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# Piecewise Hermite Interpolation in One and Two Variables with Applications to Partial Differential Equations

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### § 2. Piecewise Hermite Interpolation in One Variable

In this section, we derive upper bounds for the errors in *piecewise* Hermite interpolation in one variable. We start by obtaining (Theorem 1) an integral representation for the error for Hermite interpolation for an unpartitioned interval by means of the Peano Kernel Theorem [9, 14, 18]. Because of the local character of piecewise Hermite interpolation on a partitioned interval, we are then able to obtain global error bounds for piecewise Hermite interpolation (Theorem 2).

To begin, we introduce some notation which will be useful for subsequent generalizations. We shall throughout denote any closed intervals $[a, b]$ by $I$ and $[0, 1]$ by $I_u$.

**Definition 1.** For $m$ a positive integer, let $H^{(m)}(I)$ be the set of all real polynomials of degree at most $2m-1$, defined on the interval $I$.

**Definition 2.** Given any real-valued function $f(x) \in C^{m-1}(I)$, let its $H^{(m)}(I)$-*interpolate*, for $m$ a positive integer, be any element $f_m(x)$ of $H^{(m)}(I)$ such that

$$D^k f(a) = D^k f_m(a); \quad D^k f(b) = D^k f_m(b), \quad 0 \leqq k \leqq m-1, \quad D \equiv \frac{d}{dx}. \tag{2.1}$$

Since (2.1) is a special case of Hermite interpolation, it follows [9, p. 67] that each real-valued function in $C^{m-1}(I)$ possesses a *unique* $H^{(m)}(I)$-interpolate.

**Definition 3.** For any positive integer $p$ and any extended real number $r$, $1 \leqq r \leqq +\infty$, let $K^{p,r}(I)$ be the set of all real-valued functions $f(x)$ defined on $I$, such that $D^{p-1}f(x)$ is absolutely continuous on $I$ and $D^p f(x) \in L^r(I)$.

We remark that $K^{p,r}(I)$ is actually a subset of the Sobolev space $W^{p,r}(I)$, and that the particular set $K^{p,1}(I)$ is used extensively in the work of Sard [18, p. 12].

Because of the local character of piecewise Hermite interpolation, we can focus our attention on the unit interval $I_u = [0, 1]$. Now, it is clear that each element $f(x) \in K^{p,r}(I_u)$ for $p \geqq m$ possesses a unique $H^{(m)}(I_u)$-interpolate $f_m(x)$, and that the error $D^j f(x_0) - D^j f_m(x_0)$, for fixed $x_0$ with $0 \leqq x_0 \leqq 1$, is a linear functional on $K^{p,r}(I_u)$. We call this error functional

$$F(f) \equiv D^j f(x_0) - D^j f_m(x_0), \quad 0 \leqq j \leqq p-1, \quad f \in K^{p,r}(I_u), \tag{2.2}$$

*which depends on* $j, x_0, m, p,$ *and* $r$. Next, by the definition of $F$, we can also express $F(f)$ as

$$F(f) = \sum_{i=0}^{p-1} \int_0^1 D^i f(x)\, d\mu_i(x; x_0), \quad f \in K^{p,r}(I_u), \tag{2.3}$$

where each $\mu_i(x; x_0)$ is of *bounded variation* with respect to $x$ in $I_u$ for each $x_0 \in I_u$. To give an explicit representation of these functions $\mu_i(x; x_0)$, let $S_{0,k}(x; m)$ and $S_{1,k}(x; m)$ be the polynomials of degree $2m-1$ defined by

$$\begin{aligned} &D^\ell S_{0,k}(0; m) = \delta_{\ell,k}; \quad D^\ell S_{0,k}(1; m) = 0 \quad \text{for all} \quad 0 \leqq k, \ell \leqq m-1, \\ &D^\ell S_{1,k}(0; m) = 0; \quad D^\ell S_{1,k}(1; m) = \delta_{\ell,k} \quad \text{for all} \quad 0 \leqq k, \ell \leqq m-1, \end{aligned} \tag{2.4}$$

where $\delta_{\ell,k}$ is the Kronecker delta function. Then, the $H^{(m)}(I_u)$-interpolate $f_m(x)$ of any $f \in C^{m-1}(I_u)$ can be uniquely expressed as

$$f_m(x) = \sum_{k=0}^{m-1} \{(D^k f(0))\, S_{0,k}(x; m) + (D^k f(1))\, S_{1,k}(x; m)\}. \tag{2.5}$$

With this, it follows that the functions $\mu_i(x; x_0)$ of (2.3) can be defined, for $0 < x_0 < 1$, by

$$\text{(2.6)}\quad \mu_i(x; x_0) = \begin{cases} 0, & x = 0, \\ -D^j S_{0,i}(x_0; m), & 0 < x < x_0, \\ \delta_{i,j} - D^j S_{0,i}(x_0; m), & x_0 \leqq x < 1, \\ \delta_{i,j} - D^j S_{0,i}(x_0; m) - D^j S_{1,i}(x_0; m), & x = 1, \quad 0 \leqq i \leqq m-1, \end{cases}$$

and

$$\text{(2.6')}\quad \mu_i(x; x_0) = \begin{cases} 0, & 0 \leqq x \leqq x_0 \\ \delta_{i,j}, & x_0 < x \leqq 1 \end{cases}, \quad m \leqq i \leqq p-1.$$

Now, we observe that if $g(x)$ is any polynomial of degree at most $2m-1$, then $g(x)$ is identical with its $H^{(m)}(I_u)$-interpolate $g_m(x)$, i.e., $g(x) \equiv g_m(x)$. Consequently, $D^j g(x) \equiv D^j g_m(x)$ for any $j \geqq 0$. This shows that

(2.7) $F(g) = 0$ for any polynomial $g$ of degree at most $p-1$, where $1 \leqq p \leqq 2m$.

