq n$, there exists a constant K independent of i such that

$$\inf_{w\in H_0^{(m)}(\pi_i)}\|D^n(w-\varphi)\|_{L^2}\geqq K(\underline{h}(\pi_i))^{2m-n},\quad \text{for all } i\geqq 1. \tag{6.12}$$

Proof. First, select any particular partition π_i of the sequence. Then, we can write

$$\|D^n(w-\varphi)\|_{L^2}^2=\sum_{j=0}^{N_i}\left(\frac{h_j}{2}\right)\int_{-1}^{+1}\{D^n[\varphi(\alpha_j+\beta_j t)-w(\alpha_j+\beta_j t)]\}^2\,dt \tag{6.13}$$

for any $w\in H_0^{(m)}(\pi_i)$, where $\alpha_j=(x_{j+1}+x_j)/2$ and $\beta_j=h_j/2$ both depend upon π_i. Because $\varphi(x)$ is a fixed polynomial of degree $2m$, the Legendre expansion of $D^n\varphi(\alpha_j+\beta_j t)$ in $[-1, +1]$ can be expressed as

$$D^n\varphi(\alpha_j+\beta_j t)=c_0 L_{2m-n}(t)+\sum_{j=0}^{2m-n-1}c_j L_j(t) \tag{6.14}$$

for suitable coefficients c_j. As is readily seen, the coefficient c_0 can be expressed as

$$c_0=M(h_j(\pi_i))^{2m-n},$$

where M depends on m and n, but is independent however of π_i. Next, any element w of $H_0^{(m)}(\pi_i)$ is a piecewise-polynomial function of degree $2m-1$ in each subinterval of π_i. Hence, the best choice of $D^n w$ in *each* subinterval of π_i which minimizes the integral

$$T_j = \int_{-1}^{+1} \{D^n[\varphi(\alpha_j + \beta_j t) - w(\alpha_j + \beta_j t)]\}^2 dt$$

is, from (6.14), just $\sum_{j=0}^{2m-n-1} c_j L_j(t)$. Thus, for any $w \in H_0^{(m)}(\pi_i)$, we have

$$T_j \geqq \frac{2M^2 (h_j(\pi_i))^{4m-2n}}{(4m-2n+1)} = 2M' (h_j(\pi_i))^{4m-2n}.$$

This inequality applied to (6.13) gives, with the hypothesis of this lemma,

$$\|D^n(w-\varphi)\|_{L^2}^2 \geqq M' \sum_{j=0}^{N_i} (h_j(\pi_i))^{4m-2n+1} \geqq M' (\underline{h}(\pi_i))^{4m-2n}.$$

As this is valid for any $w \in H_0^{(m)}(\pi_i)$ and any partitioning π_i, we have the desired result of (6.12). Q.E.D.

Theorem 11. Assuming the special case of (1.5'): $\|D^n v\|_{L^2} \leqq K \|v\|_\gamma$ for all $v \in S$, let $\{\pi_i\}_{i=1}^\infty$ be any sequence of partitions of $[0, 1]$ let $\hat{w}(x; \pi_i)$ be the unique function which minimizes $F[w]$ over $H_0^{(m)}(\pi_i)$, and let $\varphi(x) = x^n(1-x)^{2m-n}$ where $m \geqq n$. Then, there exists a constant M, independent of the partitions, such that

$$\|\hat{w}(x, \pi_i) - \varphi(x)\|_\gamma \geqq \inf_{w \in H_0^{(m)}(\pi_i)} \|w - \varphi\|_\gamma \geqq M(\underline{h}(\pi_i))^{2m-n}, \qquad i \geqq 1. \tag{6.15}$$

Thus, for partitionings π_i satisfying $\sigma\,\underline{h}(\pi_i) \geqq \bar{h}(\pi_i)$ for all $i \geqq 1$ for some constant $\sigma > 0$, the exponent of $\bar{h}(\pi)$ in (6.7) of Theorem 10 cannot in general be improved.

Proof[2]. The first inequality of (6.15) is obvious. Then, the hypothesis $K\|v\|_\gamma \geqq \|D^n v\|_{L^2}$ for all $v \in S$, coupled with the result of Lemma 5, give the final inequality of (6.15). Q.E.D.

Another interesting observation can be deduced from the inequality of (6.15). Instead of minimizing $F[w]$ over $H_0^{(m)}(\pi)$ to find an approximation to φ, let Z_m be the element of best approximation in $H_0^{(m)}(\pi)$ to φ in the norm $\|\cdot\|_\gamma$, i.e.,

$$\|Z_m - \varphi\|_\gamma = \inf_{w \in H_0^{(m)}(\pi)} \|w - \varphi\|_\gamma.$$

Since S is an inner product space with respect to the inner product $\langle\,,\,\rangle_\gamma$ of (4.4), then Z_m is uniquely determined in $H_0^{(m)}(\pi)$. Although Z_m is not in general $\hat{w}_m$, the inequalities of (6.15) of Theorem 11 and (6.7) of Theorem 10 nevertheless show us that no improvement in the exponent of $\bar{h}(\pi)$ can in general be made in the error bound by considering Z_m instead of $\hat{w}_m$ in $H_0^{(m)}(\pi)$.

The piecewise-Hermite subspaces $H_0^{(m)}(\pi)$ just considered are but rather special piecewise-polynomial subspaces of S, and we can generalize the preceding material in two essentially different directions. First, we observe that increasing the parameter m in $H_0^{(m)}(\pi)$ forces us in applications of Theorem 9 to consider both higher order interpolations at the joints of π, as well as smoother functions $r(x)$, i.e., $r(x) \in C^t[0, 1]$, $t \geqq 2m$. In one direction of generalization, we shall consider low

[2] We are indebted to Professor GARRETT BIRKHOFF for having pointed out to us this result for the special case $m=2$, $n=1$.

order interpolation by very smooth piecewise-polynomial functions, and this leads naturally to the consideration of *spline functions*. This will be treated in the next section. Another direction of generalization of the preceding material is to consider subspaces of piecewise polynomial functions which are less smooth then the smooth Hermite polynomials, but sufficiently differentiable so that they are subspaces of S. This may be described as follows. Let $\pi\colon 0=x_0<x_1<\cdots<x_{N+1}=1$ be any partition of $[0, 1]$, and m and k are positive integers with $m\geqq 2k$. Then, the *Hermite space*[3] $H(\pi; k; m)$ is the collection of all piecewise-polynomial functions $w(x)$ such that $w(x)$ is a polynomial of degree $m-1$ on each subinterval $[x_i, x_{i+1}]$ of π, and if $d_i^{(l)}$, $0\leqq l\leqq k-1$, $0\leqq i\leqq N+1$, are any real interpolation parameters, then

$$D^l w(x_i+)=D^l w(x_i-)=d_i^{(l)}, \qquad 0\leqq l\leqq k-1, \qquad 0\leqq i\leqq N+1.$$

The elements of $H(\pi; k; m)$ are thus of class $C^{k-1}[0, 1]$. Moreover, specifying the parameters $d_0^{(l)}$ and $d_{N+1}^{(l)}$ to be zero, $0\leqq l\leqq n-1$ gives us the subspace $H_0(\pi; k; m)$ of S. We remark that for even greater generality, one can similarly vary m and k at each joint of π. The basic results we shall obtain for $H(\pi; k; m)$ extend easily to this case, but for notational simplicity, we shall consider only $H(\pi; k; m)$ in what is to follow.

The elements of $H(\pi; k; m)$ can be represented as follows. Let $h_i(x)$ denote the unique Hermite interpolation polynomial defined on $[x_i, x_{i+1}]$ of degree $2k-1$ which interpolates all the parameters $d_i^{(l)}$ and $d_{i+1}^{(l)}$, i.e.,

$$D^l h_i(x_i)=d_i^{(l)}, \qquad D^l h_i(x_{i+1})=d_{i+1}^{(l)}, \qquad 0\leqq l\leqq k-1.$$

Then, the elements in $H(\pi; k; m)$ associated with the interpolation parameters $d_i^{(l)}$ can be represented as

$$w(x)=h_i(x)+(x-x_i)^k(x-x_{i+1})^k\sum_{j=0}^{m-2k-1} a_j x^j, \qquad x\in[x_i, x_{i+1}]. \tag{6.16}$$

It is thus clear that $H(\pi; k; m)$ is of dimension $\tau=m(N+1)-Nk$. Similarly, $H_0(\pi; k; m)$, a subspace of S, is of dimension $\tau-2n$.

Two convenient ways of selecting a basis for $H(\pi; k; m)$ suggest themselves. The first is to supplement the obvious basis for the Hermite interpolation piecewise-polynomial functions $h_i(x)$ by the basis functions $(x-x_i)^k(x-x_{i+1})^k x^j$, $0\leqq j\leqq m-2k-1$. The second is to *add* arbitrary (but fixed) new distinct points y_i to the joints x_i of π to define a new partition π' of $[0, 1]$, and to assign positive integers k_i' to these new joints so that the sum of these integers in any subinterval $[x_i, x_{i+1}]$ of π is precisely m. Then, one can view the elements of $H(\pi; k; m)$ in terms of general Hermite interpolation (cf. [*44*, p. 1]).

