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L-Splines*

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Abstract. In this paper, we study the problem of unique interpolation and approximation by a class of spline functions, L-splines, containing as special cases the deficient and generalized spline functions of AHLBERG, NILSON, and WALSH [3, 5, 6], the Chebyshevian spline functions of KARLIN and ZIEGLER [27], and the piecewise Hermite polynomial functions, as considered in [17]. We first give sufficient conditions for unique interpolation by L-spline functions in Section 2. Then, we obtain new L^∞ and L^2 error estimates for interpolation by L-splines in Section 4, and show that these error estimates are, in a certain sense, sharp. In addition, we make a similar study for the g-splines of SCHOENBERG, cf. [44, 3], in Section 5. In Section 6, an application of these new error estimates is made to the analysis of the error made in the use of finite dimensional subspaces of L-splines and g-splines in the Rayleigh-Ritz procedure for the class of nonlinear two-point boundary value problems studied in [17].

Because of the rapid growth of the number of papers devoted to or connected with the topic of splines, we believe that a compilation of papers on splines for the reader's use is desirable, and such a list is found in the References at the end of this paper[1].

1. Introduction

For each positive integer m, let $K_2^m[a, b]$ denote the collection of all real-valued functions $u(x)$ defined on $[a, b]$ such that $u \in C^{m-1}[a, b]$, and such that $D^{m-1}u(x) \equiv u^{(m-1)}(x)$ is absolutely continuous, with $D^m u(x) \in L^2[a, b]$. Let L be the m-th order linear differential operator defined by

$$L[u(x)] = \sum_{j=0}^{m} a_j(x)\, D^j u(x) \tag{1.1}$$

for any $u \in C^m[a, b]$. We assume that $a_j(x)$ is in $K_2^m[a, b]$ for all $0 \leqq j \leqq m$, and we further assume that there exists a positive real number ω such that

$$a_m(x) \geqq \omega > 0 \quad \text{for all} \quad x \in [a, b]. \tag{1.2}$$

It is a well-known result (cf. [3', p. 63]) from the classical theory of ordinary differential equations that the equation

$$L[u] = 0 \tag{1.3}$$

possesses m linearly independent solutions $u_1(x), u_2(x), \ldots, u_m(x)$ in $C^m[a, b]$, and the m-th order Wronskian

$$W(x; u_1, u_2, \ldots, u_m) = \det \begin{bmatrix} u_1(x) & u_2(x) & \ldots & u_m(x) \\ D u_1(x) & D u_2(x) & \ldots & D u_m(x) \\ \vdots & & & \vdots \\ D^{m-1} u_1(x) & & \cdots & D^{m-1} u_m(x) \end{bmatrix} \tag{1.4}$$

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[1] Papers not specifically concerned with splines are referred to in the text by [1', 2'], etc.

is not zero for any $x \in [a, b]$. The formal adjoint, L^*, of L is defined by

$$L^*[v(x)] = \sum_{j=0}^{m} (-1)^j D^j \{a_j(x)\, v(x)\}. \tag{1.5}$$

Associated with L is the bilinear concomitant $P(u, v)$ (cf. [1', p. 86]), defined by

$$P(u, v) = \sum_{j=0}^{m-1} D^{m-j-1} u(x) \sum_{k=0}^{j} (-1)^k D^k \{a_{m-j+k}(x)\, v(x)\}, \tag{1.6}$$

$u, v \in K_2^m[a, b]$. As a direct consequence of the definition of $P(u, v)$, we have Green's formula

$$\int_{\alpha}^{\beta} \{v L[u] - u L^*[v]\}\, dx = P\big(u(\beta), v(\beta)\big) - P\big(u(\alpha), v(\alpha)\big) \tag{1.7}$$

for any $\alpha, \beta \in [a, b]$, and any $u, v \in K_2^m[a, b]$.

Let π: $a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b$ be a partition of the interval $[a, b]$, and let $z = (z_1, z_2, \ldots, z_N)$, the *incidence vector associated with* π, be an N-vector with positive integer components, each less than or equal to m, i.e., $1 \leq z_i \leq m$ for $1 \leq i \leq N$.

Definition 1. The real-valued function $s(x)$ defined on $[a, b]$ is said to be an *L-spline* for π and z if and only if $s(x) \in K_2^{2m}[x_i, x_{i+1}]$ for each i, $0 \leq i \leq N$,

$$L^* L[s(x)] = 0 \tag{1.8}$$

for almost all $x \in (x_i, x_{i+1})$, for each i, $0 \leq i \leq N$, and

$$D^k s(x_i-) = D^k s(x_i+) \quad \text{for} \quad 0 \leq k \leq 2m - 1 - z_i, \qquad 1 \leq i \leq N. \tag{1.9}$$

The class of all L-splines for fixed π and z is denoted by $Sp(L, \pi, z)$.

We remark that if $z_1 = z_2 = \cdots = z_N = 1$, then $Sp(L, \pi, z)$ coincides with the space of *generalized spline functions* of AHLBERG, NILSON, and WALSH [5], while if $z_1 = z_2 = \cdots = z_N = q$, $Sp(L, \pi, z)$ coincides with the space of *deficient spline functions* of deficiency q of AHLBERG, NILSON, and WALSH [6]. Furthermore, when $z_1 = z_2 = \cdots = z_m = m$ and $L = D^m$, then $Sp(L, \pi, z)$ coincides with the Hermite space $H^{(m)}(\pi)$ of piecewise-polynomial functions (cf. [17]).

If $f(x)$ is a given function in $C^{m-1}[a, b]$, we can define four basic types of interpolates of $f(x)$ in $Sp(L, \pi, z)$. In so doing, it is convenient to *augment* the incidence vector z with positive integer components z_0 and z_{N+1}, where $1 \leq z_0$, $z_{N+1} \leq m$.

Definition 2. Given $f(x) \in C^{m-1}[a, b]$, a function $s(x) \in Sp(L, \pi, z)$ is said to be a *$Sp(L, \pi, z)$-interpolate of $f(x)$*

of Type I if (i) $D^k s(x_i) = D^k f(x_i)$, $\quad 0 \leq k \leq z_i - 1$, $\quad 1 \leq i \leq N$, and

(ii) $D^k s(x_i) = D^k f(x_i)$, $\quad 0 \leq k \leq m - 1$, $\quad i = 0$ and $N + 1$;

of Type II if (i) $D^k s(x_i) = D^k f(x_i)$, $\quad 0 \leq k \leq z_i - 1$, $\quad 1 \leq i \leq N$,

(ii) $D^k s(x_i) = D^k f(x_i)$, $\quad 0 \leq k \leq z_i - 1$, $\quad i = 0$ and $N + 1$,

(iii) if $z_i < m$, then $\sum_{k=0}^{j} (-1)^k D^k \{a_{m-j+k}(x_i)\, L[s(x_i)]\} = 0$

for $0 \leq j \leq m - 1 - z_i$, $\quad i = 0$ and $N + 1$, of Type III if (i) $D^k s(x_i) = D^k f(x_i), \quad 0 \leqq k \leqq z_i - 1, \quad 1 \leqq i \leqq N,$

(ii) $D^k s(x_i) = D^k f(x_i), \quad 0 \leqq k \leqq z_i - 1, \quad i = 0 \text{ or } N+1,$

(iii) if $z_i < m$, then $\sum_{k=0}^{j} (-1)^k D^k \left(a_{m-j+k}(x_i) \{L[f(x_i)] - L[s(x_i)]\}\right)$
$= 0$ for $0 \leqq j \leqq m-1-z_i, \quad i = 0 \text{ or } N+1$, and

of Type IV if (i) $f \in C_p^{2m-1}[a, b]$, i.e., $f(x) \in C^{2m-1}[a, b]$, and
$D^k f(a) = D^k f(b)$ for $0 \leqq k \leqq 2m-1$,

(ii) $a_j \in C_p^{m-1}[a, b]$ for all $0 \leqq j \leqq m$,

(iii) $D^k s(x_i) = D^k f(x_i), \quad 0 \leqq k \leqq z_i - 1, \quad 1 \leqq i \leqq N,$

(iv) $D^k s(x_i) = D^k f(x_i), \quad 0 \leqq k \leqq z_i - 1, \quad i = 0$, and

(v) $D^k s(a) = D^k s(b), \quad z_0 \leqq k \leqq 2m - 1.$

In essence, interpolation of Type I fixes the augmented incidence components z_0 and z_{N+1} to be m, i.e., we adopt the convention that $z_0 = z_{N+1} = m$ in this case. For Type II and Type III interpolation, z_0 and z_{N+1} are arbitrary with $1 \leqq z_0$, $z_{N+1} \leqq m$, while for Type IV interpolation, z_0 is arbitrary with $1 \leqq z_0 \leqq m$, and $z_0 = z_{N+1}$.

Definition 3. Given a real-valued function $f(x) \in C^{m-1}[a, b]$, and given L, π, and z, the *L-Hermite problem of Type I* (resp. Type II, III, or IV) is to find a function $u(x) \in Sp(L, \pi, z)$ such that

$$L[u(x)] = 0 \quad \text{for all} \quad x \in [a, b], \tag{1.10}$$

and such that $u(x)$ is an $Sp(L, \pi, z)$-interpolate of f of Type I (resp. Type II, III, or IV). The L-Hermite problem is said to be *well-posed*[2] for $Sp(L, \pi, z)$ if and only if it has *at most* one solution.

We now determine sufficient conditions for the L-Hermite problem of Type I (resp. Type II, III, or IV) to be well-posed. The well-posed nature of the L-Hermite problem is, as we shall see in Section 2, fundamental to the question of unique interpolation in $Sp(L, \pi, z)$.

Theorem 1. If the coefficients $a_j(x)$ of (1.1) are not all identically zero in $[a, b]$ for $0 \leqq j \leqq m-1$, let c be the positive zero of

$$\frac{M_m c^m}{m!} \left\{ \frac{(m-1)^{m-1}}{m^m} \right\} + \frac{M_{m-1} c^{m-1}}{(m-1)!} + \cdots + M_1 c - 1 = 0, \tag{1.11}$$

where

$$M_j \equiv \sup_{x \in [a, b]} \left| \frac{a_{m-j}(x)}{a_m(x)} \right|, \quad 1 \leqq j \leqq m. \tag{1.12}$$

Otherwise, define $c \equiv b - a$. Given π and z, assume that the components of the augmented incidence vector z for Type I (resp. Type II, III, or IV) interpolation,

[2] In the special case $L = D^m$, we remark that our term "well-posed" corresponds to the term "m-poised" of Schoenberg [47].

for some $0 \leqq p \leqq q \leqq N+1$, satisfy the inequality

$$\sum_{l=p}^{q} z_l \geqq m \quad \text{where} \quad x_q - x_p \leqq c. \tag{1.13}$$

Then, the L-Hermite problem of Type I (resp. Type II, III, or IV) is well-posed for $Sp(L, \pi, z)$.

Proof. It suffices to show that if $f(x) \equiv 0$ in $[a, b]$, then the only solution of the L-Hermite problem of Type I (resp. Type II, III, or IV) for $Sp(L, \pi, z)$ is $u(x) \equiv 0$. Using a result from the theory of ordinary differential equations [3', p. 346; 4'], any solution $u(x)$ of $L[u]=0$ having m zeros (counting multiplicities) in any subinterval of $[a, b]$ of length c or less must be identically zero. By definition, if $L[u]=0$ and $u(x)$ is a $Sp(L, \pi, z)$-interpolate of Type I (resp. Type II, III, or IV) of $f(x) \equiv 0$, then $u(x)$ has a zero of order z_l at each x_l, $0 \leqq l \leqq N+1$. Thus, by hypothesis (1.13), $u(x)$ has at least m zeros in an interval of length c or less, and consequently $u(x) \equiv 0$. Q.E.D.