But with the expressions of (2.3) and (2.7), the *Peano Kernel Theorem*[1] [18, p. 25] can be applied, and we thus have

**Theorem 1.** Given any $f(x) \in K^{p,r}(I_u)$ where $p \geqq m$, then for any fixed $x_0$ with $0 \leqq x_0 \leqq 1$ the functional of (2.2) can be expressed as

$$\text{(2.8)}\qquad D^j f(x_0) - D^j f_m(x_0) = \int_0^1 D^s f(t)\, k_{j,m,s}(t; x_0)\, dt, \qquad s = \min(p, 2m), \quad 0 \leqq j \leqq s-1,$$

where

$$\text{(2.9)}\qquad k_{j,m,s}(t; x_0) \equiv F_x\left\{\frac{(x-t)_+^{s-1}}{(s-1)!}\right\} = \sum_{i=0}^{s-1} \int_t^1 d\mu_i(x; x_0)\, \frac{(x-t)^{s-i-1}}{(s-i-1)!}.$$

We remark that $F_x$ in (2.9) means the application of $F$ to $\left\{\frac{(x-t)_+^{s-1}}{(s-1)!}\right\}$ considered as a function of $x$ for fixed $t$, and, as usual [9, p. 70],

$$(x-t)_+^{s-1} = \begin{cases} (x-t)^{s-1} & \text{for } t \leqq x, \\ 0 & \text{for } x < t. \end{cases}$$

The explicit representations of (2.5) and (2.6) allow us to determine the kernels $k_{j,m,s}(t, x_0)$, and we do this in two particular cases to illustrate the result of Theorem 1. For the case of *linear interpolation*, i.e., $m = 1$, consider the particular choices of $j = 0$ and $p = 2$ of (2.2). Then, the associated kernel $k_{0,1,2}(t; x_0)$ is explicitly given by the Green's function for the two-point boundary value problem, $D^2 u(x) = g(x)$, $0 < x < 1$, $u(0) = u(1) = 0$:

$$k_{0,1,2}(t; x_0) = \begin{cases} (x_0 - 1)\, t, & 0 \leqq t \leqq x_0, \\ x_0 (t-1), & x_0 \leqq t \leqq 1. \end{cases}$$

Similarly, for the case of *cubic interpolation*, i.e., $m = 2$, consider the particular choices of $j = 0$ and $p = 4$ in (2.2). Then, the associated kernel $k_{0,2,4}(t; x_0)$ is

[1] For a proof of a slightly less general result, see [9, p. 70].

explicitly given by the Green's function for the two-point boundary value problem, $D^4 u(x) = g(x)$, $0 < x < 1$, $u(0) = Du(0) = u(1) = Du(1) = 0$:

$$6k_{0,2,4}(t;\, x_0) = \begin{cases} (x_0 - t)^3 - (1-t)^3\,[3x_0 - 2x_0^3] - 3(1-t)^2\,[-x_0^2 + x_0^3], & 0 \leqq t \leqq x_0, \\ -(1-t)^3\,[3x_0^2 - 2x_0^3] - 3(1-t)^2\,[-x_0^2 + x_0^3], & x_0 \leqq t \leqq 1. \end{cases}$$

From formula (2.6), it is clear that each $\mu_i(x;\, x_0)$ is of bounded variation on $I_u$, *uniformly* with respect to $x_0 \in I_u$, i.e., there exists a constant $K$, dependent on $j$ and $m$ but independent of $x_0$, such that

$$\operatorname{Var} \mu_i(x;\, x_0) \leqq K \quad \text{for all} \quad x, x_0 \in I_u, \quad \text{all} \quad 0 \leqq i \leqq s-1.$$

Thus, as $|(x-t)^{s-i-1}|$ is bounded in $I_u \times I_u$, it follows from (2.9) that the kernel $k_{j,m,s}(t;\, x_0)$ is uniformly bounded in $I_u \times I_u$. Consequently, if $\frac{1}{r} + \frac{1}{r'} = 1$, then the function

$$g_{j,m,s,r}(x_0) \equiv \left\{ \int_0^1 |k_{j,m,s}(t;\, x_0)|^{r'}\, dt \right\}^{1/r'}$$

is an element of $L^q[0,1]$ for *any* $q$ with $1 \leqq q \leqq +\infty$, and we can define the constants $c_{j,m,s,r,q}$ by

$$c_{j,m,s,r,q} \equiv \left\{ \int_0^1 |g_{j,m,s,r}(x_0)|^q\, dx_0 \right\}^{1/q}. \tag{2.10}$$

This can be used as follows. First, applying Hölder's inequality to (2.8) gives

$$|D^j(f - f_m)(x_0)| \leqq \|D^s f\|_{L^r(I_u)} \cdot g_{j,m,s,r}(x_0), \qquad 0 \leqq j \leqq s-1, \tag{2.11}$$

and integrating the $q$-th power of both sides with respect to $x_0$ gives, with the definition of $c_{j,m,s,r,q}$, the result of

**Corollary 1.** Given any $f \in K^{p,r}(I_u)$ with $p \geqq m$, then with $s \equiv \min(p, 2m)$,

$$\|D^j(f - f_m)\|_{L^q(I_u)} \leqq c_{j,m,s,r,q} \|D^s f\|_{L^r(I_u)} \tag{2.12}$$

for all $0 \leqq j \leqq s-1$ and all $1 \leqq q \leqq +\infty$.

The inequality of (2.12) is sharp for the set $K^{p,r}(I_u)$ when $q = +\infty$, i.e., if $m \leqq p \leqq 2m$, then given any integer $j$ with $0 \leqq j \leqq p-1$, there is a function $f(x) \in K^{p,r}(I_u)$ for which equality is valid in (2.12) when $q = +\infty$. Thus, the particular constants $c_{j,m,s,r,\infty}$ are best possible in (2.12), and they can in principle be computed from (2.10). Upper bounds for these constants are more easily determined, and in fact it was recently shown in [7] that

$$c_{0,m,2m,\infty,\infty} = \frac{1}{2^{2m}(2m)!}; \quad c_{j,m,2m,\infty,\infty} < \frac{1}{2^{2m-2j}j!\,(2m-2)}, \quad 1 \leqq j \leqq m. \tag{2.13}$$

More recently, the exact values for $c_{j,m,2m,\infty,\infty}$ have also been determined in [4] for the cases $m = 1, 2, 3$.

To deduce results analogous to those of Corollary 1 for an arbitrary interval $I \equiv [a, b]$ is now easy. For any $f(x) \in K^{p,r}(I)$ with $p \geqq m$, Eq. (2.8) can be written as

$$\text{(2.14)} \quad \begin{aligned} &D^j\{f[a+x_0(b-a)] - f_m[a+x_0(b-a)]\} \\ &= (b-a)^{s-j} \int_0^1 k_{j,m,s}(t)\, D^s f[a+t(b-a)]\, dt, \quad 0 \leqq j \leqq s-1, \quad s = \min(p, 2m), \end{aligned}$$

where $0 \leqq x_0 \leqq 1$. Thus, we have

**Corollary 2.** Given any $f \in K^{p,r}(I)$ with $p \geqq m$, then with $s \equiv \min(p, 2m)$,

$$\text{(2.15)} \qquad \|D^j(f - f_m)\|_{L^q(I)} \leqq (b-a)^{s-j-\frac{1}{r}+\frac{1}{q}} c_{j,m,s,r,q} \|D^s f\|_{L^r(I)}$$

for all $0 \leqq j \leqq s-1$ and all $1 \leqq q \leqq +\infty$.