Since $H_0^{(m)}(\pi)\subset H_0(\pi; k; 2m)$ we may use the result of Theorem 10 to estimate the error involved in using the subspaces $H_0(\pi; k; 2m)$.

Theorem 12. Let $\varphi(x)$, the solution of (1.1)—(1.2) subject to the conditions of (1.4) and (1.7), be of class $C^t[0, 1]$, with $t\geqq 2m\geqq 2k\geqq 2n$, let π be any partition of $[0, 1]$, and let $\hat{w}_{k,m}(x)$ be the unique function which minimizes $F[w]$ over

[3] It is worth pointing out that both the smooth Hermite spaces $H^{(m)}(\pi)$ and the Hermite spaces $H(\pi; k; m)$ can be regarded as generalized spline subspaces [*38*].

$H_0(\pi; k; 2m)$. Then, there exists a constant M, which can be determined a priori and is independent of π, such that

(6.17) $$\|\hat{w}_{k,m} - \varphi\|_{L^\infty} \leqq K \|\hat{w}_{k,m} - \varphi\|_\gamma \leqq KM \|D^{2m}\varphi\|_{L^\infty} (\bar{h}(\pi))^{2m-n}.$$

If the stronger hypothesis of (1.5) or (1.5′) is valid, then there exists a similar constant M' which can also be determined a priori and is independent of π, such that

(6.18) $$\|D^j(\hat{w}_{k,m} - \varphi)\|_{L^\infty} \leqq KM' \|D^{2m}\varphi\|_{L_\infty} (\bar{h}(\pi))^{2m-n/2^{l-j}}$$

for all integers j with $0 \leqq j \leqq l$.

Corollary 6. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4) and (1.7), be of class $C^t[0, 1]$ with *fixed* $t \geqq 2m \geqq 2k \geqq 2n$, and let $\{\pi_i\}_{i=1}^\infty$ be any sequence of partitions of $[0, 1]$ such that $\lim_{i\to\infty} \bar{h}(\pi_i) = 0$. If the sequence of function $\{\hat{w}_{k,m}(x; \pi_i)\}_{i=1}^\infty$ is obtained by minimizing $F[w]$ over $H_0(\pi; k; m)$, then $\{\hat{w}_{k,m}(x; \pi_i)\}_{i=1}^\infty$ converges uniformly to $\varphi(x)$.

As with the smooth Hermite subspace, we may show that the exponent of $\bar{h}$ in Theorem 12 cannot be improved in general. The proof of the following result is exactly the same as the proof of Lemma 5.

Lemma 6. Let $\{\pi_i\}_{i=1}^\infty$ be any sequence of partitions of $[0, 1]$ and let $m \geqq k \geqq n$. Then, for $\varphi(x) = x^n(1-x)^{2m-n}$, there exists a constant K independent of i such that

(6.19) $$\inf_{w \in H_0(\pi i, k, m)} \|D^n(w - \varphi)\|_{L^2} \geqq K(\underline{h}(\pi_i))^{2m-n}, \quad \text{for all } i \geqq 1.$$

When the degree, $m-1$, of the Hermite polynomials under consideration is greater than or equal to t, the degree of differentiability of the solution, φ, of (1.1)—(1.2), we may invoke the approximation theory results of § 5 and obtain improved error estimates. We now discuss an analogue of Theorem 5.

Theorem 13. Let $r(x) \in C^t[0, 1]$, $t \geqq n$, where $r(x)$ satisfies the boundary conditions of (1.2), let π be a fixed partition of $[0, 1]$, and let k be a fixed positive integer with $k \geqq n$. Then, there exists a sequence $\{\tilde{p}_i(x)\}_{i=\max(t,2k)}^\infty$ of piecewise-polynomial functions with $\tilde{p}_i(x) \in H_0(\pi; k; i+1)$ such that

(6.20) $$\|D^j(u - \tilde{p}_i)\|_{L^\infty} \leqq \frac{K}{(i-k)^{t-k}} (\bar{h}(\pi))^{t-j} \omega\left(D^t r; \frac{1}{i-k}\right)$$

for all $0 \leqq j \leqq n$, where K depends only on n and t, but not on π or i, and ω is the modulus of continuity.

Proof. Consider the subinterval $[x_j, x_{j+1}]$ of π. Because we can express any element of $H_0(\pi; k; i)$ in the form (6.16) with $m = i$, it follows that we can regard this problem as the approximation of $(r(x) - h_i(x))$ by polynomials of the form $(x - x_j)^k (x - x_{j+1})^k \sum_{j=0}^{i-2k-1} a_j x^j$. As such, we can apply (5.1) of Theorem 5, reinterpreted for the subinterval $[x_j, x_{j+1}]$ rather than for $[0, 1]$. But this gives rise to the additional factor $(x_{j+1} - x_j)^{t-l} \leqq (\bar{h}(\pi))^{t-l}$ which gives (6.20). Q.E.D.

As an application of this result, we have

Theorem 14. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions (1.4) and (1.7), be of class $C^t[0, 1]$, $t \geqq 2n$, k be a fixed positive integer, π be any partition of $[0, 1]$, and let $\widehat{w}_{k,i}(x)$ be the unique element which minimizes $F[w]$ over $H_0(\pi; k; i)$ where $i \geqq \max(t, 2k)$. Then there exists a constant depending only on γ, k, t, and n such that

$$\|\widehat{w}_{k,i} - \varphi\|_{L^\infty} \leqq K \|\widehat{w}_{k,i} - \varphi\|_\gamma \leqq \frac{M(\gamma)\, \|D^t \varphi\|_{L^\infty}}{(i-k)^{t-k}} (\bar{h}(\pi))^{t-n}. \tag{6.21}$$

Moreover, if in addition, either (1.5) or (1.5′) is valid, then

$$\|D^j(\widehat{w}_{k,i} - \varphi)\|_{L^\infty} \leqq \frac{M(\gamma)}{2^{l-j}(i-k)^{t-k}} \|D^t \varphi\|_{L^\infty} (\bar{h}(\pi))^{t-n}, \qquad 0 \leqq j \leqq l. \tag{6.22}$$

It is now clear that uniform convergence of the $\widehat{w}_i$ to φ can be obtained by either selecting a sequence of partitions π_j with $\bar{h}(\pi_j) \to 0$ or fixing the partition π and letting i in $H_0(\pi; k; i)$ tend to infinity. This gives us the following corollaries.

Corollary 7. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4) and (1.7), be of class $C^t[0, 1]$ with $t \geqq 2n$ and let $\{\pi_j\}_{j=1}^\infty$ be any sequence of partitions of $[0, 1]$ such that $\lim_{j\to\infty} \bar{h}(\pi_j) = 0$. If the sequence of functions $\{\widehat{w}_j(x)\}_{j=1}^\infty$ is obtained by minimizing $F[w]$ over $H_0(\pi_j; k; i)$, with $i \geqq \max(t, 2k)$ and $k \geqq n$, then $\{\widehat{w}_j(x)\}_{j=1}^\infty$ converges uniformly to $\varphi(x)$.

Corollary 8. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4) and (1.7), be of class $C^t[0, 1]$ with $t \geqq 2n$, π be a fixed partition of $[0, 1]$, and $\widehat{w}_i(x)$ be the unique element that minimizes $F[w]$ over $H_0(\pi; k; i)$ where k is fixed with $k \geqq n$ and $i \geqq \max(t, 2k)$. Then, $\{\widehat{w}_i(x)\}_{i=\max(t,2k)}^\infty$ converges uniformly to $\varphi(x)$.

As an example of the importance of Theorem 14, consider the following result. If the solution $\varphi(x)$ of the linear problem:

$$D^2 u(x) = f(x), \qquad 0 < x < 1, \quad \text{with} \quad u(0) = u(1) = 0,$$

is *only* of class $C^3[0, 1]$, then the results of (6.21) with $t = i = 3$, $k = n = 1$ give

$$\|\widehat{w}_{1,3} - \varphi\|_{L^\infty} \leqq \frac{M(\gamma)\|D^3 \varphi\|_{L^\infty}}{4} (\bar{h}(\pi))^2,$$

i.e., if $\{\pi_i\}_{i=1}^\infty$ is any sequence of partitions of $[0, 1]$ such that $\lim_{i\to\infty} \bar{h}(\pi_j) = 0$ then the sequence of functions $\{\widehat{w}_{k,i}(x; \pi_j)\}_{i=1}^\infty$ converges at least *quadratically* (in $\bar{h}(\pi_j)$) and uniformly to $\varphi(x)$.