For any z_i with $z_i = m$ where $0 \leqq i \leqq N+1$, then (1.13) is trivially satisfied, which gives us the

Corollary 1. Given $Sp(L, \pi, z)$, then the L-Hermite problem of some particular type is well-posed for $Sp(L, \pi, z)$ if some component z_i of the associated augmented incidence vector satisfies $z_i = m$. In particular, the L-Hermite problem of Type I is always well-posed for $Sp(L, \pi, z)$.

If we have a sequence of partitions $\{\pi_i\}_{i=1}^{\infty}$ on $[a, b]$, where π_i: $a = x_0^{(i)} < x_1^{(i)} < \cdots < x_{N_i+1}^{(i)} = b$, and if we define $\bar{\pi}_i = \max_{0 \leqq j \leqq N_i} (x_{j+1}^{(i)} - x_j^{(i)})$, suppose that $\lim_{i \to \infty} \bar{\pi}_i = 0$. If $\{z^{(i)}\}_{i=1}^{\infty}$ is *any* sequence of augmented incidence vectors associated with $\{\pi_i\}_{i=1}^{\infty}$, then, since the positive zero c of (1.11) is fixed, independent of i, and any component of any incidence vector is at least unity, then it is clear that there exist q_i and p_i with $0 \leqq p_i \leqq q_i \leqq N_i + 1$ such that

$$\sum_{l=p_i}^{q_i} z_l^{(i)} \geqq m \quad \text{with} \quad x_{q_i}^{(i)} - x_{p_i}^{(i)} \leqq c \tag{1.14}$$

for all i sufficiently large. This gives us

Corollary 2. Given any sequence of partitions $\{\pi_i\}_{i=1}^{\infty}$ with $\lim_{i \to \infty} \bar{\pi}_i = 0$, and any sequence $\{z^{(i)}\}_{i=1}^{\infty}$ of augmented incidence vectors associated with $\{\pi_i\}_{i=1}^{\infty}$, the L-Hermite problem of Type I, Type II, Type III, or Type IV is well-posed for $Sp(L, \pi_i, z^{(i)})$ for all i sufficiently large.

If we assume that the differential operator L has Pólya's property W of (1.15), we obtain a result analogous to (1.13) of Theorem 1.

Theorem 2. Let the differential operator L have property W on some interval $(\alpha, \beta) \subset [a, b]$, i.e., $L[u]=0$ has m solutions $u_1(x), u_2(x), \ldots, u_m(x)$ such that

$$W(x; u_1, \ldots, u_k) = \det \begin{bmatrix} u_1(x) & u_2(x) & \ldots & u_k(x) \\ D u_1(x) & D u_2(x) & \ldots & D u_k(x) \\ \vdots & & & \vdots \\ D^{k-1} u_1(x) & & \ldots & D^{k-1} u_k(x) \end{bmatrix} \neq 0 \tag{1.15}$$

$$\text{for all} \quad x \in (\alpha, \beta), \quad \text{for all} \quad 1 \leqq k \leqq m.$$

Given π and z, let the components of the augmented incidence vector for Type I (resp. Type II, III, or IV) interpolation, for some $0 \leqq p \leqq q \leqq N+1$, satisfy the inequality

$$(1.16) \qquad \sum_{\ell=p}^{q} z_\ell \geqq m \quad \text{where} \quad x_p, x_q \in (\alpha, \beta).$$

Then, the L-Hermite problem of Type I (resp. Type II, III, or IV) is well-posed for $Sp(L, \pi, z)$.

Proof. The argument is like that of Theorem 1. If $L[u]=0$ and $u(x)$ is a $Sp(L, \pi, z)$-interpolate of Type I (resp. Type II, III, or IV) of $f(x)\equiv 0$, then $u(x)$ has a zero of order z_ℓ at each x_ℓ, $0 \leqq \ell \leqq N+1$. With the hypothesis of (1.16), it is known [3′, p. 67; 5′] that if $u(x)$ has m zeros in (α, β), then $u(x)\equiv 0$. Q.E.D.

If we choose any point μ interior to $[a, b]$ and let $u_1(x), \ldots, u_m(x)$ be solutions of $L[u]=0$ with $D^\ell u_j(\mu)=\delta_{j-1,\ell}$, $1 \leqq j \leqq m$, $0 \leqq \ell \leqq m-1$, then $W(\mu; u_1, \ldots, u_k)=1$ for all $1 \leqq k \leqq m$. Hence, by continuity, there always exists an interval $(\alpha, \beta) \subset [a, b]$ with $\alpha < \mu < \beta$ such that L has property W on (α, β). If $\beta - \alpha > c$ where c is defined by (1.11), then (1.16) is a weaker condition than that of (1.13), and Theorem 2 gives an improved result.

2. Existence and Uniqueness

The following result is a generalization of Theorem 4 of [5].

Theorem 3. Let π, z, and $f \in C^{m-1}[a, b]$ be given. If the L-Hermite problem of Type I (resp. Type II, III, or IV) is well-posed for $Sp(L, \pi, z)$, then there exists a unique function $s(x) \in Sp(L, \pi, z)$ which is the $Sp(L, \pi, z)$-interpolate of $f(x)$ of Type I (resp. Type II, III, or IV).

Proof. Recalling our original assumption that $a_j(x) \in K_2^m[a, b]$, $0 \leqq j \leqq m$, it follows that the coefficients $\beta_j(x)$ in $L^*L[v(x)] = \sum_{j=0}^{2m} \beta_j(x)\, D^j v(x)$ are all elements of $L^2[a, b]$. Thus (cf. [3′, p. 43]), there exist $2m$ linearly independent functions $v_j(x) \in K_2^{2m}[a, b]$, $1 \leqq j \leqq 2m$, with $L^*L[v_j(x)]=0$ almost everywhere in $[a, b]$, such that if $s(x)$ is a $Sp(L, \pi, z)$-interpolate of Type I (resp. Type II, III, or IV), then on each subinterval (x_i, x_{i+1}), $0 \leqq i \leqq N$, $s(x)$ can be expressed as $s(x) = \sum_{j=1}^{2m} \alpha_{i,j} v_j(x)$, i.e., $s(x)$ is determined by $2m$ coefficients $\alpha_{i,j}$ in each subinterval. Thus, the total number of coefficients determining $s(x)$ in $[a, b]$ is $2m(N+1)$.

We now calculate the number of *linear* equations which constrain these coefficients. At each interior mesh point x_i, the differentiability condition (1.9) yields $2m - z_i$ homogeneous conditions, and thus there are $\sum_{i=1}^{N}(2m - z_i)$ such equations in all. Next, if $s(x)$ is a $Sp(L, \pi, z)$-interpolate of $f(x)$, the conditions of Definition 2 impose $\sum_{i=1}^{N} z_i + 2m$ constraints in all, independent of type. Hence, the total number of constraint equations is $2mN - \sum_{i=1}^{N} z_i + \sum_{i=1}^{N} z_i + 2m = 2m(N+1)$. In other words, if $s(x)$ exists, it is obtained from a solution of $2m(N+1)$ linear equations in $2m(N+1)$ unknowns. To establish both the existence and uniqueness of $s(x)$, it suffices to show that if $f(x)\equiv 0$ on $[a, b]$, then $s(x)\equiv 0$ also for all $x\in[a, b]$.

Consider the integral

$$J=\int_a^b (L[s(x)])^2\,dx=\sum_{i=0}^{N}\int_{x_i}^{x_{i+1}}(L[s(x)])^2\,dx, \tag{2.1}$$

where $s(x)$ is any $Sp(L, \pi, z)$-interpolate of $f(x)\equiv 0$ of any type. With $u=s$ and $v=L[s]$ in (1.7), we can express J as

$$J=\sum_{i=0}^{N}\int_{x_i}^{x_{i+1}} s(x)\left(L^* L[s(x)]\right)dx+\sum_{i=0}^{N}\left\{P\left(s(x), L[s]\right)\Big|_{x=x_i}^{x=x_{i+1}}\right\}. \tag{2.2}$$

The first sum of (2.2) is zero since $s(x)$ is an L-spline (cf. (1.8)), and, using (1.6) and Definition 2, it can be verified that the second sum vanishes for all four types of interpolation. Thus, J is zero and consequently $L[s](x)=0$ almost everywhere in $[a, b]$. But, as $s(x)\in Sp(L, \pi, z)$ implies that $s\in K_2^{2m}[x_i, x_{i+1}]$ for each i, $0\leqq i\leqq N$, then surely $L[s](x)=0$ for $x\in(x_i, x_{i+1})$, $0\leqq i\leqq N$. Now, letting $u_1(x), u_2(x), \ldots, u_m(x)$ denote m linearly independent solutions of $L[u]=0$, we have $s(x)=\sum_{k=1}^{m}\beta_{i,k}u_k(x)$ for all $x\in(x_i, x_{i+1})$, $i=0, \ldots, N$. To show that $L[s(x)]=0$ for *all* $x\in[a, b]$, we must show that $\beta_{i,k}=\beta_{i+1,k}$, $i=0, \ldots, N-1$, $k=1, \ldots, m$. But, it follows from (1.9) of Definition 1 that $D^k s(x_i-)=D^k s(x_i+)$, $k=0, \ldots, m-1$, $i=1, \ldots, N$, since each z_i is at most m. Hence for each i, $0\leqq i\leqq N-1$, the m differences $\beta_{i,k}-\beta_{i+1,k}$ satisfy the m homogeneous linear equations

$$\sum_{k=1}^{m}(\beta_{i,k}-\beta_{i+1,k})\,D^j u_k(x_{i+1})=0, \qquad j=0, \ldots, m-1. \tag{2.3}$$

But the determinant of the coefficient matrix in (2.3) is exactly the Wronskian $W(x; u_1, u_2, \ldots, u_m)$ of (1.4), evaluated at the point $x=x_{i+1}$, which we know does not vanish. Hence, $\beta_{i,k}=\beta_{i+1,k}$, $i=0, \ldots, N-1$, $k=1, \ldots, m$, and $L[s(x)]=0$ for *all* $x\in[a, b]$. But, since the L-Hermite problem is well-posed, then $s(x)\equiv 0$ for all $x\in[a, b]$. Q.E.D.

In the special case that the differential operator L has property W on the interval (a, b), Theorems 2 and 3 give us that interpolation in $Sp(L, \pi, z)$ is unique, if $\sum_{l=1}^{N} z_l\geqq m$. Thus, Theorem 3 is a generalization of the basic results of KARLIN and ZIEGLER [27, Theorems 3 and 3′].

3. Integral Relations

The results of this section generalize the first and second integral relations of [6].

Theorem 4. Let $f(x)\in K_2^m[a, b]$, π, and z be given. If $s(x)$ is a $Sp(L, \pi, z)$-interpolate of f of Type I, II, or IV, then the following *first integral relation* is valid:

$$\int_a^b (L[f])^2\,dx=\int_a^b (L[f-s])^2\,dx+\int_a^b (L[s])^2\,dx. \tag{3.1}$$

Proof. Clearly,

$$(3.2)\qquad \int_a^b (L[f])^2\,dx = \int_a^b (L[f-s])^2\,dx + 2\int_a^b L[f-s]\cdot L[s]\,dx + \int_a^b (L[s])^2\,dx,$$

and the first integral relation of (3.1) follows if we can show that the middle term on the right-hand side of (3.2) vanishes. With $u=f-s$ and $v=L[s]$ in (1.7), we then have

$$(3.3)\qquad \begin{aligned}\int_a^b L[f-s]\,L[s]\,dx &= \sum_{i=0}^{N}\int_{x_i}^{x_{i+1}} (f-s)\,(L^*L[s])\,dx\\ &\quad+\sum_{i=0}^{N}\left\{P\,(f(x)-s(x),\,L[s])\Big|_{x_i}^{x_{i+1}}\right\}.\end{aligned}$$

As in the proof of Theorem 3, the first sum on the right-side vanishes since $s\in Sp(L,\pi,z)$, and the second sum vanishes if $s(x)$ is a $Sp(L,\pi,z)$-interpolate of f of Type I, Type II, or Type IV. Q.E.D.