We are now in a position to estimate the *global* error for piecewise Hermite interpolation for the general case of a partitioned interval. Let $\pi$: $a = x_0 < x_1 < \cdots < x_{N+1} = b$ denote any partition of the interval $I$. The following definitions are the analogues for a partitioned interval of Definitions 1 and 2.

**Definition 4.** For $m$ a positive integer and $\pi$ a partition of $I$, let $H^{(m)}(\pi; I)$ be the set of all real-valued piecewise-polynomial functions $w(x)$ defined on $I$ such that $w(x) \in C^{m-1}(I)$, and $w(x)$ is a polynomial of degree $2m-1$ on each subinterval $[x_i, x_{i+1}]$ of $I$ defined by $\pi$.

**Definition 5.** Given any real-valued function $f(x) \in C^{m-1}(I)$, and any partition $\pi$ of $I$, let its (unique) $H^{(m)}(\pi; I)$-interpolate be the element $f_{m,\pi}$ of $H^{(m)}(\pi; I)$ such that

$$\text{(2.16)} \quad D^k f(x_i) = D^k f_{m,\pi}(x_i) \quad \text{for all} \quad 0 \leqq k \leqq m-1, \quad 0 \leqq i \leqq N+1.$$

If $\bar{\pi} \equiv \max_{0 \leqq i \leqq N} (x_{i+1} - x_i)$, then we prove

**Theorem 2.** Let $\pi$ be any partition of $I$, $f(x) \in K^{p,r}(I)$ where $p \geqq m \geqq 1$, and $f_{m,\pi}(x)$ be the $H^{(m)}(\pi; I)$-interpolate of $f(x)$. Then with $s \equiv \min(p, 2m)$,

$$\text{(2.17)} \qquad \|D^j(f - f_{m,\pi})\|_{L^q(I)} \leqq c_{j,m,s,r,q} (\bar{\pi})^{s-j-\frac{1}{r}+\frac{1}{q}} \|D^s f\|_{L^r(I)},$$

for any $q \geqq r$, for any $0 \leqq j \leqq m-1$, and also for $j = m$ if $p > m$ or $q = r$, and

$$\text{(2.17')} \qquad \|D^j(f - f_{m,\pi})\|_{L^q(I)} \leqq c_{j,m,s,r,r} (\bar{\pi})^{s-j} (b-a)^{\frac{(r-q)}{rq}} \|D^s f\|_{L^r(I)},$$

for any $1 \leqq q \leqq r$, for any $0 \leqq j \leqq m$.

*Proof.* With the above hypotheses, it is clear that $D^j(f - f_{m,\pi}) \in L^q(I)$ for any $0 \leqq j \leqq m$ and any $1 \leqq q \leqq +\infty$ if $p > m$, while if $p = m$, then $D^j(f - f_{m,\pi}) \in L^q(I)$ for any $0 \leqq j \leqq m$ and any $1 \leqq q \leqq r$ and $D^j(f - f_{m,\pi}) \in L^q(I)$ for any $0 \leqq j \leqq m-1$ and any $r < q \leqq +\infty$. With these restrictions on $j$ and $q$, let us define the quantities

$$v_i = \left( \int_{x_i}^{x_{i+1}} |D^j(f - f_{m,\pi})(t)|^q dt \right)^{1/q}, \qquad w_i = \left( \int_{x_i}^{x_{i+1}} |D^s f(t)|^r dt \right)^{1/r}, \quad 0 \leqq i \leqq N.$$

From (2.15) of Corollary 2, we have that

$$(2.18)\qquad v_i \leqq c_{j,m,s,r,q}(x_{i+1}-x_i)^{s-j-\frac{1}{r}+\frac{1}{q}}\,w_i,\qquad 0\leqq i\leqq N.$$

Hence, as $(x_{i+1}-x_i)\leqq\bar{\pi}$, it follows that

$$(2.19)\qquad \|D^j(f-f_m)\|_{L^q(I)}=\Big(\sum_{i=0}^{N}v_i^q\Big)^{1/q}\leqq c_{j,m,s,r,q}(\bar{\pi})^{s-j-\frac{1}{r}+\frac{1}{q}}\Big(\sum_{i=1}^{N}w_i^q\Big)^{1/q}.$$

For $q\geqq r$, Jensen's inequality [2, p. 18] gives

$$(2.20)\qquad \Big(\sum_{i=0}^{N}w_i^q\Big)^{1/q}\leqq\Big(\sum_{i=0}^{N}w_i^r\Big)^{1/r}=\|D^s f\|_{L^r(I)},$$

and combining this inequality with that of (2.19) establishes (2.17). To establish the inequality of (2.17′), consider the case $q=r$ of (2.17):

$$\|D^j(f-f_m)\|_{L^r(I)}\leqq c_{j,m,s,r,r}(\bar{\pi})^{s-j}\|D^s w\|_{L^r(I)}.$$

For $1\leqq q\leqq r$, the (integral) Hölder inequality applied to the left-hand side of the above inequality then gives the inequality of (2.17′). Q.E.D.

As a direct consequence of Theorem 2, we have the

**Corollary.** Let $\{\pi_i\}_{i=1}^{\infty}$ be any sequence of partitions of $I$ such that $\lim_{i\to\infty}\bar{\pi}_i=0$. If $f(x)\in K^{p,r}(I)$ where $p\geqq m\geqq 1$, and $f_i(x)$ denotes the unique $H^{(m)}(\pi_i; I)$-interpolate of $f(x)$, then the sequence $\{D^j f_i(x)\}_{i=1}^{\infty}$ converges *uniformly* to $D^j f(x)$ in $I$ if $0\leqq j<s-\frac{1}{r}$ and $0\leqq j\leqq m-1$ in the case that $p=m$, and if $0\leqq j\leqq s-\frac{1}{r}$ and $0\leqq j\leqq m$ in the case that $p>m$.

## § 3. Best Exponents in One Variable

It is natural to ask if the exponents of $\bar{\pi}$ in (2.17) and (2.17′) are best possible for the set $K^{p,r}(I)$. The main result of this section (Theorem 3) is a proof that these exponents are indeed best possible.

As in [19], let $q$ be any extended real number with $1\leqq q\leqq+\infty$, let $m$ be a positive integer, and consider any function $x^\mu$ on $I_u\equiv[0,1]$ such that $\mu>m-1-\frac{1}{q}$ and such that $x^\mu\notin V\equiv\operatorname{span}(1, x, \ldots, x^{2m-1})$. For each $h$ with $0\leqq h\leqq 1$, form

$$(3.1)\qquad \sigma_q(h; j, \mu, m)\equiv\inf_{r\in V}\|D_x^j(x^\mu-r(hx))\|_{L^q(I_u)},\qquad 0\leqq j\leqq m-1.$$

As $\sigma_q(h; j, \mu, m)$ is positive and continuous with respect to $h$ in $I_u$, then

$$(3.2)\qquad \min_{0\leqq h\leqq 1}\sigma_q(h; j, \mu, m)\equiv c_q(j, \mu, m)>0.$$

This will be used in the construction to follow.