§ 7. Spline Subspaces

As another example of subspaces of S which have connections with practical computation, we consider now the *spline interpolation spaces* $Sp^{(m)}(\pi)$, $m \geqq 1$, first considered by SCHOENBERG [36]. With the partition $\pi: 0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ of the interval $[0, 1]$, we define $Sp^{(m)}(\pi)$ to be the collection of all piecewise-polynomial functions $w(x)$ defined on $[0, 1]$ such that $w(x)$ is a polynomial of degree $2m-1$ in each subinterval $[x_i, x_{i+1}]$, $0 \leqq i \leqq N$, determined by π, and such that $w(x) \in C^{2m-2}[0, 1]$. As such, $Sp^{(m)}(\pi)$ is a subspace of $H^{(m)}(\pi)$. More precisely, if $m = 1$, then $Sp^{(1)}(\pi) = H^{(1)}(\pi)$ is the collection of all continuous piecewise-linear functions on $[0, 1]$. For $m > 1$, $Sp^{(m)}(\pi)$ can be described as follows. Given the $m(N+2)$ numbers $d_i^{(l)}$, $0 \leqq l \leqq m-1$, $0 \leqq i \leqq N+1$, we know that there is a unique function $w(x) \in H^{(m)}(\pi)$ which interpolates these numbers

$d_i^{(l)}$ in the joints x_i of π. Regarding the $2m+N$ numbers $d_0^{(l)}, d_{N+1}^{(l)}, 0 \leqq l \leqq m-1$, $d_i^{(0)}$, $1 \leqq i \leqq N$, as parameters, we now *determine* the remaining numbers $d_i^{(l)}$, $1 \leqq l \leqq m-1$, $1 \leqq i \leqq N$, so that $w(x) \in C^{2m-2}[0,1]$, i.e., $D^{2m-2}w(x)$ is to be continuous at all the joints x_i. This gives rise to a system of linear equations in these numbers $d_i^{(1)}, d_i^{(2)}, \ldots, d_i^{(m-1)}$, $1 \leqq i \leqq N$, whose associated matrix is a nonsingular block-tridiagonal matrix having principal submatrices of order $(m-1)$. In other words, given the parameters for the end joints $d_0^{(k)}, d_{N+1}^{(k)}$, $0 \leqq k \leqq m-1$, and the $d_i^{(0)}$, $1 \leqq i \leqq N$, there is a *unique* interpolation function (cf. [5]) $w(x)$ in $Sp^{(m)}(\pi)$ with $D^k w(0) = d_0^{(k)}$, $D^k w(1) = d_{N+1}^{(k)}$, $0 \leqq k \leqq m-1$, and $w(x_i) = d_i^{(0)}$, $1 < i < N$. As such, $Sp^{(m)}(\pi)$ is a linear space of dimension $N+2m$, and for $m \geqq n$, we denote by $Sp_0^{(m)}(\pi)$ the subspace of $Sp^{(m)}(\pi)$ whose elements satisfy the boundary conditions of (1.2). Thus, $Sp_0^{(m)}(\pi)$ is an $N+2(m-n)$ dimensional subspace of S. For a comprehensive coverage of the topic of splines which includes an extensive bibliography, we recommend [4] and [38].

In analogy with Theorems 5, 7, 9, and 13, we consider the approximation error in such spline interpolation. Because each element $w \in Sp^{(m)}(\pi)$ satisfies $D^{2m}w(x) = 0$ on every open subinterval of π, the following result is a special case of results of Ahlberg, Nilson, and Walsh [1].

Theorem 15. Let $r(x) \in C^t[0,1]$, $t \geqq 2m$, let $\{\pi_i\}_{i=1}^{\infty}$ be any sequence of partitions of $[0,1]$ with $\lim_{i\to\infty} \bar{h}(\pi_i) = 0$, and let $\tilde{w}_i(x)$ be the unique interpolation of $r(x)$ in $Sp^{(m)}(\pi_i)$, i.e., $D^k r(0) = D^k \tilde{w}_i(0)$; $D^k r(1) = D^k \tilde{w}_i(1)$, $0 \leqq k \leqq m-1$, and $r(x_{j(i)}) = w_i(x_{j(i)})$, $1 \leqq j(i) \leqq N(i)$. Then,

$$(7.1) \qquad \|D^k(r - \tilde{w}_i)\|_{L^\infty} \leqq K \|D^{2m} r\|_{L^\infty} (\bar{h}(\pi_i))^{2m-1-k}, \qquad 0 \leqq k \leqq m-1,$$

where K is dependent on m but is independent of the π_i. Moreover, if the partitions $\{\pi_i\}_{i=1}^{\infty}$ are such that $\sigma\, \underline{h}(\pi_i) \geqq \bar{h}(\pi_i)$ for all $i \geqq 1$ for some $\sigma > 0$, then

$$(7.2) \qquad \|D^k(r - \tilde{w}_i)\|_{L^\infty} \leqq K' \|D^{2m} r\|_{L^\infty} (\bar{h}(\pi_i))^{2m-1-k}, \qquad 0 \leqq k \leqq 2m-2,$$

where K' is dependent on m but is independent of the π_i.

As an application of these inequalities, we have

Theorem 16. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4) and (1.7), be of class $C^t[0,1]$ with $t \geqq 2m > 2n$, let $\{\pi_i\}_{i=1}^{\infty}$ be any sequence of partitions of $[0,1]$ with $\lim_{i\to\infty} \bar{h}(\pi_i) = 0$, and let $\hat{w}_i(x)$ be the unique function which minimizes $F[w]$ over $Sp_0^{(m)}(\pi_i)$. Then, there exists a constant M, independent of the π_i, such that

$$(7.3) \qquad \|\hat{w}_i - \varphi\|_{L^\infty} \leqq K \|\hat{w}_i - \varphi\|_{\gamma} \leqq KM \|D^{2m}\varphi\|_{L^\infty} (\bar{h}(\pi_i))^{2m-1-n}, \qquad i \geqq 1,$$

If the stronger hypothesis of (1.5) or (1.5′) is valid, then there exists a similar constant M' independent of the π_i such that

$$(7.4) \qquad \|D^k(\hat{w}_i - \varphi)\|_{L^\infty} \leqq KM' \|D^{2m}\varphi\|_{L^\infty} (\bar{h}(\pi_i))^{2m-1-n/2^{l-k}}, \qquad 0 \leqq k \leqq l,$$

for all $i \geqq 1$. The conclusions of (7.3) and (7.4) remain valid if $m = n$, provided that there exists a positive constant σ such that $\sigma\, \underline{h}(\pi_i) \geqq \bar{h}(\pi_i)$ for all $i \geqq 1$.

The results of (7.1) and (7.2) of Theorem 16 are similar to that of (6.4) of Theorem 9, the difference being that the exponent of $\bar{h}(\pi)$ in (7.1) and (7.2) is one less than that in (6.4). What is probably true in general is that (7.1) and (7.2) are also valid for the exponent $2m-k$. In fact, for $m=1$, we already know that $Sp_0^{(1)}(\pi)=H_0^{(1)}(\pi)$, and the result of (6.4) is sharper than that of (7.1) or (7.2) in this case. For the case $m=2$ of cubic splines, BIRKHOFF and DE BOOR [3] have established (7.2) with the exponent $2m-k$. This work has recently been extended by SHARMA and MEIR [40] who have proved the following rather sharp analogues of (7.1) and (7.2) for the case $m=2$. If $r(x)\in C^2[0,1]$, then

$$\|D^k(r-\tilde{w}_i)\|_{L^\infty} \leqq 5\,(\bar{h}(\pi_i))^{2-k}\,\omega\,(D^2 r; \bar{h}(\pi_i)), \qquad 0\leqq k\leqq 2, \tag{7.5}$$

where ω is the modulus of continuity. If the partitions π_i are such that $\sigma\,\underline{h}(\pi_i)\geqq \bar{h}(\pi_i)$ for all $i\geqq 1$ for some $\sigma>0$, then for $r(x)\in C^3[0,1]$,

$$\|D^k(r-\tilde{w}_i)\|_{L^\infty} \leqq [1+\sigma(1+\sigma)^2]\,(\bar{h}(\pi_i))^{3-k}\,\omega\,(D^3 r; \bar{h}(\pi_i)), \qquad 0\leqq k\leqq 3. \tag{7.6}$$

The importance of these last two results lies in the following observations. If we minimize the functional $F[w]$ over the subspaces $Sp_0^{(2)}(\pi_i)$ of cubic splines where $\{\pi_i\}_{i=1}^\infty$ is any sequence of partitions of $[0,1]$ with $\lim_{i\to\infty}\bar{h}(\pi_i)=0$, then we need only assume that the solution $\varphi(x)$ of (1.1)–(1.2) for the case $n=1$ is of class $C^2[0,1]$ in order to establish that the sequence $\{\hat{w}_i(x)\}_{i=1}^\infty$ converges linearly and uniformly to $\varphi(x)$. As an application, if the solution $\varphi(x)$ of the linear problem $D^2u(x)=f(x)$, $0<x<1$, $u(0)=u(1)=0$, is only of class $C^2[0,1]$, the use of cubic splines nevertheless gives a sequence of functions which converge *linearly* in $\bar{h}(\pi_i)$ to $\varphi(x)$. Similarly, if the solution $\varphi(x)$ of the linear thin beam problem: $D^4u(x)=f(x)$, $0<x<1$, $D^ku(0)=D^ku(1)=0$, $0\leqq k\leqq 1$, is only of class $C^4[0,1]$, the use of spline functions gives a sequence of functions which converges *quadratically* in $\bar{h}(\pi_i)$ to $\varphi(x)$. Such results, as far as we know, are not obtainable from Taylor series and Gerschgorin-type convergence arguments for discrete methods (cf. [*12*, p. 348], [*41*, p. 165]).