In much the same manner, we obtain the following *second integral relation* if $f(x)\in K_2^{2m}[a,b]$.

Theorem 5. Let $f(x)\in K_2^{2m}[a,b]$, π, and z be given. If $s(x)$ is a $Sp(L,\pi,z)$-interpolate of $f(x)$ of Type I, III, or IV, then the following *second integral relation* is valid:

$$(3.4)\qquad \int_a^b (L[f-s])^2\,dx = \int_a^b (f-s)\,(L^*L[f])\,dx.$$

Proof. With $u=f-s$ and $v=L[f-s]$ in (1.7), we have

$$(3.5)\qquad \begin{aligned}\int_a^b (L[f-s])^2\,dx &= \int_a^b (f-s)\,(L^*L[f])\,dx\\ &\quad+\sum_{i=0}^{N}\left\{P\,(f(x)-s(x),\,L[f]-L[s])\Big|_{x_i}^{x_{i+1}}\right\}\end{aligned}$$

since $L^*L[s]\equiv 0$ almost everywhere on each interval (x_i, x_{i+1}), $i=0,\ldots,N$. But, as before in the proofs of Theorems 3 and 4, the second term on the right-hand side of (3.5) vanishes since $s(x)$ is a $Sp(L,\pi,z)$-interpolate of $f(x)$ of Type I, Type III, or Type IV. Q.E.D.

It is worth noting in Definition 2 that *only* the periodic boundary conditions of Type IV couple the boundary conditions at one end with the other. This means that we can in fact independently assign boundary conditions of Types I, II, or III at either end. Thus, it is clear that the result of Theorem 4 is equally valid for a hybrid interpolate of f in $Sp(L,\pi,z)$ with a Type I boundary condition at one end and a Type II boundary condition at the other end. A similar remark is also valid for Theorem 5.

4. Error Bounds

Let $\{\pi_i\}_{i=1}^{\infty}$ be any sequence of partitions of $[a,b]$, with $\lim_{i\to\infty}\bar{\pi}_i=0$. We then have the following error bounds which generalize Theorem 1 of [6] and Theorem 9 of [17].

Theorem 6. Let $f \in K_2^m[a, b]$, let $\{\pi_i\}_{i=1}^{\infty}$ be a sequence of partitions of $[a, b]$ with $\lim_{i\to\infty} \bar{\pi}_i = 0$, and let $\{z^{(i)}\}_{i=1}^{\infty}$ be any sequence of incidence vectors associated with $\{\pi_i\}_{i=1}^{\infty}$. Then, there exists a positive integer i_0 such that the $Sp(L, \pi_i, z^{(i)})$-interpolate $s_i(x)$ of f of Type I, II, or IV exists and is unique for any $i \geqq i_0$, and moreover, there exists a constant M_1, dependent on j and m but independent of i, such that

$$
(4.1)\qquad \begin{aligned} \|D^j(f - s_i)\|_{L^\infty[a,b]} &\leqq M_1 (\bar{\pi}_i)^{m-j-(\frac{1}{2})} \|L(f - s_i)\|_{L^2[a,b]} \\ &\leqq M_1 (\bar{\pi}_i)^{m-j-(\frac{1}{2})} \|Lf\|_{L^2[a,b]}, \end{aligned}
$$

for any j with $0 \leqq j \leqq m-1$, and any $i \geqq i_0$.

Proof. The first part of this result follows directly from Corollary 1 of Theorem 1 and Theorem 3. For the remainder of the proof, fix i large enough, say $i \geqq i_0$, so that $s_i(x)$, the $Sp(L, \pi_i, z^{(i)})$-interpolate of f of Type I, II, or IV, exists and is unique. If π_i: $a = x_0 < x_1 < \cdots < x_{N_i+1} = b$, then, as each component z_i of any augmented vector is at least unity, $f(x_j) - s_i(x_j) = 0$ for all $0 \leqq j \leqq N_i + 1$. Because $(f - s_i) \in C^{m-1}[a, b]$, we can apply Rolle's Theorem to $f(x) - s(x)$. Hence, there exist points $\{\xi_l^{(j)}\}_{l=0}^{N_i+1-j}$ in $[a, b]$ such that

$$
(4.2)\qquad \begin{cases} D^j f(\xi_l^{(j)}) - D^j s_i(\xi_l^{(j)}) = 0, \quad 0 \leqq l \leqq N_i + 1 - j, \quad 0 \leqq j \leqq m-1, \text{ where} \\ a \leqq \xi_0^{(j)} < \xi_1^{(j)} < \cdots < \xi_{N_i+1-j}^{(j)} \leqq b, \text{ and} \\ \xi_l^{(j)} \leqq \xi_l^{(j+1)} < \xi_{l+1}^{(j)} \quad \text{for all } 0 \leqq l \leqq N_i + 1 - j, \quad 0 \leqq j \leqq m-1. \end{cases}
$$

It is readily verified by induction that $|\xi_{l+1}^{(j)} - \xi_l^{(j)}| \leqq (j+1)\bar{\pi}_i$, $|a - \xi_0^{(j)}| \leqq (j+1)\bar{\pi}_i$, and $|b - \xi_{N_i+1-j}^{(j)}| \leqq (j+1)\bar{\pi}_i$ for any $0 \leqq j \leqq m-1$. Now, for each j with $0 \leqq j \leqq m-1$, let $x_j \in [a, b]$ be such that

$$
(4.3)\qquad |D^j(f(x_j) - s_i(x_j))| = \|D^j(f - s_i)\|_{L^\infty[a,b]}, \qquad 0 \leqq j \leqq m-1.
$$

Again, it is easily seen that there is a $\xi_k^{(j)}$ such that $|x_j - \xi_k^{(j)}| \leqq (j+1)\bar{\pi}_i$. Then, as $D^j[f(\xi_k^{(j)}) - s_i(\xi_k^{(j)})] = 0$, we have from (4.3) that

$$
\|D^j(f - s_i)\|_{L^\infty[a,b]} = \left| \int_{\xi_k^{(j)}}^{x_j} D^{j+1}(f(t) - s_i(t))\, dt \right|.
$$

For $j < m-1$, this integral is bounded above by $(j+1)\bar{\pi}_i \|D^{j+1}(f - s_i)\|_{L^\infty[a,b]}$. By repeating this argument, we obtain

$$
(4.4)\qquad \|D^j(f - s_i)\|_{L^\infty[a,b]} \leqq \frac{(m-1)!}{j!} (\bar{\pi}_i)^{m-1-j} \|D^{m-1}(f - s_i)\|_{L^\infty[a,b]}, \quad 0 \leqq j \leqq m-1.
$$

Similarly, using the Schwarz inequality, we obtain

$$
(4.5)\qquad \begin{aligned} \|D^{m-1}(f - s_i)\|_{L^\infty[a,b]} &= \left| \int_{\xi_k^{(m-1)}}^{x_{m-1}} D^m(f(t) - s_i(t))\, dt \right| \\ &\leqq (x_{m-1} - \xi_k^{(m-1)})^{\frac{1}{2}} \|D^m(f - s_i)\|_{L^2[a,b]} \\ &\leqq \sqrt{m\bar{\pi}_i}\, \|D^m(f - s_i)\|_{L^2[a,b]}. \end{aligned}
$$

Combining (4.4) and (4.5), we have

$$\|D^j(f-s_i)\|_{L^\infty[a,b]} \leqq \frac{m!}{\sqrt{m}\, j!} (\bar{\pi}_i)^{m-j-(\frac{1}{2})} \|D^m(f-s_i)\|_{L^2[a,b]}, \quad 0 \leqq j \leqq m-1. \tag{4.6}$$

Writing

$$a_m(x)\, D^m(f(x)-s_i(x)) = L[f(x)-s_i(x)] - \sum_{j=0}^{m-1} a_j(x)\, D^j(f(x)-s_i(x))$$

and recalling the lower bound for $a_m(x)$ of (1.2), the triangle inequality gives us

$$\|D^m(f-s_i)\|_{L^2[a,b]} \leqq \frac{1}{\omega}\left\{\|L(f-s_i)\|_{L^2[a,b]} + \sum_{j=0}^{m-1} \|a_j\|_{L^\infty[a,b]} \cdot \|D^j(f-s_i)\|_{L^2[a,b]}\right\}.$$

Since

$$(b-a)^{\frac{1}{2}} \|D^j(f-s_i)\|_{L^\infty[a,b]} \geqq \|D^j(f-s_i)\|_{L^2[a,b]},$$

the bounds of (4.6) thus yield

$$\left\{1 - \frac{1}{\omega}\sum_{j=0}^{m-1} \|a_j\|_{L^\infty[a,b]} \frac{(b-a)^{\frac{1}{2}}\, m!}{\sqrt{m}\, j!} (\bar{\pi}_i)^{m-j-(\frac{1}{2})}\right\} \|D^m(f-s_i)\|_{L^2[a,b]} \leqq \frac{1}{\omega} \|L(f-s_i)\|_{L^2[a,b]}.$$

Thus, there exists a positive integer i_1 and a positive constant H such that

$$\|D^m(f-s_i)\|_{L^2[a,b]} \leqq H\, \|L(f-s_i)\|_{L^2[a,b]} \quad \text{for all} \quad i \geqq i_1. \tag{4.7}$$

Combining (4.7) and (4.6), we obtain

$$\|D^j(f-s_i)\|_{L^\infty[a,b]} \leqq \frac{H\, m!\, (\bar{\pi}_i)^{m-j-(\frac{1}{2})}}{\sqrt{m}\, j!} \|L(f-s_i)\|_{L^2[a,b]} \quad \text{for} \quad 0 \leqq j \leqq m-1, \tag{4.8}$$

for all $i \geqq \max(i_0, i_1)$, which establishes the first inequality of (4.1). But, the second inequality of (4.1) is a direct consequence of the first integral relation of (3.1). Q.E.D.

If we are interested in $L^2[a,b]$-type rather than $L^\infty[a,b]$-type error bounds, the result of Theorem 6 can be improved by the following new result.

Theorem 7. Let $f \in K_2^m[a,b]$, let $\{\pi_i\}_{i=1}^\infty$ be a sequence of partitions of $[a,b]$ with $\lim_{i\to\infty} \bar{\pi}_i = 0$, and let $\{z^{(i)}\}_{i=1}^\infty$ be any sequence of incidence vectors associated with $\{\pi_i\}_{i=1}^\infty$. Then, there exists a positive integer i_0 such that the $Sp(L, \pi_i, z^{(i)})$-interpolate $s_i(x)$ of f of Type I, II, or IV exists and is unique for any $i \geqq i_0$, and moreover, there exists a constant M_2, dependent on j and m but independent of i, such that

$$\|D^j(f-s_i)\|_{L^2[a,b]} \leqq M_2 (\bar{\pi}_i)^{m-j} \|L(f-s_i)\|_{L^2[a,b]} \leqq M_2 (\bar{\pi}_i)^{m-j} \|Lf\|_{L^2[a,b]} \tag{4.9}$$

for any j with $0 \leqq j \leqq m$ and any $i \geqq i_0$.