Choose any positive integer $p$ with $m\leqq p\leqq 2m$, and choose any two extended real numbers $r$ and $q$ with $1\leqq r, q\leqq+\infty$. For each positive integer $n$ and any

$\varepsilon$ with $0<\varepsilon<1$ such that $\varepsilon \neq 1/r$, define the function $f_n(x)$ in $C^{p-1}(I_u)$ by

$$f_n(x) = x^{p-\frac{1}{r}+\varepsilon}, \quad 0 \leqq x \leqq 1/n, \quad \varepsilon > 0, \tag{3.3}$$

and

$$D^p f_n(x) \equiv 0, \quad 1/n \leqq x \leqq 1, \tag{3.4}$$

i.e., $f_n(x)$ is a polynomial of degree $p-1$ in $[1/n, 1]$. Note that $f_n \in K^{p,r}(I_u)$ for all $n \geqq 1$. For each positive integer $n$, we form the uniform partition $\pi_n$ of $I_u$, with mesh length $1/n$, so that $\bar{\pi}_n = 1/n$. It follows that for *any* function $s(x)$ in $H^{(m)}(\pi_n; I_u)$, we have

$$\begin{aligned} \|D^j(f_n(x) - s(x))\|_{L^q(I_u)} &\geqq \|D^j(f_n(x) - s(x))\|_{L^q[0,1/n]} \\ &\geqq \inf_{w \in V} \|D^j(f_n(x) - w(x))\|_{L^q[0,1/n]} \end{aligned} \tag{3.5}$$

for $0 \leqq j \leqq m-1$ if $q \geqq r$ and $p = m$, and $0 \leqq j \leqq m$ otherwise. But, with the change of variables $x = t/n$, the definitions of (3.2) and (3.3) give

$$\|D^j(f_n(x) - s(x))\|_{L^q(I_u)} \geqq \left(\frac{1}{n}\right)^{p-j-\frac{1}{r}+\varepsilon+\frac{1}{q}} c_q\left(j, p - \frac{1}{r} + \varepsilon, m\right) > 0 \tag{3.6}$$

for all $n \geqq 1$. On the other hand, we directly compute from (3.3)—(3.4) that

$$\|D^p f_n\|_{L^r(I_u)} = \frac{1}{n^\varepsilon} B_1(p, r, \varepsilon), \quad n \geqq 1, \tag{3.7}$$

where $B_1(p, r, \varepsilon)$ is independent of $n$. Since $\bar{\pi}_n = 1/n$, it follows that

$$\left(\frac{\|D^j(f_n - s)\|_{L^q(I_u)}}{(\bar{\pi}_n)^{p-j-\frac{1}{r}+\frac{1}{q}} \|D^p f_n\|_{L^q(I_u)}}\right) \geqq M_1(j, p, r, \varepsilon, m) > 0 \quad \text{for all} \quad n \geqq 1, \tag{3.8}$$

for any function $s(x)$ in $H^{(m)}(\pi_n; I_u)$, and $M_1$ is independent of $n$. In particular, this inequality must be valid with $s(x)$ equal to the $H^{(m)}(\pi_n; I_u)$-interpolate of $f_n(x)$. Thus, if we choose $q \geqq r$, we see that the exponent of $\bar{\pi}$ in (2.17) cannot in general be improved.

Similarly, for each positive integer $p$ with $m \leqq p \leqq 2m$ and for each positive integer $n$, we define the function $g_n(x) \in K^{p,r}(I_u) \cap C^{p-1}(I_u)$ by

$$g_n(x) = x^{p-\frac{1}{r}+\varepsilon}, \quad 0 \leqq x \leqq \frac{1}{n}, \quad \varepsilon > 0, \tag{3.9}$$

and

$$D^p g_n(x) = D^p g_n\left(x - \frac{i}{n}\right) \quad \text{for} \quad \frac{i}{n} \leqq x \leqq i + \frac{1}{n}, \quad 0 \leqq i \leqq n-1, \tag{3.10}$$

i.e., the $p$-th derivative of $g_n(x)$ is periodically extended over intervals of length $1/n$. It follows from (3.9) and (3.10) that

$$\|D^p g_n\|_{L^r(I_u)} = \left(\frac{1}{n}\right)^{\varepsilon - \frac{1}{r}} B_2(p, r, \varepsilon) > 0, \quad n \geqq 1. \tag{3.11}$$

Consider now any $s(x) \in H^{(m)}(\pi_n; I_u)$. By definition, for any $0 \leqq j \leqq m-1$ if $q \geqq r$ and $p = m$, and $0 \leqq j \leqq m$ otherwise, we have

$$(\|D^j(g_n - s)\|_{L^q(I_u)})^q = \sum_{\ell=0}^{n-1} \int_{\ell/n}^{(\ell+1)/n} |D^j(g_n(x) - s(x))|^q \, dx, \tag{3.12}$$

but it is clear from (3.9) and (3.10) that

$$D^j(g_n(x)) = D^j\left(\left(x - \frac{\ell}{n}\right)^{p-\frac{1}{r}+\varepsilon} + \sigma_\ell(x)\right), \tag{3.13}$$
$$0 \leqq \ell \leqq n-1, \quad \frac{\ell}{n} \leqq x \leqq \frac{\ell+1}{n},$$

where $\sigma_\ell(x)$ is a polynomial of degree $p-1$, determined in each subinterval $\left[\frac{\ell}{n}, \frac{\ell+1}{n}\right]$ so as to make $g_n(x) \in C^{p-1}[0, 1]$. Substituting (3.13) into each integral of (3.12) gives

$$\int\limits_{\ell/n}^{(\ell+1)/n} \left| D^j\left(\left(x - \frac{\ell}{n}\right)^{p-\frac{1}{r}+\varepsilon} + \sigma_\ell(x) - s(x)\right|^q dx,$$

and as $\sigma_\ell(x) - s(x)$ is a polynomial of degree at most $2m-1$, each of these integrals can be bounded below by $\left(\frac{1}{n}\right)^{\left(p-\frac{1}{r}+\varepsilon-j\right)q+1} \left[c_q\left(j, p - \frac{1}{r} + \varepsilon, m\right)\right]^q$, using the definitions of (3.1) and (3.2). In this way, we can establish that, for any $s(x) \in H^{(m)}(\pi_n; I_u)$,

$$\|D^j(g_n - s)\|_{L^q(I_u)} \geqq \left(\frac{1}{n}\right)^{p-\frac{1}{r}+\varepsilon-j} c_q\left(j, p - \frac{1}{r} + \varepsilon, m\right) > 0 \tag{3.14}$$

for all $n \geqq 1$. With (3.11), we then obtain

$$\frac{\|D^j(g_n - s)\|_{L^q(I_u)}}{(\bar{\pi})^{p-j} \|D^p g_n\|_{L^r(I_u)}} \geqq M_2(j, p, r, \varepsilon, m) > 0 \quad \text{for all} \quad n \geqq 1. \tag{3.15}$$