Returning to the previous comment that the exponent of $\bar{h}(\pi)$ in (7.1) and (7.2) is probably $2m-k$ instead of $2m-1-k$, we remark that the proof of Lemma 5 in § 6 is necessarily valid for functions of $Sp_0^{(m)}(\pi)$, since $Sp_0^{(m)}(\pi)$ is a subset of $H_0^{(m)}(\pi)$. Consequently, the exponent of $\bar{h}(\pi)$ cannot in general be greater than $2m-k$. A similar conclusion was reached in [*1*], but by a different proof.

In analogy with Corollary 5, let π now be a *fixed* partition of $[0,1]$. Again, recognizing that $P_0^{(N)}$ is a subset of $Sp_0^{(m)}(\pi)$ for *any* partition of $[0,1]$ if $2m-1\geqq N$, then we have the

Corollary 9. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4) and (1.7), be of class $C^t[0,1]$ where $t\geqq 2n$, and let $\{m_i\}_{i=1}^\infty$ be any sequence of positive integers with $m_i\geqq t$ and $\lim_{i\to\infty} m_i=+\infty$. If the sequence of functions $\{\hat{w}_i(x)\}_{i=1}^\infty$ is obtained from minimizing $F[w]$ over $Sp_0^{(m_i)}(\pi)$, where π is a fixed partition of $[0,1]$, then $\{\hat{w}_i(x)\}_{i=1}^\infty$ converges uniformly to $\varphi(x)$.

For practical computations, the choice of a basis for $Sp^{(m)}(\pi)$ is very important. The most natural choice for a basis for $Sp^{(m)}(\pi)$ is probably the *cardinal functions* $C_i(x)$ (cf. [*4*, p. 168]), which are the analogue of the basis functions $s_{i,k}(x;m;\pi)$ for the smooth Hermite space $H^{(m)}(\pi)$, defined in (6.2).

If π: $0=x_0<x_1<\cdots<x_{N+1}=1$ is a partition of $[0, 1]$, then $C_i(x)\in Sp^{(m)}(\pi)$ is defined by

$$
\begin{aligned}
&C_i(x_j)=\delta_{i,j}, \quad 0\leqq j\leqq N+1, \quad D^k C_i(0)=D^k C_i(1)=0, \quad 0\leqq k\leqq m-1 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad 1\leqq i\leqq N, \\
(7.7)\quad &C_{N+l}(x_j)=0, \quad 0\leqq j\leqq N+1, \quad D^k C_{N+l}(0)=\delta_{l,k}, \quad D^k C_{N+l}(1)=0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\leqq k\leqq m-1, \ \text{for} \ 1\leqq l\leqq m, \\
&C_{N+m+l}(x_j)=0, \quad 0\leqq j\leqq N+1, \quad D^k C_{N+m+l}(0)=0, \quad D^k C_{N+m+l}(1)=\delta_{l,k}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\leqq k\leqq m-1, \ \text{for} \ 1\leqq l\leqq m.
\end{aligned}
$$

It is easy to see that the support of each $C_i(x)$, $1\leqq i\leqq N+2m$, is the *entire* interval $[0, 1]$. Thus, the inner products $\langle C_i, C_j\rangle_0$ are in general nonzero. Consequently, the use of these cardinal functions as basis elements in $Sp^{(m)}(\pi)$ results in nonlinear matrix equations (cf. (5.11) and (5.12)) for which the coefficient matrix A is full, i.e., certainly not sparse. This necessarily complicates data storage and generally means slower convergence of iterative techniques on digital computers. Because of this, the following basis elements in $Sp^{(m)}(\pi)$ are more suitable for practical computation. Since any element $w(x)$ of $Sp^{(m)}(\pi)$ is a polynomial of degree $2m-1$ on each subinterval of π with $w(x)\in C^{2m-2}[0,]$, then $D^{2m-2}w(x)$ is a continuous piecewise linear function on $[0, 1]$. The object then is to select a particular continuous piecewise linear function which, when integrated yields an element of $Sp^{(m)}(\pi)$ with *minimal* support in $[0, 1]$. In general, the support is on $2m$ adjacent subintervals of $[0, 1]$ (cf. [*37*]). For the case of $m=2$ of cubic splines for a uniform partition π, i.e., $x_i=ih$, $0\leqq i\leqq N+1$, these functions $s_i(x)$ are given explicitly by

$$
(7.8)\quad s_i(x)=\begin{cases}
(x-x_{i-2})^3, \quad x\in[x_{i-2}, x_{i-1}], \\
h^3+3h^2(x-x_{i-1})+3h(x-x_{i-1})^2-3(x-x_{i-1})^3, \quad x\in[x_{i-1}, x_i], \\
h^3+3h^2(x_{i+1}-x)+3h(x_{i+1}-x)^2-3(x_{i+1}-x)^3, \quad x\in[x_i, x_{i+1}], \\
(x_{i+2}-x)^3, \quad x\in[x_{i+1}, x_{i+2}], \\
\text{zero otherwise},
\end{cases}
$$

for $2\leqq i\leqq N-1$. If we extend the uniform partitioning π outside of $[0, 1]$, we can then use (7.8) to define the functions $s_{-1}(x)$, $s_0(x)$, $s_1(x)$, and $s_N(x)$, $s_{N+1}(x)$, $s_{N+2}(x)$ in $[0, 1]$. Then $\{s_i(x)\}_{i=-1}^{N+2}$ is a basis for $Sp^{(m)}(\pi)$.

In order to obtain a convenient basis for $Sp_0^{(2)}(\pi)$ for computations, it is necessary to modify these basis functions somewhat. Let $\tilde{s}_0(x)\equiv s_0(x)-4s_{-1}(x)$; $\tilde{s}_1(x)\equiv s_1(x)-s_{-1}(x)$, $\tilde{s}_i(x)=s_i(x)$, $2\leqq i\leqq N-1$, $s_N(x)\equiv s_N(x)-s_{N+2}(x)$; $\tilde{s}_{N+1}(x)=s_{N+1}(x)-4s_N(x)$. Then, $\{\tilde{s}_i(x)\}_{i=0}^{N+2}$ is a basis for $Sp_0^{(2)}(\pi)$, and the support of each $\tilde{s}_i(x)$ is on at most four adjacent subintervals of $[0, 1]$.

Finally, we must remark that the spline interpolation spaces $Sp^{(m)}(\pi)$ can themselves be generalized substantially. Noting that $w(x)\in Sp^{(m)}(\pi)$ implies that $w(x)$ satisfies the differential equation $M^*M[w]=D^{2m}w=0$ on each open subinterval of π, one can then define *generalized splines* [*1*] through the solution of the linear self-adjoint differential equation $M^*M[w]=0$ on each subinterval of π, where M is a differential operator of order m. This is closely related to the ROSE's use of a *patch basis* [34] to solve two-point second order linear boundary value problems, in which one uses *local* solutions of (1.1) on subintervals of π. We simply state that the result of Theorem 15 is valid for such generalized spline spaces, as is its application in Theorem 16.

§ 8. Extensions

In this section, we extend our previous results to cover more general nonlinear boundary value problems. Moreover, we show how to obtain a posteriori error bounds.

First, it is easy to verify that all of our previous results hold if we make the following weakened assumptions about $f(x, u)$. We again assume that $f(x, u) \in C^0([0, 1] \times R)$, but in place of (1.7), we assume that there exists a constant γ such that

$$\frac{f(x,u)-f(x,v)}{u-v} \geqq \gamma > -\Lambda, \quad \text{for all } x \in [0,1], \quad \text{and all } -\infty < u, v < +\infty, \quad \text{with } u \neq v, \tag{8.1}$$

and for each $c > 0$, there exists a number $M(c)$ such that

$$u \neq v, \quad |u| \leqq c, \quad |v| \leqq c \quad \text{implies} \quad \frac{f(x,u)-f(x,v)}{u-v} \leqq M(c) < \infty \quad \text{for all } x \in [0,1]. \tag{8.2}$$

Second, there is an important class of problems which do not satisfy a condition of the form (8.1), but can however be treated via the techniques we have thus far described.

For the boundary value problem (1.1)—(1.2), we assume that

$$\frac{f(x,u)-f(x,v)}{u-v} \geqq 0 \quad \text{if} \quad u, v \geqq 0, \quad u \neq v, \tag{8.3}$$

in place of the assumption of (8.1). A typical example is $f(x, u) = u^{2m}$, $m \geqq 1$. In addition, we assume that Λ of (1.6) is positive and that (8.2) is valid for nonnegative u and v.

As is the case for the second order problem, i.e., $n=1$ in (1.1), (cf. [31]), the general problem (1.1)—(1.2) under the condition (8.3) may have a unique nonnegative solution. If this is the case and we are interested in only that particular solution, we can solve a modified problem where $f(x, u)$ will be replaced by $g(x, u) \equiv f(x, \max\{u, 0\})$. The new function $g(x, u)$ satisfies (8.1), and the new boundary value problem:

$$L[u] = g(x, u), \qquad 0 < x < 1, \tag{8.4}$$

and

$$D^k u(0) = D^k u(1) = 0, \qquad 0 \leqq k \leqq n-1 \tag{8.5}$$

has a unique solution which, as is easily verified, is the unique nonnegative solution of (1.1)—(1.2).