Proof. The first part of this proof has already been established in Theorem 6. Now, for any j with $0 \leqq j \leqq m-1$, we have from (4.2) that $D^j(f(\xi_l^{(j)}) - s_i(\xi_l^{(j)})) = 0$ for $0 \leqq l \leqq N_i + 1 - j$. Hence, applying the Rayleigh-Ritz inequality [2', p. 184], we have

$$(4.10)\qquad \int_{\xi_\ell^{(j)}}^{\xi_{\ell+1}^{(j)}} \{D^j(f(t)-s_i(t))\}^2\,dt \leq \left[\frac{(j+1)\,\bar{\pi}_i}{\pi}\right]^2 \int_{\xi_\ell^{(j)}}^{\xi_{\ell+1}^{(j)}} \{D^{j+1}(f(t)-s_i(t))\}^2\,dt$$

since $|\xi_{\ell+1}^{(j)}-\xi_\ell^{(j)}| \leq (j+1)\,\bar{\pi}_i$, and this inequality holds for ℓ with $0\leq \ell \leq N_i-j$. Thus, summing both sides of this inequality with respect to ℓ gives

$$(4.11)\qquad (\|D^j(f-s_i)\|_{L^2[\xi_0^{(j)},\,\xi_{N_i-j+1}^{(j)}]})^2 \leq \left[\frac{(j+1)\,\bar{\pi}_i}{\pi}\right]^2 \cdot (\|D^{j+1}(f-s_i)\|_{L^2[\xi_0^{(j)},\,\xi_{N_i-j+1}^{(j)}]})^2$$

for all $0\leq j\leq m-1$. Since $[\xi_0^{(m-1)}, \xi_{N_i-m+2}^{(m-1)}]\subset[a,b]$, the special case $j=m-1$ and the inequality of (4.7) give

$$(4.11')\qquad (\|D^{m-1}(f-s_i)\|_{L^2[\xi_0^{(m-1)},\,\xi_{N_i-m+2}^{(m-1)}]})^2 \leq H^2\left[\frac{(j+1)\,\bar{\pi}_i}{\pi}\right]^2 (\|L(f-s_i)\|_{L^2[a,b]})^2$$

for all $i\geq i_0$. Next, the inequality of (4.8) of Theorem 6 gives us that

$$(4.12)\qquad \begin{aligned}\int_a^{\xi_0^{(j)}} [D^j(f(t)-s_i(t))]^2\,dt &\leq (\|D^j(f-s_i)\|_{L^\infty[a,b]})^2\cdot|\xi_0^{(j)}-a|\\ &\leq M'(\bar{\pi}_i)^{2m-2j}(\|L(f-s_i)\|_{L^2[a,b]})^2 \quad \text{for all}\quad 0\leq j\leq m-1,\qquad i\geq i_0,\end{aligned}$$

since $|\xi_0^{(j)}-a|\leq (j+1)\,\bar{\pi}_i$. Similarly, we have

$$(4.12')\qquad \begin{aligned}\int_{\xi_{N_i-j+1}}^{b} [D^j(f(t)-s_i(t))]^2\,dt \leq M'(\bar{\pi}_i)^{2m-2j}(\|L(f-s_i)\|_{L^2[a,b]})^2,\\ 0\leq j\leq m-1,\quad i\geq i_0.\end{aligned}$$

Thus, summing the inequality of (4.11′) with the inequalities of (4.12)—(4.12′) for the case of $j=m-1$ gives

$$(4.13)\qquad \|D^{m-1}(f-s_i)\|_{L^2[a,b]} \leq M''(\bar{\pi}_i)\,\|L(f-s_i)\|_{L^2[a,b]},\qquad i\geq i_0.$$

Continuing this argument, we can use (4.11) in conjunction with (4.12)—(4.13) to establish the desired result of (4.9). Q.E.D.

If we make the stronger assumption that $f\in K_2^{2m}[a,b]$, we can materially improve the above error estimates. The following result generalizes and improves Theorem 2 of [6].

Theorem 8. Let $f\in K_2^{2m}[a,b]$, let $\{\pi_i\}_{i=1}^\infty$ be a sequence of partitions of $[a,b]$ with $\lim\limits_{i\to\infty}\bar{\pi}_i=0$, and let $\{z^{(i)}\}_{i=1}^\infty$ be any sequence of incidence vectors associated with $\{\pi_i\}_{i=1}^\infty$. Then, there exists a positive integer i_0 such that the $Sp(L,\pi_i,z^{(i)})$-interpolate $s_i(x)$ of f of Type I, III, or IV exists and is unique for any $i\geq i_0$, and moreover, there exists a constant M_3, dependent on j and m but not on i, such that

$$(4.14)\qquad \|D^j(f-s_i)\|_{L^\infty[a,b]} \leq M_3(\bar{\pi}_i)^{2m-j-(\frac{1}{2})}\,\|L^*L[f]\|_{L^2[a,b]}$$

for any j with $0\leq j\leq m-1$, and any $i\geq i_0$.

Proof. Schwarz's inequality applied to the second integral relation of (3.4) yields

$$(4.15)\qquad (\|L(f-s_i)\|_{L^2[a,b]})^2 \leq \|f-s_i\|_{L^2[a,b]}\cdot\|L^*L[f]\|_{L^2[a,b]}.$$

Using the first inequality of (4.9) for the case $j=0$ then gives

(4.16) $$\|L(f-s_i)\|_{L^2[a,b]} \leqq M_2(\bar{\pi}_i)^m \|L^* L[f]\|_{L^2[a,b]}.$$

Combining this inequality with that of (4.8) produces

(4.17) $$\|D^j(f-s_i)\|_{L^\infty[a,b]} \leqq \frac{M_2 H m!}{\sqrt{m}\, j!} (\bar{\pi}_i)^{2m-j-(\frac{1}{2})} \|L^* L[f]\|_{L^2[a,b]} \quad \text{for} \quad 0 \leqq j \leqq m-1,$$

the desired result. Q.E.D.

If we are again interested in $L^2[a,b]$-type error bounds, the result of Theorem 8 can be improved as follows.

Theorem 9. If the hypotheses of Theorem 8 hold, then there exists a positive integer i_0 and a constant M_4, dependent on j and m but not on i, such that

(4.18) $$\|D^j(f-s_i)\|_{L^2[a,b]} \leqq M_4(\bar{\pi}_i)^{2m-j} \|L^* L[f]\|_{L^2[a,b]}$$

for any j with $0 \leqq j \leqq m$ and any $i \geqq i_0$.

Proof. Just combine (4.13) and (4.16). Q.E.D.

The result of Theorem 9 here generalizes and improves the results of Theorems 2 and 6 of [6], and Theorem 9 of [17].

In [6], AHLBERG, NILSON, and WALSH obtain the convergence of derivatives of higher order of generalized splines to the corresponding derivatives of f. This can also be generalized and improved as follows. With the hypotheses of Theorem 8, we know from Theorem 8 that there is a positive integer i_0 such that the $Sp(L, \pi_i, z^{(i)})$-interpolate of f, called $s_i(x)$, of Type I, III, or IV exists and is unique for any $i \geqq i_0$. If $\{v_j(x)\}_{j=1}^{2m}$ is any linearly independent set of functions in $K_2^{2m}[a,b]$ such that $L^* L[v_j(x)]=0$ for almost all $x \in [a,b]$, and if π_i: $a = x_0^{(i)} < x_1^{(i)} < \cdots < x_{N_i+1}^{(i)} = b$, then $s_i(x)$ for any $i \geqq i_0$ can be represented on each subinterval of π_i by

(4.19) $$s_i(x) = \sum_{j=1}^{2m} A_j^{(i,k)} v_j(x), \quad x \in [x_k^{(i)}, x_{k+1}^{(i)}], \quad 0 \leqq k \leqq N_i, \quad i \geqq i_0.$$

Fixing our attention on the particular subinterval $[x_k^{(i)}, x_{k+1}^{(i)}]$ of π_i, divide this subinterval into $2m$ equal parts by means of $\xi_j = x_k^{(i)} + \frac{j(x_{k+1}^{(i)} - x_k^{(i)})}{2m}$, $0 \leqq j \leqq 2m$. Just as in [6], we form the divided differences

(4.20) $$s_i[\xi_0, \xi_1, \ldots, \xi_\ell] \equiv \frac{1}{h^\ell} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} s_i(\xi_j), \quad h \equiv \frac{x_{k+1}^{(i)} - x_k^{(i)}}{2m}, \quad 0 \leqq \ell \leqq 2m-1,$$

so that from (4.19)

(4.21) $$\ell!\, s_i[\xi_0, \xi_1, \ldots, \xi_\ell] = \sum_{j=1}^{2m} A_j^{(i,k)} \{\ell!\, v_j[\xi_0, \xi_1, \ldots, \xi_\ell]\}, \quad 0 \leqq \ell \leqq 2m-1.$$

Thus, regarding (4.21) as $2m$ linear equations in the unknowns $A_j^{(i,k)}$, the $2m \times 2m$ coefficient matrix $C = (c_{p,q})$ for these unknowns has its entries given by $c_{p,q} = (p-1)!\, v_q[\xi_0, \xi_1, \ldots, \xi_{p-1}]$, $1 \leqq p, q \leqq 2m$. On the other hand (cf. (1.4)), the

$2m \times 2m$ Wronskian matrix $W(x; v_1, v_2, \ldots, v_{2m}) = (b_{p,q})$ has its entries given by $b_{p,q} = v_q^{(p-1)}(x)$, $1 \leqq p, q \leqq 2m$, and the connection between these two matrices comes from the well-known result which relates divided differences to derivatives, i.e., if $g(x) \in C^\ell[x_k^{(i)}, x_{k+1}^{(i)}]$, then

$$\ell!\, g[\xi_0, \xi_1, \ldots, \xi_\ell] = D^\ell g(\tau_\ell), \quad \text{where} \quad x_k^{(i)} = \xi_0 < \tau_\ell < \xi_\ell \leqq x_{k+1}^{(i)}. \tag{4.22}$$

Now, we make use of the fact that $W(x; v_1, v_2, \ldots, v_{2m})$ is nonsingular for any $x \in [a, b]$. By choosing $\bar{\pi}_i$ sufficiently small, i.e., $i \geqq i_1$, it follows from (4.22) that the entries of C can be made uniformly close to the entries of $W(x_k^{(i)}; v_1, \ldots, v_{2m})$, and as such, C is nonsingular. Moreover, for $i \geqq i_1$ there is a constant $K_1 > 0$ which uniformly bounds the entries in modulus of the inverse of C for any $0 \leqq k \leqq N_i$. Thus, from (4.21) we have for all j and k that

$$|A_j^{(i,k)}| \leqq K_1 \sum_{\ell=0}^{2m-1} \{\ell!\, s_i[\xi_0, \xi_1, \ldots, \xi_\ell]\} \quad \text{for any} \quad i \geqq i_1. \tag{4.23}$$

It remains to show that the $s_i[\xi_0, \ldots, \xi_\ell]$ are all bounded for $0 \leqq \ell \leqq 2m-1$.