Thus, choosing $q \leqq r$ in (3.15) shows that the exponent of $\bar{\pi}$ in (2.17′) cannot be improved. This proves

**Theorem 3.** For any fixed choice of $p$ with $m \leqq p \leqq 2m$, any extended real numbers $1 \leqq r, q \leqq +\infty$, and any $j$ with $0 \leqq j \leqq m-1$ if $q > r$ and $p = m$ and $0 \leqq j \leqq m$ otherwise, there exist sequences of functions $\{f_n(x)\}_{n=1}^\infty$ and $\{g_n(x)\}_{n=1}^\infty$ in $K^{p,r}(I)$ and constants $M_1$ and $M_2$, independent of $n$, such that the inequalities of (3.8) and (3.15) are valid for all $n \geqq 1$. Thus, the exponents of $\bar{\pi}$ in (2.17) and (2.17′) cannot in general be increased for the class $K^{p,r}(I)$.

## § 4. Piecewise Bivariate Hermite Interpolation: Method of Sard

In this section, we now give upper bounds for errors in piecewise Hermite interpolation in two variables. To obtain these bounds we use the results of SARD [18], which are extensions of the Peano Kernel Theorem to higher dimensions.

**Definition 6.** Let $R$ be the rectangle $[a, b] \times [c, d]$. For any positive integer $m$, let $H^{(m)}(R)$ be the collection of all real polynomials in the variables $x$ and $y$ of degree $2m-1$ in each variable, i.e., $g(x, y) \in H^{(m)}(R)$ if and only if

$$g(x, y) = \sum_{j=0}^{2m-1} \sum_{i=0}^{2m-1} \alpha_{i,j} x^i y^j, \qquad (x, y) \in R.$$

**Definition 7.** Given any real function

$$f(x, y) \in C^{m-1,m-1}(R), \quad \text{i.e.,} \quad D^{(p,q)} f(x, y) \equiv \frac{\partial^{p+q} f(x, y)}{\partial x^p \, \partial y^q}$$

is continuous in $R$ for all $0 \leqq p, q \leqq m-1$, let its $H^{(m)}(R)$-interpolate be any element $f_m \in H^{(m)}(R)$ such that

$$D^{(p,q)} f(x_i, y_j) = D^{(p,q)} f_m(x_i, y_j) \quad \text{for all} \quad 0 \leqq p, q \leqq m-1, \tag{4.1}$$

where $x_i = a$ or $b$ and $y_j = c$ or $d$.

The $H^{(m)}(R)$-interpolate $f_m(x, y)$ of an element of $C^{m-1,m-1}(R)$ is determined by the $4m^2$ linear conditions of (4.1), and it is known (cf. [1]) that this interpolate is unique.

**Definition 8.** For any positive integer $p$ and any extended real number $r$ with $1 \leqq r \leqq +\infty$, let $S^{p,r}(R)$ be the set of all real-valued functions $f(x, y)$ defined on $R$ such that

$$D^{(p-i,i)} f \in L^r(R) \quad \text{for all} \quad 0 \leqq i \leqq p, \tag{4.2}$$

and

$$D^{(i,j)} f \in C^0(R) \quad \text{for all} \quad 0 \leqq i+j < p. \tag{4.3}$$

We remark that the set $S^{2p,r}(R)$ is actually a subset of the set $\boldsymbol{B}_{p,p}(R)$ of SARD [18, p. 172]. If $p \geqq 2m$, each element $f$ of $S^{p,r}(R)$ obviously possesses a unique $H^{(m)}(R)$-interpolate $f_m$, and each function $f(x, y)$ has a finite Taylor series (cf. (4.4)). For convenience, we restrict our attention to $R_u \equiv [0, 1] \times [0, 1]$. We now apply a general method of SARD [18, p. 163].

**Lemma 1.** Given any $f(x, y) \in S^{p,r}(R)$, with $p \geqq 2m$, then for almost all $(x_0, y_0) \in R_u$,

$$\begin{aligned} f(x, y) = & \sum_{i+j<2m} \frac{(x-x_0)^i}{i!} \frac{(y-y_0)^j}{j!} D^{(i,j)} f(x_0, y_0) \\ & + \sum_{j<m} \frac{(y-y_0)^j}{j!} T_x^{2m-j} D^{(2m-j,j)} f(x, y_0) \\ & + \sum_{i<m} \frac{(x-x_0)^i}{i!} T_y^{2m-i} D^{(i,2m-i)} f(x_0, y) \\ & + T_x^m T_y^m D^{(m,m)} f(x, y), \end{aligned} \tag{4.4}$$

where

$$T_x f(x, y) \equiv \int_{x_0}^{x} f(t, y)\, dt; \qquad T_y f(x, y) \equiv \int_{y_0}^{y} f(x, t)\, dt. \tag{4.5}$$

*Proof.* Since $D^{(i,2m-i)} f \in L^r(R_u)$ for each $0 \leqq i \leqq 2m$, then Fubini's Theorem gives us that $D^{(i,2m-i)} f(x, y_0)$ is integrable as a function of $x$ on $[0, 1]$ and $D^{(i,2m-i)} f(x_0, y)$ is integrable as a function of $y$ on $[0, 1]$ for all $0 \leqq i \leqq 2m$, for almost all $(x_0, y_0) \in R_u$. With this, one can directly verify that the analysis of SARD [18, p. 163] applies equally well to this case, giving the lemma.