For our third extension, we restrict (for ease of exposition) our attention to second-order problems of the form

$$D\{p_1(x)\,D\,u(x)\} = f(x, u), \qquad 0 < x < 1, \tag{8.6}$$

with the boundary conditions

$$u(0) = u(1) = 0, \tag{8.7}$$

where we have combined the linear term $-p_0(x)u$ of (1.1) with the right-hand side. We assume that the coefficient function $p_1(x)$ of (8.6), instead of being $C^1[0, 1]$, is only piecewise continuous, i.e., $p_1(x)$ is discontinuous at only a finite number of points x_i, $1 \leqq i \leqq k$, in $(0, 1)$, $\lim_{x\to x_i-} p_1(x)$ and $\lim_{x\to x_i+} p_1(x)$ both exist, and $p_1(x)$ is a strictly positive function, i.e., there exists a constant ω such that

$$p_1(x) \geqq \omega > 0, \qquad 0 \leqq x \leqq 1. \tag{8.8}$$

We must carefully define what is meant by a (classical) solution of (8.6), (8.7). Solving (8.6), (8.7) amounts to finding a function $\varphi(x)$ satisfying the following: $\varphi(x) \in C^0[0, 1]$, $D\varphi(x)$ exists and is continuous on each subinterval (x_i, x_{i+1}), $0 \leqq i \leqq k$, $\{p_1(x)\,D\,\varphi(x)\}$ is continuously differentiable on each subinterval (x_i, x_{i+1}), $0 \leqq i \leqq k$, $\lim_{x\to x_i-} \{p_1(x)\,D\,\varphi(x)\} = \lim_{x\to x_i+} \{p_1(x)\,D\,\varphi(x)\}$, $1 \leqq i \leqq k$, $D\{p_1(x)\,D\,\varphi(x)\} \equiv f(x, \varphi(x))$ on each interval (x_i, x_{i+1}), $0 \leqq i \leqq k$, and $\varphi(0) = \varphi(1) = 0$.

It is not difficult to show that Theorems 1, 2, 3, and 4 hold without modification. Moreover, unlike the use of polynomial subspaces, the use of the piecewise-polynomial subspaces will yield the same high-order error bounds, provided the discontinuities of $p_1(x)$ are always chosen to be joints of the associated partition π. We illustrate this point by considering the use of the smooth Hermite subspaces $H^{(2)}(\pi)$ for a "model problem".

Let us assume that $p_1(x)$ is of class C^3 except at some point y, $0 < y < 1$. Similarly, we assume that $\lim_{x\to 0+} D^i p_1(x)$, $\lim_{x\to y-} D^i p_1(x)$, $\lim_{x\to y+} D^i p_1(x)$, and $\lim_{x\to 1-} D^i p_1(x)$ exist for all i, $0 \leqq i \leqq 3$. Finally, let $f(x, u) \in C^2([0, 1] \times R)$. Under these assumptions, the solution $\varphi(x)$ is of class C^4 in each interval $[0, y]$ and $[y, 1]$. The vector $\boldsymbol{u}$ which represents a function of $H^{(2)}(\pi)$ has a precise definition: the first $N+2$ components of $\boldsymbol{u}$ can be chosen to represent the ordinates at each mesh point, and the last $N+2$ components represent the derivatives at each mesh point. However, as stated above, y is always chosen to be a mesh point and the derivative at the mesh point y is *not* defined in the present case. All that is required now is that

$$\lim_{x\to y-} p(x)\,D\,\varphi(x) = \lim_{x\to y+} p(x)\,D\,\varphi(x) \quad\text{or}\quad p(y-)\,D\,\varphi(y-) = p(y+)\,D\,\varphi(y+). \tag{8.9}$$

Accordingly, if the point y is chosen to be the joint x_j, the basis function $s_{j,1}(x; 2; \pi)$ will be defined as the unique piecewise polynomial of degree 3 satisfying: $s_{j,1}(x_i; 2; \pi) = 0$, $0 \leqq i \leqq N+1$; $Ds_{j,1}(x_i; 2; \pi) = 0$, $x_i \neq x_j$, $0 \leqq i \leqq N+1$; $Ds_{j,1}(x_j-; 2; \pi) = p(y+)$; and $Ds_{j,1}(x_j+; 2; \pi) = p(y-)$. All the other basis functions will remain unmodified. With the above basis functions it is easily verified that condition (8.9) is automatically satisfied for any element in this piecewise-polynomial subspace.

Thus, *the error estimates are unchanged,* basically because the bounds we developed in § 6 were obtained by an argument over each subinterval $[x_i, x_{i+1}]$, which is still possible in this case since $\varphi(x) \in C^4$ in such intervals.

As our fourth extension, suppose that $w(x)$ is any function in $C^{2n-1}[0, 1]$ satisfying the boundary conditions (1.2), such that $D^{2n-1}w(x)$ is absolutely continuous in $[0, 1]$, and $D^{2n}w \in L^2[0, 1]$. For example, it may be the minimizing function $\hat{w}_M(x)$, as defined in § 3, whenever the subspace S_M is composed of sufficiently smooth functions. Then, we can theoretically determine the function $\tau(x)$ defined by:

(8.10) $$\tau(x) \equiv L[w(x)] - f(x, w(x)), \qquad 0 < x < 1,$$

and, clearly, $\tau(x) \in L^2[0, 1]$.

The following question naturally arises. From the knowledge of $\tau(x)$, can we deduce anything about the difference $[\varphi(x) - w(x)]$, where $\varphi(x)$ is the unique solution of (1.1)—(1.2)? The answer is given by the following result.

Theorem 17. Assume that Λ of (1.6) is positive. Then, for any function $w(x)$ satisfying the boundary conditions of (1.2), such that $w(x) \in C^{2n-1}[0, 1]$ with $D^{2n-1}w(x)$ absolutely continuous in $[0, 1]$ and $D^{2n}w(x) \in L^2[0, 1]$, the following error estimate holds:

(8.11) $$\|w - \varphi\|_{L^\infty} \leqq K\|w - \varphi\|_\gamma \leqq \frac{K}{\sqrt{\Lambda}}\|\tau\|_{L^2[0,1]},$$

where $\tau(x)$ is defined by (8.10).

Proof. The first inequality of (8.11) is just (4.3). Now, define $\varepsilon(x) = \varphi(x) - w(x)$. Then, with the notation of (4.1), we have, after integration by parts,

$$\|\varepsilon\|_\gamma^2 = -\int_0^1 \varepsilon(x) L[\varepsilon(x)]\,dx + \gamma\int_0^1 \varepsilon^2(x)\,dx.$$

Using the definition of $\tau(x)$, this can be expressed as

$$\|\varepsilon\|_\gamma^2 = \int_0^1 \varepsilon^2(x)\left\{\gamma - \frac{f(x, \varphi(x)) - f(x, w(x))}{\varphi(x) - w(x)}\right\} dx + \int_0^1 \varepsilon(x)\,\tau(x)\,dx.$$

But with (8.1), this first integral is nonpositive, and thus, applying Schwarz's inequality, we have
$$\|\varepsilon\|_\gamma^2 \leqq \|\varepsilon\|_{L^2[0,1]} \cdot \|\tau\|_{L^2[0,1]}.$$
On the other hand, the positivity of Λ in (1.6) gives us that $\sqrt{\Lambda}\,\|\varepsilon\|_{L^2[0,1]} \leqq \|\varepsilon\|_\gamma$. Thus, $\|\varepsilon\|_\gamma \leqq \Lambda^{-\frac{1}{2}}\|\tau\|_{L^2[0,1]}$. Q.E.D.

As we have seen, the sequence $\{\hat{w}_{M_i}(x)\}_{i=1}^\infty$ will converge uniformly to the solution $\varphi(x)$, if the approximating subspaces, S_{M_i} are properly chosen. However, the residuals $\tau_{M_i}(x) = L[\hat{w}_{M_i}(x)] - f(x, \hat{w}_{M_i}(x))$ need *not* converge to zero. Thus, the previous result does not necessarily give a useful error bound. However, Courant [*13*] has suggested the following remedy. If the subspaces S_{M_i} consist of sufficiently differentiable functions, we may modify the functional (2.2) of the variational formulation by adding terms involving higher order derivatives which vanish for the actual solution φ.