With $f \in K_2^{2m}[a, b]$ and $i \geqq i_0$, Theorem 8 gives us that there exists a constant $M_1 > 0$ such that

$$|f(x) - s_i(x)| \leqq M_1 (\bar{\pi}_i)^{2m-(\frac{1}{2})} \quad \text{for all} \quad x \in [a, b]. \tag{4.24}$$

Using the notation $\underline{\pi}_i \equiv \min_k (x_{k+1}^{(i)} - x_k^{(i)})$, it follows from (4.24) and the definition of (4.20) that

$$|f[\xi_0, \xi_1, \ldots, \xi_\ell] - s_i[\xi_0, \xi_1, \ldots, \xi_\ell]| \leqq (4m)^\ell M_1 \frac{(\bar{\pi})^{2m-(\frac{1}{2})}}{(\underline{\pi}_i)^\ell}, \qquad 0 \leqq \ell \leqq 2m-1.$$

If we make the mesh restriction that there exists a positive constant σ such that $\sigma \underline{\pi}_i \geqq \bar{\pi}_i$ for all $i \geqq 1$, this then becomes

$$|f[\xi_0, \ldots, \xi_\ell] - s_i[\xi_0, \ldots, \xi_\ell]| \leqq (4\sigma m)^\ell M_1 (\bar{\pi}_i)^{2m-\ell-(\frac{1}{2})}, \qquad 0 \leqq \ell \leqq 2m-1. \tag{4.25}$$

But as

$$|f[\xi_0, \ldots, \xi_\ell]| = |D^\ell f(\tau_\ell)/\ell!| \leqq \|D^\ell f\|_{L^\infty[a,b]}/\ell!, \qquad 0 \leqq \ell \leqq 2m-1,$$

it is clear from (4.25) that the quantities $|s_i[\xi_0, \ldots, \xi_\ell]|$ are uniformly bounded for all $i \geqq i_0$, all $0 \leqq \ell \leqq 2m-1$, and all $0 \leqq k \leqq N_i$. Thus, with (4.23), we can assert, as in [6], that there is a constant $K_2 > 0$ such that

$$|A_j^{(i,k)}| \leqq K_2, \quad \text{for} \quad i \geqq i_2 = \max(i_0, i_1). \tag{4.26}$$

By definition, $s_i(x) \in K_2^{2m}[x_k^{(i)}, x_{k+1}^{(i)}]$ for every $0 \leqq k \leqq N_i$, and also $v_j \in K_2^{2m}[a, b]$ for all $1 \leqq j \leqq 2m$. Again, considering the particular subinterval $[x_k^{(i)}, x_{k+1}^{(i)}]$ of π_i, it follows from (4.25) that there is a τ_ℓ with $\xi_0 < \tau_\ell < \xi_\ell$ such that

$$|D^\ell(f - s_i)(\tau_\ell)| \leqq \ell!\,(4\sigma m)^\ell M_1 (\bar{\pi}_i)^{2m-\ell-(\frac{1}{2})}, \qquad 0 \leqq \ell \leqq 2m-1. \tag{4.27}$$

Thus, for any $x \in [x_k^{(i)}, x_{k+1}^{(i)}]$, we can write

$$D^\ell(f - s_i)(x) - D^\ell(f - s_i)(\tau_\ell) = \int_{\tau_\ell}^{x} D^{\ell+1}(f - s_i)(t)\, dt, \qquad 0 \leqq \ell \leqq 2m-1. \tag{4.28}$$

For the case $\ell = 2m-1$, combining an application of Schwarz's inequality to the integral of (4.28) with the inequality of (4.27) yields

$$(4.29)\qquad |D^{2m-1}(f-s_i)(x)| \leqq \{\ell!\,(4\sigma m)^\ell M_1 + \|D^{2m}(f-s_i)\|_{L^2[x_k^{(i)},\,x_{k+1}^{(i)}]}\}\,(\bar{\pi}_i)^{\frac{1}{2}}.$$

Now, the representation of $s_i(x)$ in (4.19), coupled with the uniform boundedness of the $A_j^{(i,k)}$ in (4.26), gives us that the norms $\|D^{2m}s_i\|_{L^2[x_k^{(i)},\,x_{k+1}^{(i)}]}$ are also uniformly bounded for all k and all $i \geqq i_2$, and as f is a fixed function in $K_2^{2m}[a, b]$, the same is evidently true for the norms $\|D^{2m}(f-s_i)\|_{L^2[x_k^{(i)},\,x_{k+1}^{(i)}]}$. Hence, there is a positive constant M_2 such that for $i \geqq i_2$,

$$(4.30)\qquad \|D^{2m-1}(f-s_i)\|_{L^\infty[x_k^{(i)},\,x_{k+1}^{(i)}]} \leqq M_2(\bar{\pi}_i)^{\frac{1}{2}} \quad \text{for all} \quad 0 \leqq k \leqq N_i.$$

From (4.28), it is clear that this argument can be repeated in the uniform norm for the lower order derivatives, and we have for $i \geqq i_2$ that there is a constant M_3 such that

$$(4.31)\qquad \|D^j(f-s_i)\|_{L^\infty[x_k^{(i)},\,x_{k+1}^{(i)}]} \leqq M_3(\bar{\pi}_i)^{2m-j-(\frac{1}{2})}, \qquad 0 \leqq j \leqq 2m-1,$$

for *all* $0 \leqq k \leqq N_i$. Thus, if we extend the usual definition of the L^∞-norm on $[a, b]$ by defining

$$(4.31')\qquad \|D^j(f-s_i)\|_{L^\infty[a,b]} \equiv \max_{0 \leqq k \leqq N_i} \{\|D^j(f-s_i)\|_{L^\infty[x_k^{(i)},\,x_{k+1}^{(i)}]}\},$$

we have

Theorem 10. If the hypotheses of Theorem 8 hold, and if there is a positive constant σ such that $\sigma \underline{\pi}_i \geqq \bar{\pi}_i$ for all $i \geqq 1$, then there exists a positive integer i_0 and a positive constant M_5, independent of i, such that

$$(4.32)\qquad \|D^j(f-s_i)\|_{L^\infty[a,b]} \leqq M_5(\bar{\pi}_i)^{2m-j-(\frac{1}{2})}$$

for any j with $0 \leqq j \leqq 2m-1$ and any $i \geqq i_0$.

It is worth noting that if $\max_{1 \leqq k \leqq N_i} z_k^{(i)} \equiv \sigma_i$, then the $Sp(L, \pi_i, z^{(i)})$-interpolate $s_i(x)$ of $f(x)$ is in general only of class $C^{2m-1-\sigma_i}[a, b]$ on the entire interval $[a, b]$. Thus, for any j with $0 \leqq j \leqq 2m-1-\sigma_i$, the statement of (4.32) is an inequality for the *continuous* derivatives $D^j(f-s_i)(x)$.

We now investigate in what sense the previous theorems (Theorems 6—10) are *sharp*, with respect to the exponent of $\bar{\pi}_i$. It suffices to consider the unit interval $[0, 1]$. Let $v_1(x), v_2(x), \ldots, v_{2m}(x)$ again be $2m$ linearly independent functions in $K_2^{2m}[0, 1]$ such that $L^*L[v_j(x)] = 0$ for almost all $x \in [0, 1]$, and let V be the finite dimensional linear space of all linear combinations of $v_1(x), v_2(x), \ldots, v_{2m}(x)$, where $0 \leqq x \leqq 1$. For each integer j, $0 \leqq j \leqq 2m-1$, and each h with $0 \leqq h \leqq 1$, consider the following problem of best *Chebyshev* approximation. For any $\mu > 0$ such that $D^j x^\mu \notin V$, let

$$\sigma_\infty(h) = \sigma_\infty(h; j; \mu) \equiv \inf_{r \in V} \|D^j(t^\mu - r(th))\|_{L^\infty[0,1]}.$$

Because V is finite dimensional, $\sigma_\infty(h)$ is readily seen to be continuous on $[0, 1]$, and as $D^j x^\mu \notin V$, then $\sigma_\infty(h)$ is strictly positive in $0 \leqq h \leqq 1$. Thus, we define

$\min_{0\leq h\leq 1} \sigma_\infty(h; j; \mu) \equiv c_\infty(j, \mu) > 0$. We now assert that

$$(4.33) \qquad \inf_{r\in V} \|D^j(x^\mu - r(x))\|_{L^\infty[0,h]} \geq h^{\mu-j} c_\infty(j, \mu) > 0,$$

since, with the change of variables $x = th$, $0\leq t\leq 1$, we have

$$\inf_{r\in V} \|D^j(x^\mu - r(x))\|_{L^\infty[0,h]} = h^{\mu-j} \inf_{r\in V} \left\| D^j\left(t^\mu - \frac{r(th)}{h^\mu}\right)\right\|_{L^\infty[0,1]}$$
$$= h^{\mu-j} \inf_{s\in V} \|D^j(t^\mu - s(th))\|_{L^\infty[0,1]} \geq h^{\mu-j} c_\infty(j, \mu) > 0.$$

The inequality of (4.33) can now be used to show that the results of Theorems 6 and 10 are sharp with respect to the exponents of $\bar{\pi}_i$. For Theorem 6, consider the function $f_1(x) = x^{m-(\frac{1}{2})+\varepsilon}$, $0\leq x\leq 1$. For each $\varepsilon > 0$, we see that $f_1 \in K_2^m[0, 1]$. Moreover, as any collection of functions of the form $\{x^{m-(\frac{1}{2})+\varepsilon_i}\}_{i=1}^m$ for $\varepsilon_m > \varepsilon_{m-1} > \cdots > \varepsilon_1 > 0$ are linearly independent on $[0, 1]$, it follows from the finite dimensionality of V that $D^j f_1 \notin V$ for *any* $0\leq j\leq m-1$ for *all* $\varepsilon > 0$ sufficiently small. Thus, for *any* $s_i(x)$ of $Sp(L, \pi_i, z^{(i)})$, we must have that

$$\|D^j(f_1 - s_i)\|_{L^\infty[0,1]} \geq \|D^j(f_1 - s_i)\|_{L^\infty[0, x_1^{(i)}]} \geq \inf_{r\in V} \|D^j(f_1(x) - r(x))\|_{L^\infty[0, x_1^{(i)}]}$$

for any $0\leq j\leq m-1$, where $x_1^{(i)}$ denotes the right endpoint of the first subinterval of π_i. With (4.33), and the mesh restriction of Theorem 10, i.e., $\sigma\underline{\pi}_i \geq \bar{\pi}_i$ for all $i\geq 1$, the above inequality becomes

$$(4.34) \qquad \|D^j(f_1 - s_i)\|_{L^\infty[0,1]} \geq \left(\frac{\bar{\pi}_i}{\sigma}\right)^{m-(\frac{1}{2})-j+\varepsilon} c_\infty\left(j, m - \left(\frac{1}{2}\right) + \varepsilon\right), \qquad 0\leq j\leq m-1.$$

In the same manner, we deduce for the function $f_2(x) = x^{2m-(\frac{1}{2})+\varepsilon}$ in $K_2^{2m}[0, 1]$ the inequalities

$$(4.35) \qquad \|D^j(f_2 - s_i)\|_{L^\infty[0,1]} \geq \left(\frac{\bar{\pi}_i}{\sigma}\right)^{2m-(\frac{1}{2})-j+\varepsilon} c_\infty\left(j, 2m - \left(\frac{1}{2}\right) + \varepsilon\right), \qquad 0\leq j\leq 2m-1.$$

As these inequalities are valid for *all* $\varepsilon > 0$ sufficiently small, we have proved

Theorem 11. Assuming that the hypotheses of Theorem 6 hold and that there is a positive constant σ such that $\sigma\underline{\pi}_i \geq \bar{\pi}_i$ for all $i\geq 1$, then, for each $\varepsilon > 0$ sufficiently small, there is an element $f_1(x) \in K_2^m[a, b]$ and a positive constant M_6, independent of i, such that

$$(4.36) \qquad \|D^j(f_1 - s_i)\|_{L^\infty[a,b]} \geq M_6 (\bar{\pi}_i)^{m-j-(\frac{1}{2})+\varepsilon}, \qquad 0\leq j\leq m-1, \qquad i\geq 1$$

for any $s_i \in Sp(L, \pi_i, z^{(i)})$. Similarly, assuming the hypotheses of Theorem 8 and that there is a positive constant σ such that $\sigma\underline{\pi}_i \geq \bar{\pi}_i$ for all $i\geq 1$, then, for each $\varepsilon > 0$ sufficiently small, there is an element $f(x) \in K_2^{2m}[a, b]$ and a positive constant M_7, independent of i, such that

$$(4.37) \qquad \|D^j(f_2 - s_i)\|_{L^\infty[a,b]} \geq M_7 (\bar{\pi}_i)^{2m-j-(\frac{1}{2})+\varepsilon}, \qquad 0\leq j\leq 2m-1, \qquad i\geq 1$$

for any $s_i \in Sp(L, \pi_i, z^{(i)})$. Thus, the respective exponents of $\bar{\pi}_i$ in (4.1) in Theorem 6 and (4.32) of Theorem 10 *cannot* in general be increased for the classes of functions $K_2^m[a, b]$ and $K_2^{2m}[a, b]$.