Now, consider the linear functional $F$ *depending on* $x_0, y_0, h$, *and* $\ell$, defined on $S^{p,r}(R_u)$ by

$$F(f) = D^{(h,\ell)} f(x_0, y_0) - D^{(h,\ell)} f_m(x_0, y_0), \quad (x_0, y_0) \text{ fixed in } R_u, \quad f \in S^{p,r}(R_u), \tag{4.6}$$

where

$$h, \ell \text{ nonnegative with } \quad 0 \leqq h + \ell \leqq 2m-1. \tag{4.7}$$

As in the one-dimensional case, we must show that this functional has a form (cf. (2.3)), to which Sard's extension of the Peano Kernel Theorem is applicable. To this end, we make use of the polynomials $S_{0,k}(x)$ and $S_{1,k}(x)$, of degree $2m-1$, defined in (2.4). With these polynomials, we can represent the $H^{(m)}(R_u)$-interpolate of any $f \in S^{p,r}(R_u)$, $p \geqq 2m$ as

$$f_m(x,y) = \sum_{\alpha,\beta} \sum_{i,j=0}^{m-1} D^{(i,j)} f(\alpha,\beta)\, S_{\alpha,i}(x)\, S_{\beta,j}(y),$$

where $\alpha = 0$ or $1$, $\beta = 0$ or $1$, and consequently,

$$D^{(h,\ell)} f_m(x_0,y_0) = \sum_{\alpha,\beta} \sum_{i,j=0}^{m-1} D^{(i,j)} f(\alpha,\beta)\, S_{\alpha,i}^{(h)}(x_0)\, S_{\beta,j}^{(\ell)}(y_0). \tag{4.8}$$

**Lemma 2.** Given any $f(x,y) \in S^{p,r}(R_u)$, the functional $F(f)$ of (4.6) is the sum of the Riemann-Stieltjes integrals

$$\begin{aligned} F(f) = &\sum_{i,j<m} \int_0^1 \int_0^1 D^{(i,j)} f(s,t)\, d\mu_{i,j}(s,t) \\ &+ \sum_{\substack{i+j<2m \\ i \geqq m}} \int_0^1 D^{(i,j)} f(s,y_0)\, d\mu_{i,j}(s) \\ &+ \sum_{\substack{i+j<2m \\ j \geqq m}} \int_0^1 D^{(i,j)} f(x_0,t)\, d\mu_{i,j}(t) \end{aligned} \tag{4.9}$$

for all $(x_0, y_0) \in R_u$, where the $\mu_{i,j}$ are of bounded variation.

*Proof.* From (4.8), it is clear that the term $D^{(h,\ell)} f_m(x_0,y_0)$ can be represented by the double integrals of (4.9) for all $h, \ell$ satisfying (4.7). The same is true of $D^{(h,\ell)} f(x_0,y_0)$ if $0 \leqq h, \ell < m$. Furthermore, if for example, $h \geqq m$ so that $0 \leqq \ell \leqq m-1$ from (4.7), then by means of a step function with a unit jump at the $x_0$, we can trivially write

$$D^{(h,\ell)} f(x_0,y_0) = \int_0^1 D^{(h,\ell)} f(s,y_0)\, d\mu_{h,\ell}(s),$$

which establishes (4.9).

Next, it is easily seen that

(4.10) $F(g) = 0$ for any $g(x,y)$ a polynomial of degree less than or equal to $2m-1$ in each variable.

Consequently, with Lemmas 1 and 2, we can apply the Kernel Theorem of Sard [18, p. 175]:

**Theorem 4.** Let $f \in S^{p,r}(R_u)$ where $p \geqq 2m$ and $h, \ell$ satisfy (4.7). Then for almost all $(x_0, y_0) \in R_u$, the functional $F$ of (4.6) can be expressed as

$$\begin{aligned} F(f) = D^{(h,\ell)}(f - f_m)(x_0,y_0) = &\sum_{j<m} \int_0^1 D^{(2m-j,j)} f(t,y_0)\, k_{2m-j,j}(t)\, dt \\ &+ \sum_{i<m} \int_0^1 D^{(i,2m-i)} f(x_0,t)\, k_{i,2m-i}(t)\, dt \\ &+ \int_0^1 \int_0^1 D^{(m,m)} f(t,t')\, k_{m,m}(t,t')\, dt\, dt', \end{aligned} \tag{4.11}$$

where the kernels may be expressed as

$$\begin{aligned} k_{2m-j,j}(t) &= F\left[\frac{(x-t)^{2m-j-1}}{(2m-j-1)!}\psi(x_0,t,x)\frac{(y-y_0)^j}{j!}\right], \quad 0<t<1, \quad 0\leqq j<m, \\ k_{i,2m-i}(t) &= F\left[\frac{(x-x_0)^i}{i!}\frac{(y-t)^{2m-i-1}}{(2m-i-1)!}\psi(x_0,t,y)\right], \quad 0<t<1, \quad 0\leqq i<m, \\ k_{m,m}(t,t') &= F\left[\frac{(x-t)^{m-1}}{(m-1)!}\psi(x_0,t,x)\frac{(y-t)^{m-1}}{(m-1)!}\psi(y_0,t',y)\right], \quad 0<t,\ t'<1, \end{aligned} \tag{4.12}$$

where in general

$$\psi(a,t,x) \equiv \begin{cases} 1 & \text{if } a\leqq t\leqq x \\ -1 & \text{if } x\leqq t<a \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

To illustrate the result of Theorem 4, let us first consider the case of bilinear Hermite interpolation $m=1$, with $h=1$, and $\ell=0$. Then, the kernels $k_{2,0}(t)$, $k_{0,2}(t)$, and $k_{1,1}(t,t')$ are explicitly:

$$\begin{aligned} k_{2,0}(t) = k_{0,2}(t) &= \begin{cases} t, & 0\leqq t<x_0 \\ t-1, & x_0\leqq t\leqq 1 \end{cases} \\ k_{1,1}(t,t') &= \begin{cases} 1-y_0, & 0\leqq t'\leqq y_0 \\ -y_0, & y_0\leqq t'\leqq 1. \end{cases} \end{aligned} \tag{4.14}$$

Because the kernels of (4.12) are of bounded variation, uniformly with respect to $(x_0, y_0)\in R_u$, as in the one-dimensional case described in §2, we can define for $\frac{1}{r}+\frac{1}{r'}=1$ the following constant

$$\begin{aligned} M &\equiv M(h,\ell,m,r) \\ &\equiv \sup_{\substack{(x_0,y_0)\in R_u \\ 0\leqq j\leqq 2m \\ j\neq m}} \left\{\left(\int_0^1 |k_{2m-j,j}(t;\,x_0,y_0)|^{r'}dt\right)^{1/r'};\ \left(\int_0^1\int_0^1 |k_{m,m}(t,t';\,x_0,y_0)|^{r'}dt\,dt'\right)^{1/r'}\right\}. \end{aligned} \tag{4.15}$$