To illustrate this, consider any *linear* differential equation in (1.1), i.e., where the function $f(x, u) = f(x)$ is independent of u. Assume also that Λ of (1.6) is positive. In place of the functional $F[w]$ of (2.2), consider

(8.12) $$F_1[w] = F[w] + \tfrac{1}{2}\int_0^1 \{L[w] - f(x)\}^2\,dx, \qquad w \in S,$$

for all elements of S, where S now is the set of all functions $w(x)$ defined on $[0, 1]$ with $w(x)$ satisfying the boundary conditions of (1.2), and $D^{2n-1}w(x)$ is absolutely continuous in $[0, 1]$, with $D^{2n}w(x) \in L^2[0, 1]$. In analogy with Theorems 1 and 2, it is easily seen that the solution $\varphi(x)$ of (1.1)—(1.2) uniquely minimizes $F_1[w]$ over S, and each M-dimensional subspace S_M of S possesses a unique element $\hat{w}_M$ such that $F_1[\hat{w}_M] = \inf_{w \in S_M} F_1[w]$. Writing $w_M(x) = \sum_{i=1}^{M} u_i w_i(x)$, then $\frac{\partial F_1[w_M]}{\partial u_i} = 0$ gives us, using the notation of (4.4),

$$(8.13) \qquad 0 = \langle \hat{w}_M, w_i \rangle_0 + \int_0^1 f(x)\, w_i(x)\, dx + \int_0^1 \{L[\hat{w}_M] - f(x)\}\, L[w_i]\, dx, \quad 1 \leqq i \leqq M.$$

Since $\varphi(x)$ is the solution of $L[u] = f(x)$ with boundary conditions of (1.2), we similarly have

$$(8.14) \qquad 0 = \langle \varphi, w_i \rangle_0 + \int_0^1 f(x)\, w_i(x)\, dx + \int_0^1 \{L[\varphi] - f(x)\}\, L[w_i]\, dx, \quad 1 \leqq i \leqq M,$$

and consequently, subtracting yields

$$(8.15) \qquad 0 = \langle \varphi - \hat{w}_M, w_i \rangle_0 + \int_0^1 L[\varphi - \hat{w}_M] \cdot L[w_i]\, dx, \quad 1 \leqq i \leqq M.$$

Thus, by defining the new inner product

$$(8.16) \qquad \langle\!\langle v, y \rangle\!\rangle \equiv \langle v, y \rangle_0 + \int_0^1 L[v] \cdot L[y]\, dx, \quad v, y \in S,$$

on S, then (8.15) can be simply expressed as

$$(8.17) \qquad 0 = \langle\!\langle \varphi - \hat{w}_M, w_i \rangle\!\rangle, \quad 1 \leqq i \leqq M.$$

Hence, if $|||\cdot|||$ denotes the norm on S induced by this inner product, then we obviously have that

$$(8.18) \qquad |||\hat{w}_M - \varphi||| = \inf_{w \in S_M} |||w - \varphi|||.$$

But as $|||v|||^2 \geqq \int_0^1 L^2[v]\, dx$, then

$$(8.19) \qquad |||\hat{w}_M - \varphi|||^2 \geqq \int_0^1 L^2[\hat{w}_M - \varphi]\, dx = \int_0^1 \tau_M^2(x)\, dx,$$

where $L[\hat{w}_M] \equiv f(x) + \tau_M(x)$. In this example, if $\{S_{M_i}\}_{i=1}^{\infty}$ are subspaces of S with $\bigcup_{i=1}^{\infty} S_{M_i}$ dense in S with respect to the new norm $|||\cdot|||$, then it is clear that $\|\tau_{M_i}\|_{L^2[0,1]}$ *does* tend to zero as $i \to \infty$. In this case, the bound (8.11) of Theorem 17 is a useful one.

§ 9. Numerical Results

In this section, we discuss the numerical results we have obtained for some concrete examples by using the particular subspaces described in §§ 5, 6, and 7 in the Rayleigh-Ritz procedure. Let us however first summarize the results of §§ 5, 6, and 7 by comparing the *asymptotic* error estimates in terms of the total number of parameters associated with each of the subspaces of S that we have considered. The total number of such parameters is of course the dimension of these subspaces.

Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4) and (1.7), be of class $C^t[0, 1]$, with *fixed* t, $t \geqq 2m \geqq 2n$. If $\hat{p}_N(x)$ is the unique function which minimizes $F[w]$ over the polynomial subspace $P_0^{(N)}$, $N \geqq t$, then (cf. Theorem 6),

$$(9.1) \quad \|\hat{p}_N - \varphi\|_{L^\infty} = \mathcal{O}\left\{\frac{1}{(d_N+n-1)^{t-n}}\right\} \quad \text{as} \quad N \to \infty, \qquad d_N \equiv N+1-2n,$$

d_N being the dimension of $P_0^{(N)}$. Next, consider a sequence of smooth Hermite subspaces $\{H_0^{(m)}(\pi_i)\}_{i=1}^\infty$ with *fixed* m ($m \geqq n$), and assume that the partitions π_i: $0 = x_{i,0} < x_{i,1} < \cdots < x_{i,N_i+1} = 1$ satisfy the regularity conditions that $\bar{h}(\pi_i) \leqq K/(N_i+1)$ for all $i \geqq 1$, and $\lim_{i\to\infty} N_i = +\infty$. If $\hat{u}(\pi_i)$ is the element which uniquely minimizes $F[w]$ over $H_0^{(m)}(\pi_i)$, then (cf. Theorem 10),

$$(9.2) \quad \|\hat{u}(\pi_i) - \varphi\|_{L^\infty} = \mathcal{O}\left\{\frac{1}{(d_i+2n-m)^{2m-n}}\right\} \text{ as } i \to \infty, \qquad d_i \equiv m(N_i+2) - 2n,$$

where d_i is the dimension of $H_0^{(m)}(\pi_i)$. Next, let $\hat{v}_i$ be the unique element which minimizes $F[w]$ over the Hermite space $H_0(\pi; k; i)$, where π is a *fixed* partition of $[0, 1]$, k is a fixed positive integer with $k \geqq n$, and $i \geqq \max(t; 2k)$. Then (cf. Theorem 14),

$$(9.3) \quad \|\hat{v}_i - \varphi\|_{L^\infty} = \mathcal{O}\left\{\frac{1}{(d_i-k+2n)^{t-k}}\right\} \quad \text{as} \quad i \to \infty, \qquad d_i \equiv i(N+1) - Nk - 2n,$$

where d_i is the dimension of $H_0(\pi; k; i)$. Here, N is defined from the fixed partition π: $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$. Similarly, let $\hat{w}_i$ be the unique element which minimizes $F[w]$ over the Hermite space $H_0(\pi_i; k; m)$, where $\{\pi_i\}_{i=1}^\infty$ is a sequence of partitions of $[0, 1]$, and k and m are *fixed* positive integers such that $k \geqq n$ and $m \geqq \max(t; 2k)$. Then, for $\bar{h}(\pi_i) \leqq K/(N_i+1)$ for all $i \geqq 1$, and $\lim_{i\to\infty} N_i = +\infty$, we have (cf. Theorem 14)

$$(9.4) \quad \|\hat{w}_i - \varphi\|_{L^\infty} = \mathcal{O}\left\{\frac{1}{(d_i-k+2n)^{t-n}}\right\} \text{ as } i \to \infty, \quad d_i \equiv m(N_i+1) - N_i k - 2n,$$

where d_i is the dimension of $H_0(\pi_i; k; m)$. Finally, consider a sequence of spline subspaces $\{Sp_0^{(m)}(\pi_i)\}_{i=1}^\infty$ where m is fixed ($m \geqq n$), and the partitions π_i, as above, satisfy the regularity conditions that $\bar{h}(\pi_i) \leqq K/(N_i+1)$ for all $i \geqq 1$, with $\lim_{i\to\infty} N_i = +\infty$. If $\hat{z}_i(x)$ is the unique element which minimizes $F[w]$ over $Sp_0^{(m)}(\pi_i)$, then (cf. Theorem 16),

$$(9.5) \quad \|\hat{z}_i - \varphi\|_{L^\infty} = \mathcal{O}\left\{\frac{1}{(d_i+2n-2m+1)^{2m-1-n}}\right\} \text{ as } i \to \infty, \quad d_i \equiv N_i + 2(m-n),$$

where d_i is the dimension of $Sp_0^{(m)}(\pi_i)$.

Thus, in the special case in which $t = 2m$ and $k = n$, we observe, surprisingly enough, that these theoretical error estimates asymptotically are all of the form $d^{-(2m-n)}$, where d is the dimension of the subspace of S considered. Consequently, the selection of the particular subspace for practical computation must depend upon factors such as the relative ease of programming and ease of solving the resultant nonlinear matrix Eq. (3.16).

However, in the special case in which $t > 2m$ and $k = n$, it is clear that the theoretical asymptotic error estimates favor polynomial subspaces, $P_0^{(N)}$, and the

Hermite subspaces, $H_0(\pi; k; m)$, over the smooth Hermite subspaces, $H_0^{(m)}(\pi)$, and the spline subspaces, $Sp_0^{(m)}(\pi)$. Moreover, we must report that the numerical experiments indicated that the polynomial subspaces and Hermite subspaces led to algorithms which were easier to program, nonlinear matrix equations which were easier to solve, and numerical results which were in general, for a fixed amount of computing time, more accurate than the smooth Hermite subspaces and the spline subspaces.