To investigate the sharpness of the exponents of $\bar{\pi}_i$ in Theorems 7 and 9, we similarly have the following problem of best L_2-approximation. For any $\mu > 0$

such that $D^j x^\mu \notin V$, let $\sigma_2(h) = \sigma_2(h, j, \mu) = \inf_{r \in V} \|D^j(t^\mu - r(th))\|_{L^2[0,1]}$ for each h with $0 \leqq h \leqq 1$. As before, the finite dimensionality of V gives that $\sigma_2(h)$ is continuous on $[0, 1]$, and if $D^j x^\mu \notin V_j$, then $\sigma_2(h)$ is strictly positive in $[0, 1]$. Thus, we define $\min_{0 \leqq h \leqq 1} \sigma_2(h, j, \mu) \equiv c_2(j, \mu) > 0$. This gives rise to the following inequality. For any partition π_i of $[0, 1]$, we have

$$\left(\|D^j(x^\mu - s_i(x))\|_{L^2[0,1]}\right)^2 = \sum_{k=0}^{N_i} \int_{x_k^{(i)}}^{x_{k+1}^{(i)}} |D^j(x^\mu - s_i(x))|^2 dx$$
$$\geqq \int_0^{x_1^{(i)}} |D^j(x^\mu - s_i(x))|^2 dx,$$

and with the change of variables $x = th$ where $h = x_1^{(i)}$, we have

$$\left(\|D^j(x^\mu - s_i(x))\|_{L^2[0,1]}\right)^2 \geqq h^{2(\mu-j)+1} \int_0^1 \left|D^j\left(t^\mu - \frac{s_i(th)}{h^\mu}\right)\right|^2 dt$$
$$\geqq (\underline{\pi}_i)^{2(\mu-j)+1} c_2^2(j, \mu).$$

Now, for Theorem 7, consider the function $f_3(x) = x^{m-(\frac{1}{2})+\varepsilon}$, $0 \leqq x \leqq 1$. For all $\varepsilon > 0$ sufficiently small, $f_3 \in K_2^m[0, 1]$ with $D^j f_3 \notin V$ for any $0 \leqq j \leqq m$. Thus, for *any* $s_i \in Sp(L, \pi_i, z^{(i)})$, we must have, with the mesh restriction that $\sigma \underline{\pi}_i \geqq \bar{\pi}_i$ for all $i \geqq 1$, that

$$\|D^j(f_3 - s_i)\|_{L^2[0,1]} \geqq \left(\frac{\bar{\pi}_i}{\sigma}\right)^{m-j+\varepsilon} c_2\left(j, m - \left(\frac{1}{2}\right) + \varepsilon\right), \qquad 0 \leqq j \leqq m,$$

for all $\varepsilon > 0$ sufficiently small. Similarly, with $f_4(x) = x^{2m-(\frac{1}{2})+\varepsilon}$ in $K_2^{2m}[0, 1]$, we deduce that

$$\|D^j(f_4 - s_i)\|_{L^2[a,b]} \geqq \left(\frac{\bar{\pi}_i}{\sigma}\right)^{2m-j+\varepsilon} c_2\left(j, 2m - j - \left(\frac{1}{2}\right) + \varepsilon\right), \qquad 0 \leqq j \leqq m,$$

for any $s_i \in Sp(L, \pi_i, z^{(i)})$ for all $\varepsilon > 0$ sufficiently small. This gives us

Theorem 12. Assuming that the hypotheses of Theorem 6 hold and that there is a positive constant σ such that $\sigma \underline{\pi}_i \geqq \bar{\pi}_i$ for all $i \geqq 1$, then, for each $\varepsilon > 0$ sufficiently small, there is an element $f_3(x) \in K_2^m[a, b]$ and a positive constant M_8, independent of i, such that

$$(4.38) \qquad \|D^j(f_3 - s_i)\|_{L^2[a,b]} \geqq M_8 (\bar{\pi}_i)^{m-j+\varepsilon}, \qquad 0 \leqq j \leqq m, \qquad i \geqq 1,$$

for any $s_i \in Sp(L, \pi_i, z^{(i)})$. Similarly, assuming the hypotheses of Theorem 8 and that there is a positive constant σ such that $\sigma \underline{\pi}_i \geqq \bar{\pi}_i$ for all $i \geqq 1$, then, for each $\varepsilon > 0$ sufficiently small, there is an element $f_4(x) \in K_2^{2m}[a, b]$ and a positive constant M_9, independent of i, such that

$$(4.39) \qquad \|D^j(f_4 - s_i)\|_{L^2[a,b]} \geqq M_9 (\bar{\pi}_i)^{2m-j+\varepsilon}, \qquad 0 \leqq j \leqq m, \qquad i \geqq 1,$$

for any $s_i \in Sp(L, \pi_i, z^{(i)})$. Thus, the respective exponents of $\bar{\pi}_i$ in (4.9) of Theorem 7 and (4.18) of Theorem 9 *cannot* in general be increased for the classes of functions $K_2^m[a, b]$ and $K_2^{2m}[a, b]$.

It is interesting to contrast the result of Theorem 11 with the results of [17] and of Birkhoff and de Boor [10]. For the special case of cubic natural splines, i.e., $L = D^2$ and $z_i \equiv 1$, it was shown in [10] that

$$\|D^j(f - s_i)\|_{L^\infty[a,b]} \leqq M(\bar{\pi}_i)^{4-j}, \qquad 0 \leqq j \leqq 3,$$

assuming $f \in C^4[a, b]$. Similarly, in Theorem 9 of [17], it was shown for Hermite interpolation, i.e., $L = D^n$ and $z_i \equiv m$, that

$$\|D^j(f - s_i)\|_{L^\infty[a,b]} \leqq M(\bar{\pi}_i)^{2m-j}, \qquad 0 \leqq j \leqq m,$$

for $f \in C^{2m}[a, b]$. Both of these results improve the corresponding exponent of $\bar{\pi}_i$ in (4.32) by $\frac{1}{2}$, at the expense of working with functions in $C^{2m}[a, b]$, rather than $K_2^{2m}[a, b]$. In general, Theorem 11 shows that such improvements in the exponent of $\bar{\pi}_i$ can *only* come from similar suitable restrictions in the function class $K_2^{2m}[a, b]$. This answers in the negative a conjecture of AHLBERG, NILSON, and WALSH [6].

Using Theorem 10, we may extend the result of Theorem 9 for $L^2[a, b]$-type error bounds, where, in analogy with the definition of (4.31′), we define

$$(4.31'') \quad \|D^j(f - s_i)\|_{L^2[a,b]} \equiv \left\{ \sum_{k=0}^{N_i} \int_{x_k^{(i)}}^{x_{k+1}^{(i)}} (D^j(f - s_i)(t))^2 \, dt \right\}^{\frac{1}{2}}, \qquad 0 \leqq j \leqq 2m - 1.$$

Theorem 13. If the hypothesis of Theorem 8 hold and if there is a positive constant σ such that $\sigma \underline{\pi}_i \geqq \bar{\pi}_i$ for all $i \geqq 1$, then there exists a positive integer i_0 and a constant M_{10}, dependent on j and m but not on i, such that

$$(4.40) \quad \|D^j(f - s_i)\|_{L^2[a,b]} \leqq M_{10}(\bar{\pi}_i)^{2m-j}, \qquad 0 \leqq j \leqq m, \quad i \geqq i_0,$$

and

$$(4.41) \quad \|D^j(f - s_i)\|_{L^2[a,b]} \leqq M_{10}(\bar{\pi}_i)^{2m-j-(\frac{1}{2})}, \qquad m + 1 \leqq j \leqq 2m - 1, \quad i \geqq i_0.$$

It follows from the second part of Theorem 12 that the exponent of $\bar{\pi}_i$ in (4.40) is sharp for the class $K_2^{2m}[a, b]$ for $0 \leqq j \leqq m$. It is easy to see that the exponent of $\bar{\pi}_i$ in (4.41) cannot exceed $2m - j$ even for the class $C^\infty[a, b]$, but it remains an open question[3] if the inequality of (4.40) is valid for *all* $0 \leqq j \leqq 2m - 1$ for the class $K_2^{2m}[a, b]$.

5. G-Splines

In the special, but important, case in which $L[u(x)] = D^m u(x)$, $x \in [a, b]$, our previous results may be generalized along the lines of recent work of AHLBERG and NILSON [3] and SCHOENBERG [47]. As before, let $\pi: a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b$ be a partition of the interval $[a, b]$, and let $E = (e_{i,j})$ denote the $N \times m$ *incidence matrix*, $1 \leqq i \leqq N$, $0 \leqq j \leqq m - 1$, having entries of 0's and 1's, with at least one nonzero entry in each row of E. Further, let e denote the collection of (i, j) such that $e_{i,j} = 1$. Following [47], we now generalize our Definitions 1, 2, and 3.

Definition 4. The real-valued function $s(x)$ defined on $[a, b]$ is said to be a *g-spline of order m* for π and E if and only if

(5.1) $s(x)$ is a polynomial of degree at most $2m - 1$ in each subinterval (x_i, x_{i+1}), $0 \leqq i \leqq N$, i.e., $D^{2m}[s(x)] = 0$ in each subinterval of π,

[3] It has just been shown by Mr. F. PERRIN of Case Western Reserve University that the inequality of (4.40) *is* valid for all $0 \leqq j \leqq 2m - 1$ (added in proof).

(5.2) $s(x) \in C^{m-1}[a, b]$ and if $e_{i,j}=0$, then $s^{(2m-j-1)}(x)$ is continuous at x_i, i.e., $(i, j) \notin e$ implies that $s^{(2m-j-1)}(x_i-) = s^{(2m-j-1)}(x_i+)$.

We denote the class of all g-splines of order m for π and E by $Sp(m, \pi, E)$.

As in Definition 2, if $f(x) \in C^{m-1}[a, b]$, we can define four basic types of interpolation of $f(x)$ in $Sp(m, \pi, E)$. It is similarly convenient to augment the incidence matrix E by the addition of two parameter-like *rows*, corresponding to $i=0$ and $i=N+1$, thereby forming the $(N+2)\times m$ matrix $E^* = (e_{i,j})$, $0 \leqq i \leqq N+1$, $0 \leqq j \leqq m-1$. Each of these added rows must have m entries consisting only of 0's and 1's, and at least one entry must be nonzero in each added row. We similarly denote the collection of all (i, j) in E^* such that $e_{i,j}=1$ by e^*.

Definition 5. Given $f(x) \in C^{m-1}[a, b]$, a function $s(x) \in Sp(m, \pi, E)$ is said to be a $Sp(m, \pi, E)$-interpolate of $f(x)$

of Type I if (i) $D^j s(x_i) = D^j f(x_i)$ for all $(i, j) \in e^*$,
(ii) $e_{0,j} \equiv e_{N+1,j} \equiv 1$ for all $0 \leqq j \leqq m-1$,

of Type II if (i) $D^j s(x_i) = D^j f(x_i)$ for all $(i, j) \in e^*$,
(ii) $D^{(2m-j-1)} s(x_i) = 0$ for $i=0$ or $i=N+1$, $(i, j) \notin e^*$,

of Type III if (i) $D^j s(x_i) = D^j f(x_i)$ for all $(i, j) \in e^*$,
(ii) $D^{(2m-j-1)} s(x_i) = D^{(2m-j-1)} f(x_i)$ for $i=0$ or $i=N+1$, $(i, j) \notin e^*$,

of Type IV if (i) $f \in C_p^{2m-1}[a, b]$, i.e., $f(x) \in C^{2m-1}[a, b]$ and $D^j f(a) = D^j f(b)$ for all $0 \leqq j \leqq 2m-1$,
(ii) $D^j s(x_i) = D^j f(x_i)$ for all $(i, j) \in e^*$,
(iii) $e_{0,j} \equiv e_{N+1,j} \equiv \delta_{0,j}$, $0 \leqq j \leqq m-1$,
(iv) $D^j s(a) = D^j s(b)$ for all $1 \leqq j \leqq 2m-1$.