Thus, applying Hölder's inequality to the terms of (4.11), we have from the definition of $M$ that

$$\begin{aligned} |D^{(h,\ell)}(f-f_m)(x_0,y_0)| \leqq{} & M\sum_{j<m}\left(\int_0^1 |D^{(2m-j,j)}f(t,y_0)|^r dt\right)^{1/r} \\ & + M\sum_{i<m}\left(\int_0^1 |D^{(i,2m-i)}f(x_0,t)|^r dt\right)^{1/r} \\ & + M\left(\int_0^1\int_0^1 |D^{(m,m)}f(t,t')|^r dt\,dt'\right)^{1/r}, \quad 0\leqq h+\ell\leqq 2m-1. \end{aligned} \tag{4.16}$$

Now, as $r$ is fixed by the assumption that $f\in S^{p,r}(R_u)$, $p\geqq 2m$, consider the function

$$g(y_0) \equiv \left(\int_0^1 |D^{(2m-j,j)}f(t,y_0)|^r dt\right)^{1/r}.$$

By definition

$$(\|g(y_0)\|_{L^r(R_u)})^r = \int_0^1 |g(y_0)|^r dy_0 = \int_0^1 dy_0 \int_0^1 |D^{(2m-j,j)}f(t,y_0)|^r dt = (\|D^{(2m-j,j)}f\|_{L^r(R_u)})^r,$$

i.e.,

$$\|g(y_0)\|_{L^r(R_u)} = \|D^{(2m-j,j)} f\|_{L^r(R_u)}. \tag{4.17}$$

With this, taking $L^r$-norms in (4.16) then yields

**Corollary 1.** Given any $f \in S^{p,r}(R_u)$ with $p \geqq 2m$, then

$$\|D^{(h,\ell)}(f - f_m)\|_{L^r(R_u)} \leqq M \sum_{i=0}^{2m} \|D^{(i,2m-i)} f\|_{L^r(R_u)} \tag{4.18}$$

for all $0 \leqq h + \ell \leqq 2m - 1$.

To deduce similar results for an arbitrary rectangle $R = [a, b] \times [c, d]$ in the $(x, y)$-plane is again easy. The analogue of (4.11) is simply

$$\begin{aligned} D^{(h,\ell)}&(f - f_m)\,[a + x_0(b-a),\, c + y_0(d-c)] \\ &= \sum_{j<m} (b-a)^{2m-j-h} (d-c)^{j-\ell} \int_0^1 D^{(2m-j,j)} f[a + t(b-a),\, c + y_0(d-c)] \\ &\qquad\qquad \cdot k_{2m-j,j}(t)\, dt \\ &+ \sum_{i<m} (b-a)^{i-h} (d-c)^{2m-i-\ell} \int_0^1 D^{(i,2m-i)} f[a + x_0(b-a),\, c + t(d-c)] \\ &\qquad\qquad \cdot k_{i,2m-i}(t)\, dt \\ &+ (b-a)^{m-h} (d-c)^{m-\ell} \int_0^1\!\!\int_0^1 D^{(m,m)} f[a + t(b-a),\, c + t'(d-c)] \\ &\qquad\qquad \cdot k_{m,m}(t, t')\, dt\, dt' \end{aligned} \tag{4.19}$$

for almost all $(x_0, y_0) \in [0, 1] \times [0, 1]$, $0 \leqq h + \ell \leqq 2m - 1$. Next, we note that

$$\begin{aligned} \left\{\int_0^1\!\!\int_0^1 |\mu(a + x_0(b-a),\, c + y_0(d-c))|^r\, dx_0\, dy_0\right\}^{\frac{1}{r}} \\ = [(b-a)(d-c)]^{-\frac{1}{r}} \left\{\int_a^b\!\!\int_c^d |\mu(t, t')|^r\, dt\, dt'\right\}^{\frac{1}{r}}. \end{aligned}$$

In other words, calculating the $L^r$-norm in $R_u$ introduces a factor in $(b-a)(d-c)$, relative to $R = [a, b] \times [c, d]$. Now, if we compute $L^r$-norms in (4.19) in $R$, the same factor appears in *each* term. Consequently, we have the following result.

**Corollary 2.** Given any $f \in S^{p,r}(R)$ with $p \geqq 2m$, then

$$\|D^{(h,\ell)}(f - f_m)\|_{L^r(R)} \leqq M \sum_{i=0}^{m} (b-a)^{i-h} (d-c)^{2m-i-\ell} \|D^{(i,2m-i)} f\|_{L^r(R)} \tag{4.20}$$

for all $0 \leqq h + \ell \leqq 2m - 1$.

We consider now arbitrary partitions in each coordinate direction of $R$:

$$\begin{aligned} \pi:&\ a = x_0 < x_1 < \cdots < x_{N+1} = b, \\ \pi':&\ c = y_0 < y_1 < \cdots < y_{N'+1} = d, \end{aligned} \tag{4.21}$$

where $N$ and $N'$ are nonnegative integers. We say that $\varrho \equiv \pi \times \pi'$ defines a partition on $R$.

**Definition 9.** For any positive integer $m$ and partition $\varrho \equiv \pi \times \pi'$ of $R$, let $H^{(m)}(\varrho; R)$ be the set of all real-valued piecewise-polynomial functions $w(x, y)$

defined on $R$ such that $D^{(i,j)}w \in C^0(R)$ for all $0 \leqq i, j \leqq m-1$, and such that $w(x, y)$ is a polynomial of degree $2m-1$ in both $x$ and $y$ in each subrectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ defined on $R$ by $\varrho$.

**Definition 10.** Given a real-valued function $f(x, y) \in C^{m-1,m-1}(R)$, let its $H^{(m)}(\varrho; R)$-interpolate be an element $f_{m,\varrho}$ of $H^{(m)}(\varrho; R)$ such that

$$D^{(i,j)} f(x_k, y_\ell) = D^{(i,j)} f_{m,\varrho}(x_k, y_\ell) \quad \text{for all} \quad 0 \leqq k \leqq N+1, \quad 0 \leqq \ell \leqq N'+1, \quad \text{and all} \quad 0 \leqq i, j \leqq m-1. \tag{4.22}$$

We remark that the $H^{(m)}(\varrho; R)$-interpolate of any function in $C^{m-1,m-1}(R)$ is again uniquely determined (cf. [1]). Moreover, it is again *local* in the sense that the restriction of the interpolate, $f_{m,\varrho}$, to the subrectangle $[x_k, x_{k+1}] \times [y_\ell, y_{\ell+1}]$, $0 \leqq k \leqq N$, $0 \leqq \ell \leqq N'$ depends only on the $4m^2$ numbers $D^{(i,j)} f(x_p, y_r)$, $p = k, k+1$, $r = \ell, \ell+1$, and $0 \leqq i, j \leqq m-1$.

With the result of Corollary 2 of Theorem 4, we need the following definition to obtain a two-dimensional analogue of Theorem 2.