We now consider in detail the numerical solution of particular examples of (1.1)–(1.2). As our first example, consider

(9.6) $$D^2 u(x) = e^{u(x)}, \qquad 0 < x < 1 \quad \text{with} \quad u(0) = u(1) = 0.$$

Thus, we verify that (1.4) is valid with $K = \frac{1}{2}$, $\beta = 0$. Similarly, Λ of (1.6) is π^2, and we can choose γ in (1.7) to be zero. The unique solution of (9.6) is [2, p. 41]:

(9.7) $$\varphi(x) = -\ln 2 + 2\ln\{c \sec[c(x - 1/2)/2]\}, \qquad c \doteqdot 1.3360557,$$

which minimizes the functional (cf. (2.2))

(9.8) $$F[w] = \int_0^1 \{\tfrac{1}{2}(D w(x))^2 + e^{w(x)} - 1\}\, dx, \qquad w \in S.$$

The numerical results of minimizing this functional $F[w]$ over the *polynomial subspaces* $P_0^{(N)}$ are summarized in Table 1.1. In this particular case, the basis functions $\{w_k(x)\}_{k=1}^{N-1}$ of $P_0^{(N)}$ were selected to be indefinite integrals of Legendre polynomials (cf. (5.14)), so that the matrix A of (5.11) is in this case just the identity matrix. The iterative method selected here, to approximately solve in succession the single nonlinear equation (cf. (5.11))

$$u_i + g_i(u_1, \ldots, u_i, \ldots, u_n) = 0, \qquad 1 \leqq i \leqq N-1,$$

in the single unknown u_i, was simply one step of NEWTON's method, which is known to be convergent [*30*, *35*].

The solution $\varphi(x)$ of (9.6) is from (9.7) analytic in an open set containing the interval [0, 1]. More precisely, $\varphi(x)$ is analytic in the ellipse with foci $x_0 = 0$, $x_1 = +1$ and with semi axes 4.7 and 4.6. Consequently [*26*, p. 76], the constant μ of Theorem 7 is $\mu = 0.107$. Thus, from Theorem 8, the error $\|\hat{p}_{N+1} - \varphi\|_{L^\infty}$ is roughly 0.107 times $\|\hat{p}_N - \varphi\|_{L^\infty}$ for N large, which agrees quite well with the results of Table 1.1.

Table 1.1
Polynomial subspaces $P_0^{(N)}$

N in $P_0^{(N)}$	$\dim(P_0^{(N)})$	$\|\hat{p}_N - \varphi\|_{L^\infty}$
3	2	$4.23 \cdot 10^{-4}$
5	4	$3.12 \cdot 10^{-6}$
7	6	$5.03 \cdot 10^{-8}$

Table 1.2
Smooth Hermite subspaces $H_0^{(2)}(\pi(h))$

h	$\dim(H_0^{(2)}(\pi(h)))$	$\|\hat{w}_h - \varphi\|_{L^\infty}$
1/3	6	$1.19 \cdot 10^{-5}$
1/4	8	$4.48 \cdot 10^{-6}$
1/5	10	$3.69 \cdot 10^{-6}$

The numerical results of minimizing the associated functional $F[w]$ of (9.8) over the *smooth Hermite space* $H_0^{(2)}(\pi)$, consisting of piecewise cubic polynomials, are given in Table 1.2. Here, $\pi = \pi(h)$ was chosen to be a uniform partition of [0, 1] of mesh-size h. The associated matrix problem of (5.11) was solved using a nonlinear point successive overrelaxation method. Again, one step of NEWTON's method was used to approximately solve in succession the nonlinear equations in a single unknown associated with the nonlinear point successive overrelaxation method, and this procedure is known to be convergent [35].

The numerical results of minimizing the associated functional $F[w]$ of (9.8) over the *non-smooth Hermite space* $H_0(\pi(h); 1; 4)$ and the *cubic spline subspace* $Sp_0^{(2)}(\pi(h))$ are given respectively in Tables 1.3 and 1.4. In both cases, $\pi = \pi(h)$ is a uniform partition of mesh-size h of $[0, 1]$. In both cases, the nonlinear point successive overrelaxation method, previously described, was employed to solve the resulting nonlinear matrix equations.

Table 1.3. *Non-smooth Hermite subspaces* $H_0(\pi(h); 1; 4)$

h	$\dim(H_0(\pi(h); 1; 4))$	$\|\hat{w}_h - \varphi\|_{L^\infty}$
1/2	5	$2.87 \cdot 10^{-5}$
1/3	8	$6.27 \cdot 10^{-6}$
1/4	11	$2.03 \cdot 10^{-6}$
1/5	14	$9.13 \cdot 10^{-7}$

Table 1.4. *Cubic spline subspaces* $Sp_0^{(2)}(\pi(h))$

h	$\dim(Sp_0^{(2)}(\pi(h)))$	$\|\hat{w}_h - \varphi\|_{L^\infty}$
1/4	5	$9.16 \cdot 10^{-6}$
1/6	7	$1.72 \cdot 10^{-6}$
1/8	9	$7.71 \cdot 10^{-7}$

As our second example, consider

$$(9.9) \qquad D^2 u(x) = \tfrac{1}{2}(u(x) + x + 1)^3, \quad 0 < x < 1 \quad \text{with} \quad u(0) = u(1) = 0.$$

In this example, we verify again that (1.4) is valid for $K = \frac{1}{2}$, $\beta = 0$, $\Lambda = \pi^2$, and that we can choose γ in (1.7) to be zero. The unique solution of (9.9) is

$$(9.10) \qquad \varphi(x) = \{2/(2-x)\} - x - 1, \quad 0 \leq x \leq 1.$$

The iterative techniques used to minimize the associated functional

$$(9.11) \qquad F[w] = \int_0^1 \left\{ \tfrac{1}{2}(D w(x))^2 + \int_0^{w(x)} [\tfrac{1}{2}(\eta + x + 1)^3]\, d\eta \right\} dx, \qquad w \in S,$$

over the different subspaces of S are the same as those previously described, and the numerical results are presented in Tables 2.1–2.5.

Table 2.1
Polynomial subspaces $P_0^{(N)}$

N in $P_0^{(N)}$	$\dim(P_0^{(N)})$	$\|\hat{p}_N - \varphi\|_{L^\infty}$
3	2	$3.76 \cdot 10^{-3}$
5	4	$1.10 \cdot 10^{-4}$
7	6	$3.29 \cdot 10^{-6}$

Table 2.2
Smooth Hermite subspaces $H_0^{(1)}(\pi(h))$

h	$\dim(H_0^{(1)}(\pi(h)))$	$\|\hat{w}_h - \varphi\|_{L^\infty}$
1/5	4	$1.45 \cdot 10^{-2}$
1/10	9	$4.22 \cdot 10^{-3}$
1/20	19	$1.15 \cdot 10^{-3}$

Table 2.3. *Smooth Hermite subspaces* $H_0^{(2)}(\pi(h))$

h	$\dim(H_0^{(2)}(\pi(h)))$	$\|\hat{w}_h - \varphi\|_{L^\infty}$
1/3	6	$1.89 \cdot 10^{-4}$
1/4	8	$7.43 \cdot 10^{-5}$
1/5	10	$3.59 \cdot 10^{-5}$

Table 2.4. *Non-smooth Hermite subspaces* $H_0(\pi(h); 1; 4)$

h	$\dim(H_0(\pi(h); 1; 4))$	$\|\hat{w}_h - \varphi\|_{L^\infty}$
1/3	8	$1.46 \cdot 10^{-4}$
1/4	11	$5.46 \cdot 10^{-5}$
1/5	14	$2.51 \cdot 10^{-5}$

As in our previous example, the solution $\varphi(x)$ of (9.9) is from (9.10) analytic in some open set containing the interval $[0, 1]$, and the associated constant μ of Theorem 8 can be shown to be $\mu = 0.172$. Again, this agrees well with the numerical results of Table 2.1.

Table 2.5. *Cubic spline subspaces* $Sp_0^{(2)}(\pi(h))$

h	$\dim(Sp_0^{(2)}(\pi(h)))$	$\|\hat{w}_h - \varphi\|_{L^\infty}$
1/4	5	$9.10 \cdot 10^{-5}$
1/6	7	$2.68 \cdot 10^{-5}$
1/8	9	$7.96 \cdot 10^{-6}$

To broaden the scope of these comparisons, we now take as our next example the *linear* problem

$$D^2 u(x) = 4u(x) + 4\cosh 1, \quad u(0) = u(1) = 0. \tag{9.12}$$

In this case, we again verify that (1.4) is valid with $K = \frac{1}{2}$, $\beta = 0$, that $\Lambda = \pi^2$, and that γ in (1.7) can be chosen to be zero. The unique solution of (9.12) is

$$\varphi(x) = \cosh(2x - 1) - \cosh 1, \qquad 0 \leqq x \leqq 1, \tag{9.13}$$

which minimizes the functional

$$F[w] = \int_0^1 \left\{ \tfrac{1}{2}(Dw(x))^2 + \int_0^{w(x)} [4\eta + 4\cosh 1]\, d\eta \right\} dx, \qquad w \in S. \tag{9.14}$$

The minimization of $F[w]$ over the various subspaces of S is now considerably simpler since the associated matrix equation to be solved is *linear* (cf. (3.16)). In fact, for the polynomial subspaces $P_0^{(N)}$, this linear matrix equation is *trivial* to solve for the choice of indefinite integrals of Legendre polynomials as basis elements in $P_0^{(N)}$, as this choice gives an orthonormal basis for $P_0^{(N)}$. For the other subspaces of S, either the standard linear point successive overrelaxation method or Gaussian elimination was used to solve the associated matrix equations. For band matrices which are positive definite and symmetric, it is known [*45*] that Gaussian elimination is quite stable with respect to rounding errors. The numerical results are listed in Tables 3.1—3.4.