Definition 6. Given the $C^{m-1}[a, b]$ function $f(x)$, the partition π, the incidence matrix E, and the positive integer m, the *Hermite-Birkhoff problem of order m* of Type I (resp. Type II, Type III, or Type IV) is to find a polynomial $p_{m-1}(x)$, $x \in [a, b]$, of degree at most $m-1$ such that $p_{m-1}(x)$ is an $Sp(m, \pi, E)$-interpolate of f of Type I (resp. Type II, Type III, or Type IV). The Hermite-Birkhoff problem of order m is said to be *well-posed* for $Sp(m, \pi, E)$ if and only if it has *at most* one solution.

As in Section 1, we now determine sufficient conditions for the Hermite-Birkhoff problem of order m, (henceforth abbreviated as the HB_m-problem) to be well-posed. For each integer i, $0 \leqq i \leqq N+1$, if $e_{i,0} \in e^*$, let μ_i be the greatest positive integer such that $e_{i,0}, e_{i,1}, \ldots, e_{i,\mu_i-1}$ are all in e^*. If $e_{i,0} \notin e^*$, define μ_i to be zero. Since a polynomial of degree $m-1$ with m zeros must be identically zero, we have, in analogy with Theorem 1 and Corollary 1, the result of

Theorem 14. If $\sum_{i=1}^{N+1} \mu_i \geqq m$, then the HB_m-problem of Type I (resp. Type II, III, or IV) is well-posed for $Sp(m, \pi, E)$. In particular, the HB_m-problem of Type I is always well-posed for $Sp(m, \pi, E)$.

As a partial generalization of our Theorem 1, with essentially the same proof using $L[u]\equiv D^m u$, we have the following result which is a slight generalization to different boundary conditions of results of AHLBERG and NILSON [3] and SCHOENBERG [Theorem 1, 47].

Theorem 15. Let π, E, and $f\in C^{m-1}[a,b]$ be given. If the HB_m-problem of Type I (resp. Type II, III, or IV) is well-posed for $Sp(m,\pi,E)$, then there exists a unique function $s(x)$ which is the $Sp(m,\pi,E)$-interpolate of $f(x)$ of Type I (resp. Type II, III, or IV).

Moreover, as before, the following *first* and *second* integral relations hold for g-splines of order m.

Theorem 16. Let $f\in K_2^m[a,b]$ and let $s(x)$ be a $Sp(m,\pi,E)$-interpolate of f of Type I, II, or IV. Then

$$\int_a^b [D^m f(x)]^2\,dx=\int_a^b [D^m(f(x)-s(x))]^2\,dx+\int_a^b [D^m s(x)]^2\,dx. \tag{5.3}$$

Proof. Clearly

$$\int_a^b [D^m f]^2\,dx=\int_a^b [D^m(f-s)]^2\,dx+2\int_a^b [D^m(f-s)]\,[D^m s]\,dx+\int_a^b [D^m s]^2\,dx,$$

and the first integral relation (5.3) will follow if we can establish that the middle term above vanishes. Using (1.7) in the special case $L[u]\equiv D^m u$, we have

$$\int_a^b [D^m(f-s)]\,[D^m s]\,dx=(-1)^m\sum_{i=0}^{N}\int_{x_i}^{x_{i+1}}(f-s)\,(D^{2m}s)\,dx$$
$$+\sum_{i=0}^{N}\sum_{j=0}^{m-1}(-1)^{m-j-1}D^j(f-s)\,D^{2m-j-1}s\Big|_{x_i}^{x_{i+1}}.$$

Since $s(x)$ is a polynomial of degree at most $2m-1$ in each subinterval (x_i, x_{i+1}) of π, $0\leqq i\leqq N$, the first sum clearly vanishes. Since $(f-s)\in C^{m-1}[a,b]$, the last sum can be written as

$$\sum_{j=0}^{m-1}(-1)^{m-j-1}\Big\{D^j\big(f(b)-s(b)\big)\,D^{2m-j-1}s(b)-D^j\big(f(a)-s(a)\big)\,D^{2m-j-1}s(a)$$
$$+\sum_{i=0}^{N}D^j\big(f(x_i)-s(x_i)\big)\cdot[D^{2m-j-1}s(x_i-)-D^{2m-j-1}s(x_i+)]\Big\}.$$

For any $1\leqq i\leqq N$, either $(i,j)\in e^*$, in which case $D^j\big(f(x_i)-s(x_i)\big)=0$ by Definition 5, or else $(i,j)\notin e^*$, in which case $D^{2m-j-1}s(x_i+)=D^{2m-j-1}s(x_i-)$ by (5.2) of Definition 4, so that the inner sum above vanishes. Similarly, the boundary conditions force the remaining terms to vanish. Q.E.D.

In the same manner, one readily verifies the *second integral relation* of

Theorem 17. Let $f\in K_2^{2m}[a,b]$, and let $s(x)$ be a $Sp(m,\pi,E)$-interpolate of f of Type I, III, or IV. Then

$$\int_a^b [D^m(f(x)-s(x))]^2\,dx=(-1)^m\int_a^b [f(x)-s(x)]\cdot D^{2m}f(x)\,dx. \tag{5.4}$$

To obtain error bounds for g-splines analogous to those of Section 4, we begin by considering any sequence of partitions $\{\pi_i\}_{i=1}^{\infty}$ of $[a,b]$ satisfying $\lim_{i\to\infty}\bar{\pi}_i=0$.

If π_i: $a=x_0^{(i)}<x_1^{(i)}<\cdots<x_{N_i+1}^{(i)}=b$, we further require, as in [3], that there exists a positive constant c and a positive integer i_0 such that for *each* k with $0\leqq k\leqq N_i+1$, there exists an integer $j=j(i,k)$ such that $e_{j,0}^{(i)}=1$ and

$$(5.5)\qquad |x_k^{(i)}-x_j^{(i)}|\leqq c\bar{\pi}_i \quad \text{for all} \quad i\geqq i_0, \quad \text{all} \quad 0\leqq k\leqq N_i+1.$$

This latter assumption allows us to apply Rolle's Theorem, as in Theorem 6, as well as the Rayleigh-Ritz inequality, as in Theorem 7, to the case of g-splines. Because the proofs of Section 4 carry over with little change, we now state, for brevity, the following error bounds for g-splines which partially generalize the results of Section 4.

Theorem 18. Let $f\in K_2^m[a,b]$, let $\{\pi_i\}_{i=1}^\infty$ be a sequence of partitions of $[a,b]$ with $\lim_{i\to\infty}\bar{\pi}_i=0$, and let $\{E^{(i)}\}_{i=1}^\infty$ be any sequence of incidence matrices associated with $\{\pi_i\}_{i=1}^\infty$. Assume that there exists a positive constant c and a positive integer i_0 such that for each k with $0\leqq k\leqq N_i+1$, there is an integer $j=j(i,k)$ such that $e_{j,0}^{(i)}=1$ and (5.5) is satisfied for all $i\geqq i_0$. Then, there exists a positive integer i_1 such that the $Sp(m,\pi_i,E^{(i)})$-interpolate $s_i(x)$ of f of Type I, II, or IV exists and is unique for any $i\geqq i_1$. Moreover, there exists a constant M_1, dependent on j and m, but independent of i, such that

$$(5.6)\qquad \begin{aligned}\|D^j(f-s_i)\|_{L^\infty[a,b]} &\leqq M_1(\bar{\pi}_i)^{m-j-(\frac{1}{2})}\|D^m(f-s_i)\|_{L^2[a,b]}\\ &\leqq M_1(\bar{\pi}_i)^{m-j-(\frac{1}{2})}\|D^m f\|_{L^2[a,b]}\end{aligned}$$

for any j with $0\leqq j\leqq m-1$ and any $i\geqq i_0$.

Theorem 19. If the hypotheses of Theorem 18 are satisfied, then there exists a constant M_2, dependent on j and m, but independent of i, such that

$$(5.7)\qquad \|D^j(f-s_i)\|_{L^2[a,b]}\leqq M_2(\bar{\pi}_i)^{m-j}\|D^m f\|_{L^2[a,b]}$$

for any j with $0\leqq j\leqq m$ and any $i\geqq i_0$.

Theorem 20. If the hypotheses of Theorem 18 are satisfied, and $f\in K_2^{2m}[a,b]$, then there exists a positive integer i_0 such that the $Sp(m,\pi_i,E^{(i)})$-interpolate $s_i(x)$ of f of Type I, III, or IV, exists and is unique for any $i\geqq i_0$. Moreover, there exists a constant M_3, dependent on j and m but independent of i, such that

$$(5.8)\qquad \|D^j(f-s_i)\|_{L^\infty[a,b]}\leqq M_3(\bar{\pi}_i)^{2m-j-(\frac{1}{2})}\|D^{2m}f\|_{L^2[a,b]}$$

for any j with $0\leqq j\leqq m-1$ and any $i\geqq i_0$. If, in addition, there is a positive constant σ such that $\sigma\underline{\pi}_i\geqq\bar{\pi}_i$ for all $i\geqq 1$, then there exists a positive integer i_1 and a constant M_4, independent of i, such that

$$(5.9)\qquad \|D^j(f-s_i)\|_{L^\infty[a,b]}\leqq M_4(\bar{\pi}_i)^{2m-j-(\frac{1}{2})}$$

for any j with $0\leqq j\leqq 2m-1$ and any $i\geqq i_1$.

Theorem 21. If the hypotheses of Theorem 18 are satisfied and $f\in K_2^{2m}[a,b]$, then there exists a positive integer i_0 and a constant M_5, independent of i, such that the $Sp(m,\pi_i,E^{(i)})$-interpolate $s_i(x)$ of f of Type I, III, or IV satisfies

$$(5.10)\qquad \|D^j(f-s_i)\|_{L^2[a,b]}\leqq M_5(\bar{\pi}_i)^{2m-j}\|D^{2m}f\|_{L^2[a,b]}$$

for any j with $0 \leqq j \leqq m$ and any $i \geqq i_0$. If, in addition, there is a positive constant σ such that $\sigma \underline{\pi}_i \geqq \bar{\pi}_i$ for any $i \geqq 1$, then there exists a positive integer i_1 and a constant M_6, independent of i, such that

(5.11) $$\|D^j(f-s_i)\|_{L^2[a,b]} \leqq M_6(\bar{\pi}_i)^{2m-j} \quad \text{for any } 0 \leqq j \leqq m, \quad i \geqq i_0,$$

and

(5.12) $$\|D^j(f-s_i)\|_{L^2[a,b]} \leqq M_6(\bar{\pi}_i)^{2m-j-(\frac{1}{2})} \quad \text{for any } m+1 \leqq j \leqq 2m-1, \quad i \geqq i_0.$$

We remark that the error bounds of Theorem 18 slightly generalize those of [3], while those of Theorems 17, 19, 20, and 21 are apparently entirely new.

Finally, because L-splines with $L=D^m$ are special cases of g-splines, it is clear from the results of Theorems 11 and 12 that the exponents of $\bar{\pi}_i$ *cannot* in general be increased in (5.6)—(5.11) for the corresponding function classes, and our results are thus *sharp*.

6. An Application

In this section, we shall consider the numerical approximation of the solution of the following real nonlinear two-point boundary value problem, studied in [17],

(6.1) $$P[u(x)] = f(x, u(x)), \qquad 0 < x < 1,$$

with boundary conditions

(6.2) $$D^k u(0) = D^k u(1) = 0, \qquad D = \frac{d}{dx}, \qquad 0 \leqq k \leqq n-1,$$

where the linear differential operator P is defined by

(6.3) $$P[u(x)] = \sum_{j=0}^{n} (-1)^{j+1} D^j [p_j(x) D^j u(x)], \qquad n \geqq 1.$$

The coefficient functions $p_j(x)$ are assumed to be of class $C^j[0,1]$, $j=0, 1, \ldots, n$.

Let S denote the linear space of all functions in $K_2^n[0,1]$ which satisfy the boundary conditions of (6.2). We assume that there exist two real constants β and K such that

(6.4) $$\|w\|_{L^\infty} \equiv \sup_{x\in[0,1]} |w(x)| \leqq K \left\{ \int_0^1 \left[\sum_{j=0}^{n} p_j(x) \left(D^j w(x)\right)^2 + \beta \left(w(x)\right)^2 \right] dx \right\}^{\frac{1}{2}}$$

for all $w \in S$. We introduce the finite quantity (cf. Lemma 1 of [17]):

(6.5) $$\Lambda \equiv \inf_{\substack{w\in S\\ w\not\equiv 0}} \frac{\int_0^1 \left\{ \sum_{j=0}^{n} p_j(x) [D^j w(x)]^2 \right\} dx}{\int_0^1 [w(x)]^2\, dx}.$$

We assume that the functions $f(x,u)$ and $\frac{\partial f(x,u)}{\partial u}$ are real and continuous in both variables, i.e., $f(x,u)$, $\frac{\partial f(x,u)}{\partial u} \in C^0([0,1]\times R)$, and that there exists a constant γ such that

(6.6) $$\frac{\partial f(x,u)}{\partial u} \equiv f_u(x,u) \geqq \gamma > -\Lambda \quad \text{for all} \quad x \in [0,1], \quad \text{and all real} \quad u.$$

Finally, we assume that a classical solution of (6.1)—(6.2) exists.

The goal of this section is to estimate the error made in applying the classical Rayleigh-Ritz procedure (cf. [17] and the references given there) to the variational formulation of (6.1)—(6.2), by minimizing over subspaces of L-spline functions. In so doing, we generalize and improve the results of [17].

The following fundamental result summarizes Theorems 1, 2, and 3 of [17].

Theorem 22. (i) If $\varphi(x)$ is a classical solution of (6.1)—(6.2), then $\varphi(x)$ strictly minimizes the following functional

$$F[w] \equiv \int_0^1 \left\{ \tfrac{1}{2} \sum_{j=0}^{n} p_j(x) \left(D^j w(x)\right)^2 + \int_0^{w(x)} f(x, \eta)\, d\eta \right\} dx \tag{6.7}$$

over the space S, and $\varphi(x)$ is thus the unique solution of (6.1)—(6.2).

(ii) If S_M is any finite dimensional subspace of S, then there exists a unique function, $\hat{w}_M(x)$, in S_M which minimizes the functional $F[w]$ over S_M.

(iii) There exists a constant C, which is independent of the choice of S_M, such that the following error bound is valid

$$\|\hat{w}_M - \varphi\|_{L^\infty} \leqq K \|\hat{w}_M - \varphi\|_\gamma \leqq C \inf_{w \in S_M} \|w - \varphi\|_\gamma, \tag{6.8}$$

where

$$\|u\|_\gamma \equiv \left\{ \int_0^1 \left[\sum_{j=0}^{n} p_j(x) \left(D^j u(x)\right)^2 + \gamma \left(u(x)\right)^2 \right] dx \right\}^{\frac{1}{2}} \quad \text{for all} \quad u \in S. \tag{6.9}$$

It follows from Theorem 22 that, to bound the error in the Rayleigh-Ritz procedure, it suffices to bound the quantity $\inf_{w \in S_M} \|w - \varphi\|_\gamma$. For the subspaces under consideration, we do this by using the fact that

$$\inf_{w \in S_M} \|w - \varphi\|_\gamma \leqq \|\tilde{w} - \varphi\|_\gamma, \tag{6.10}$$

where $\tilde{w}$ is the "interpolation" of φ in the subspace S_M.

If the L-Hermite problem of Type I is well-posed for $Sp(L, \pi, z)$, then, given any constants α_i^k, $0 \leqq k \leqq z_i - 1$, $0 \leqq i \leqq N+1$, there exists a unique function $u(x) \in Sp(L, \pi, z)$ with

$$D^k u(x_i) = \alpha_i^k, \qquad 0 \leqq k \leqq z_i - 1, \qquad 0 \leqq i \leqq N+1.$$

Let $Sp^{\mathrm{I}}(L, \pi, z)$ denote the finite-dimensional subspace of $Sp(L, \pi, z)$ of all such functions $u(x)$. We attach a similar meaning to the subspaces $Sp^{\mathrm{II}}(L, \pi, z)$, $Sp^{\mathrm{III}}(L, \pi, z)$, and $Sp^{\mathrm{IV}}(L, \pi, z)$, and remark that these may have different dimensions. If the order of the differential operator L of (1.1) is such that $m \geqq n$ and z_0, $z_{N+1} \geqq n$, then $Sp_0^{\mathrm{I}}(L, \pi, z), \ldots, Sp_0^{\mathrm{IV}}(L, \pi, z)$ denote subspaces whose elements satisfy the boundary conditions of (6.2). Thus, if the L-Hermite problem of Type I (resp. Type II, III, or IV) is well-posed for $Sp(L, \pi, z)$, there is a unique $Sp(L, \pi, z)$-interpolate of $\varphi(x)$, the solution of (6.1)—(6.2), of Type I which is necessarily in $Sp_0^{\mathrm{I}}(L, \pi, z)$. Similarly, we consider finite dimensional subspaces $Sp(m, \pi, E)$ of g-splines with $m \geqq n$, subject to the condition that $e_{0,j} = e_{N+1,j} = 1$ for all $0 \leqq j \leqq n-1$, and $Sp_0^{\mathrm{I}}(m, \pi, E)$ and $Sp_0^{\mathrm{II}}(m, \pi, E)$ denote subspaces of such g-splines satisfying the boundary conditions of (6.2). Because Hermite and natural spline piecewise-polynomial functions are just special cases of such L-splines or g-splines, the following result, obtained directly from Theorems 7 and 19, generalizes and improves Theorems 10 and 16 of [17].

Theorem 23. Let $\varphi(x)$, the solution of (6.1)—(6.2), be of class $K_2^t[0, 1]$ with $t \geqq m > n$, let $\{\pi_i\}_{i=1}^{\infty}$ be any sequence of partitions of $[0, 1]$ with $\lim_{i\to\infty} \bar{\pi}_i = 0$, let $\{z_i\}_{i=1}^{\infty}$ be any sequence of corresponding incidence vectors, let L be a differential operator of the form (1.1), and let $\hat{w}_i(x)$ be the unique function which minimizes the functional $F[w]$ over the subspace $Sp_0^{\mathrm{I}}(L, \pi_i, z^{(i)})$ or $Sp_0^{\mathrm{II}}(L, \pi_i, z^{(i)})$. Then, there exists a positive integer i_0 and a positive constant M_1, independent of i, such that

$$\|\hat{w}_i - \varphi\|_{L^\infty[0,1]} \leqq K \|\hat{w}_i - \varphi\|_\gamma \leqq K M_1 (\bar{\pi}_i)^{m-n} \|L\varphi\|_{L^2[0,1]} \tag{6.11}$$

for all $i \geqq i_0$. Similarly, if $\{E^{(i)}\}_{i=1}^{\infty}$ is any sequence of incidence matrices which, with the partitions $\{\pi_i\}_{i=1}^{\infty}$, satisfies the hypothesis of Theorem 18, let $\hat{w}_i(x)$ be the unique function which minimizes $F[w]$ over the subspace $Sp_0^{\mathrm{I}}(m, \pi_i, E^{(i)})$ or $Sp_0^{\mathrm{II}}(m, \pi_i, E^{(i)})$. Then, there exists a positive integer i_1 and a positive constant M_2, independent of i, such that

$$\|\hat{w}_i - \varphi\|_{L^\infty[0,1]} \leqq K \|\hat{w}_i - \varphi\|_\gamma \leqq K M_2 (\bar{\pi}_i)^{m-n} \|D^m \varphi\|_{L^2[0,1]} \tag{6.12}$$

for all $i \geqq i_1$.

The next result follows directly from Theorems 9 and 21. It generalizes and improves Theorems 10 and 16 of [17].

Theorem 24. Let $\varphi(x)$, the solution of (6.1)—(6.2), be of class $K_2^t[0, 1]$ with $t \geqq 2m \geqq 2n$, let $\{\pi_i\}_{i=1}^{\infty}$ be any sequence of partitions of $[0, 1]$ with $\lim_{i\to\infty} \bar{\pi}_i = 0$, let $\{z^{(i)}\}_{i=1}^{\infty}$ be any sequence of corresponding incidence vectors, let L be a differential operator of the form (1.1), and let $\hat{w}_i(x)$ be the unique function which minimizes $F[w]$ over the subspace $Sp_0^{\mathrm{I}}(L, \pi_i, z^{(i)})$ or $Sp_0^{\mathrm{III}}(L, \pi_i, z^{(i)})$. Then, there exists a positive integer i_0 and a positive constant M_3, independent of i, such that

$$\|\hat{w}_i - \varphi\|_{L^\infty[0,1]} \leqq K \|\hat{w}_i - \varphi\|_\gamma \leqq K M_3 (\bar{\pi}_i)^{2m-n} \|L^* L[\varphi]\|_{L^2[0,1]} \tag{6.13}$$

for all $i \geqq i_0$. Similarly, if $\{E^{(i)}\}_{i=1}^{\infty}$, is any sequence of incidence matrices which, with the partitions $\{\pi_i\}_{i=1}^{\infty}$, satisfy the hypothesis of Theorem 18, let $\hat{w}_i(x)$ be the unique function which minimizes $F[w]$ over the subspace $Sp_0^{\mathrm{I}}(m, \pi_i, E^{(i)})$ or $Sp_0^{\mathrm{II}}(m, \pi_i, E^{(i)})$. Then, there exists a positive integer i_1 and a positive constant M_4, independent of i, such that

$$\|\hat{w}_i - \varphi\|_{L^\infty[0,1]} \leqq K \|\hat{w}_i - \varphi\|_\gamma \leqq K M_4 (\bar{\pi}_i)^{2m-n} \|D^{2m} \varphi\|_{L^2[0,1]} \tag{6.14}$$

for all $i \geqq i_1$.

We remark that the asymptotic error estimates given in (6.11) and (6.13) are independent of the choice of incidence vectors, and independent of the choice of the particular differential operator L.

As a particular example, consider the case in which the solution $\varphi(x)$ of the linear problem $D^2 u(x) = f(x)$, $0 < x < 1$, $u(0) = u(1) = 0$, is only of class $K_2^2[0, 1]$. In this case, to satisfy the hypotheses of Theorem 23, m must be chosen to be at least 2, and we obtain for $m = 2$ a sequence of functions which converges *linearly* in $\bar{\pi}_i$ to $\varphi(x)$. Furthermore, to satisfy the hypothesis of Theorem 24, m can be chosen to be 1 and we again obtain a sequence of functions which converges *linearly* in $\bar{\pi}_i$ to $\varphi(x)$. Such results, as far as we know, are not obtainable from Taylor series and Gerschgorin-type convergence arguments for discrete methods applied to such two-point boundary value problems.

For results of numerical experiments obtained from applying the Rayleigh-Ritz procedure to Hermite and natural spline subspaces for two-point nonlinear boundary value problems (6.1)—(6.2), we refer the reader to [17].

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