Table 3.1
Polynomial subspaces $P_0^{(N)}$

N in $P_0^{(N)}$	$\dim(P_0^{(N)})$	$\|\hat{w}_N - \varphi\|_{L^\infty}$
3	2	$8.02 \cdot 10^{-3}$
5	4	$6.72 \cdot 10^{-5}$
7	6	$3.17 \cdot 10^{-7}$

Table 3.2
Smooth Herite subspaces $H_0^{(2)}(\pi(h))$

h	$\dim(H_0^{(2)}(\pi(h)))$	$\|\hat{w}_h - \varphi\|_{L^\infty}$
1/5	10	$3.66 \cdot 10^{-5}$
1/10	20	$2.66 \cdot 10^{-6}$
1/20	40	$2.31 \cdot 10^{-7}$

Table 3.3. *Non-smooth Hermite subspaces* $H_0(\pi(h); 1; 4)$

h	$\dim(H_0(\pi(h); 1; 4))$	$\|\hat{w}_h - \varphi\|_{L^\infty}$
1/3	8	$1.26 \cdot 10^{-4}$
1/4	11	$4.20 \cdot 10^{-5}$
1/5	14	$1.78 \cdot 10^{-5}$

Table 3.4. *Cubic spline subspaces* $Sp_0^{(2)}(\pi(h))$

h	$\dim(Sp_0^{(2)}(\pi(h)))$	$\|\hat{w}_h - \varphi\|_{L^\infty}$
1/5	6	$4.23 \cdot 10^{-5}$
1/7	8	$1.71 \cdot 10^{-5}$
1/9	10	$5.80 \cdot 10^{-6}$

For this linear one-dimensional boundary value problem, there are a number of known discrete methods, such as Collatz's *Mehrstellenverfahren* [12, p. 164] (referred to in [12] as a *Hermite* method) and the Bramble and Hubbard five-point scheme [7] which have $\mathcal{O}(h^4)$ accuracy at the mesh points. For comparison, we have solved the corresponding discrete approximations for these methods, and the results are given in Tables 3.5 and 3.6.

Table 3.5. Collatz Mehrstellenverfahren

h	Unknowns	$\max_i \lvert\varphi(ih)-w_i\rvert$
1/5	4	$2.56\cdot 10^{-5}$
1/10	9	$1.65\cdot 10^{-6}$
1/20	19	$4.10\cdot 10^{-8}$

Table 3.6. Bramble-Hubbard

h	Unknowns	$\max_i \lvert\varphi(ih)-w_i\rvert$
1/5	4	$2.06\cdot 10^{-3}$
1/10	9	$1.64\cdot 10^{-4}$
1/20	19	$1.20\cdot 10^{-5}$

For this linear problem, the relatively high accuracy of these discrete methods at the mesh points is not surprising. In fact, if the right-hand side of (9.12) were independent of u, for the piecewise linear smooth Hermite subspace $H_0^{(1)}(\pi)$, π arbitrary, the unique element of $\hat{w}(x)$ which minimizes $F[w]$ over $H_0^{(1)}(\pi)$ would be *infinitely accurate* at each joint of π, i.e., if $\pi\colon 0=x_0<x_1<\cdots<x_{N+1}=1$, then Rose [34] proved that

$$\hat{w}(x_i)=\varphi(x_i),\qquad 0\leqq i\leqq N+1. \tag{9.15}$$

To give a somewhat different proof of this, let $\varphi(x)$ denote the unique solution of the linear problem $D^2u(x)=f(x)$, $0<x<1$, where $u(0)=u(1)=0$, and let $\hat{w}(x)\equiv\sum_{i=1}^{N}u_i w_i(x)$ be the unique function which minimizes the associated functional $F[w]$ over the subspace $H_0^{(1)}(\pi)$ of piecewise linear functions. Here, the $w_i(x)$ are the „chapeau" functions of Fig. 1A. With the partitioning π, then (4.10) gives us

$$\int_0^1\{D(\varphi(x)-\hat{w}(x))\}\,Dw_i(x)\,dx=0=\int_{x_{i+1}}^{x_{i+1}}\{D(\varphi(x)-\hat{w}(x))\}\,Dw_i(x)\,dx,\qquad 1\leqq i\leqq N, \tag{9.16}$$

since $w_i(x)$ is zero outside of $[x_{i-1},x_{i+1}]$. As $Dw_i(x)$ is a constant and $D^2w_i(x)\equiv 0$ on each open subinterval of π, then integration by parts in (9.16) yields

$$-\frac{1}{h_i}\varepsilon_{i+1}+\left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)\varepsilon_i-\frac{1}{h_{i-1}}\varepsilon_{i-1}=0,\qquad 1\leqq i\leqq N, \tag{9.17}$$

where $\varepsilon_j=\varphi(x_j)-\hat{w}(x_j)$ and $h_j=x_{j+1}-x_j$, $0\leqq j\leqq N$. But this can be expressed as the homogeneous vector equation $A\boldsymbol{\varepsilon}=\mathbf{0}$, where the $N\times N$ matrix A is the matrix of (3.16), i.e., it has entries $a_{i,j}\equiv\langle w_i,w_j\rangle_0$. As such, we know from the results of § 3 that A is nonsingular, and consequently $\varepsilon_j=0$ for all $0\leqq j\leqq N+1$, which establishes (9.15). We remark that the basis elements $\{w_i(x)\}_{i=1}^N$ of $H_0^{(1)}(\pi)$ form a *patch basis* in the sense of Rose [34]. See also [23].

It is also worth pointing out that the infinite accuracy at the mesh points in (9.15) further gives us the improved result for linear problems that

$$\|\varphi-\hat{w}\|_{L^\infty}\leqq KM\|D^2\varphi\|_{L^\infty}(\bar{h}(\pi))^2,\qquad \hat{w}\in H_0^{(1)}(\pi), \tag{9.18}$$

as opposed to the result of Theorem 10 for $m=1$, $n=1$ which establishes the same result but with the exponent of $\bar{h}(\pi)$ one less. To prove this, the result of (9.15) gives $\hat{w}(x)$ as the *interpolation* of $\varphi(x)$ in $H_0^{(1)}(\pi)$, and as such, we simply apply the case $k=0$, $m=1$ in (6.3) of Theorem 9 to establish (9.18). Similar results can be obtained for some nonlinear problems as well [9].

As our final example, we consider the fourth order linear problem

(9.19) $$D^4 u(x) = (x^4 + 14x^3 + 49x^2 + 32x - 12)\,e^x, \qquad 0 < x < 1,$$

with

(9.19′) $$u(0) = Du(0) = u(1) = Du(1) = 0,$$

which corresponds to the bending of a thin beam, clamped at both ends. In this case, we verify that (1.5′) is valid with $l=1$, $K=1/\pi$ and $\beta=0$, that Λ is approximately 500, and that γ in (1.7) can be chosen to zero, since the problem is linear. The unique solution of (9.19)—(9.19′) is

(9.20) $$\varphi(x) = x^2(1-x)^2 e^x, \qquad 0 \leqq x \leqq 1,$$

which minimizes the functional

(9.21) $$F[w] = \int_0^1 \{\tfrac{1}{2}(D^2 w(x))^2 + (x^4 + 14x^3 + 49x^2 + 32x - 12)\,e^x w(x)\}\,dx, \quad w \in S.$$

To indicate some of the applications of the previous sections to this particular problem, consider the smooth Hermite subspace $H_0^{(2)}(\pi)$ of piecewise cubic polynomials. Then, Theorem 10 gives us for $m=2=n$ that

(9.22) $$\|\hat{w}_2(\pi) - \varphi\|_{L^\infty} \leqq KM\|D^4\varphi\|_{L^\infty}(\bar{h}(\pi))^2,$$

and

(9.23) $$\|D(\hat{w}_2(\pi) - \varphi)\|_{L^\infty} \leqq KM'\|D^4\varphi\|_{L^\infty}(\bar{h}(\pi))^2,$$

where $\hat{w}_2(\pi)$ is the unique element in $H_0^{(2)}(\pi)$ which minimizes $F[w]$ of (9.21) over $H_0^{(2)}(\pi)$. In Table 4.1, a uniform partition was used $(h=1/(N+1))$.

Table 4.1. *Smooth Hermite Subspace* $H_0^{(2)}(\pi(h))$

h	$\dim(H_0^{(2)}(\pi(h)))$	$\|\hat{w}_2(\pi(h)) - \varphi\|_{L^\infty}$	$\|D(\hat{w}_2(\pi(h)) - \varphi)\|_{L^\infty}$
1/5	8	$6.945 \cdot 10^{-4}$	$1.090 \cdot 10^{-2}$
1/7	12	$1.980 \cdot 10^{-4}$	$4.325 \cdot 10^{-3}$
1/9	16	$7.614 \cdot 10^{-5}$	$2.130 \cdot 10^{-3}$

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