**Definition 11.** A collection, $C$, of partitions, $\varrho \equiv \pi \times \pi'$, of $R$ is said to be *regular* if and only if there exist three positive constants $\sigma, \tau, \eta$ such that

$$\underline{\pi} \geqq \sigma \overline{\pi}, \quad \underline{\pi}' \geqq \sigma \overline{\pi}' \quad \text{for all} \quad \varrho \in C \tag{4.23}$$

and

$$\eta \leqq \frac{\overline{\pi}'}{\overline{\pi}} \leqq \tau \quad \text{for all} \quad \varrho \in C, \tag{4.24}$$

where

$$\overline{\pi} \equiv \max_i (x_{i+1} - x_i), \quad \overline{\pi}' \equiv \max_j (y_{j+1} - y_j),$$
$$\underline{\pi} \equiv \min_i (x_{i+1} - x_i), \quad \underline{\pi}' \equiv \min_j (y_{j+1} - y_j).$$

Let $C$ be any regular collection of partitions of $R$ and let $R_{p,q}$ be any subrectangle of a fixed $\varrho \equiv \pi \times \pi' \in C$. Applying (4.20) to $R_{p,q}$ gives

$$\|D^{(h,\ell)}(f - f_{m,\varrho})\|_{L^r(R_{p,q})} \leqq M' (\overline{\pi})^{2m-h-\ell} \sum_{j=0}^{2m} \|D^{(j,2m-j)} f\|_{L^r(R_{p,q})}, \tag{4.25}$$

where $M'$ depends on $M$, $\sigma$, $\tau$, and $\eta$. Raising (4.25) to the power $r$ yields

$$(\|D^{(h,\ell)}(f - f_{m,\varrho})\|_{L^r(R_{p,q})})^r \leqq (M')^r (\overline{\pi})^{r(2m-h-\ell)} \left( \sum_{j=0}^{2m} \|D^{(j,2m-j)} f\|_{L^r(R_{p,q})} \right)^r. \tag{4.26}$$

Applying the discrete form of the Hölder inequality to the last term of the above inequality then yields

$$\begin{aligned} &(\|D^{(h,\ell)}(f - f_{m,\varrho})\|_{L^r(R_{p,q})})^r \\ &\quad \leqq (M')^r (2m+1)^{r/r'} (\overline{\pi})^{r(2m-h-\ell)} \left\{ \sum_{j=0}^{2m} (\|D^{(j,2m-j)} f\|_{L^r(R_{p,q})})^r \right\}. \end{aligned} \tag{4.27}$$

Assuming $f \in S^{p,r}(R)$ with $p \geqq 2m$, then $f_{m,\varrho}$, its $H^{(m)}(\varrho; R)$-interpolate, is such that $D^{(h,\ell)} f_{m,\varrho} \in L^r(R)$ *provided that* $0 \leqq h, \ell \leqq m$ and $0 \leqq h + \ell \leqq 2m-1$, and the

$L^r(R)$-norm of $D^{(h,\ell)}(f-f_{m,\varrho})(x,y)$ in these cases can be represented by

$$(4.28)\qquad (\|D^{(h,\ell)}(f-f_{m,\varrho})\|_{L^r(R)})^r = \sum_{p,q} (\|D^{(h,\ell)}(f-f_{m,\varrho})\|_{L^r(R_{p,q})})^r.$$

Thus, adding the inequality of (4.27) over *all* rectangles $R_{p,q}$ of $R$ and making use of (4.28) for the cases when $0\leqq h,\ \ell\leqq m$ and $0\leqq h+\ell\leqq 2m-1$, yields

$$(4.29)\quad (\|D^{(h,\ell)}(f-f_{m,\varrho})\|_{L^r(R)})^r \leqq (M')^r(\bar{\pi})^{r(2m-h-\ell)}(2m+1)^{r/r'}\sum_{j=0}^{2m}(\|D^{(j,2m-j)}f\|_{L^r(R)})^r.$$

Then, taking $r$-th roots in (4.29) gives us

**Theorem 5.** Given any regular collection, $C$, of partitions $\varrho\equiv\pi\times\pi'$, of $R$ and any $f\in S^{p,r}(R)$, where $p\geqq 2m$, if $f_{m,\varrho}$ is the $H^{(m)}(\varrho;R)$-interpolate of $f$, then there exists a constant $K$ such that

$$(4.30)\qquad \|D^{(h,\ell)}(f-f_{m,\varrho})\|_{L^r(R)} \leqq K(\bar{\pi})^{2m-h-\ell},$$

for all $\varrho\in C$ and for all $0\leqq h,\ l\leqq m$ with $0\leqq h+\ell\leqq 2m-1$.

We remark that we could have used $\pi'$ in place of $\pi$ in (4.30), because of the assumption of regularity.

Next, we make use of the fact that Hermite interpolation is *local*. Thus, the results of Theorem 5 apply to any rectangular polygon, i.e., any polygon whose sides are parallel to the coordinate axes in the plane, such as an $L$-shaped region. We remark that any rectangular polygon can be expressed as a union of rectangles $\bigcup_{i=1}^{k} R_i$ such that $R_i\cap R_j$, $1\leqq i,j\leqq k$ is either void, or a subset of an edge of $R_i$ and an edge of $R_j$. In this case, we say that the rectangular polygon is *composed* of the rectangles $R_i$.

**Definition 12.** Let $T$ be a rectangular polygon, composed of the rectangles $R_i=[a_i,b_i]\times[c_i,d_i]$, $1\leqq i\leqq k$, in the $(x,y)$-plane and $C$ be a collection of partitions of $T$, i.e., each $\varrho\equiv\pi\times\pi'\in C$ defines a partition $\pi_i\times\pi_i'$ of each rectangle $R_i$ of $T$. Then, the collection $C$ is said to be *regular* if and only if there exists three positive constants $\sigma,\tau,\eta$ such that

$$(4.31)\qquad \underline{\pi}_i\geqq\sigma\bar{\pi}_i \quad\text{and}\quad \underline{\pi}_i'\geqq\sigma\bar{\pi}_i \quad\text{for all}\quad 1\leqq i\leqq k \quad\text{and for all}\quad \varrho\in C,$$

and

$$(4.32)\qquad \eta\leqq\frac{\bar{\pi}_i'}{\bar{\pi}_i}\leqq\tau, \qquad\text{for all}\quad 1\leqq i\leqq k \quad\text{and for all}\quad \varrho\in C.$$

From Theorem 5 and Definition 12, we have in a similar fashion

**Theorem 6.** Let $T$ be a rectangular polygon composed of the rectangles $R_i=[a_i,b_i]\times[c_i,d_i]$, $1\leqq i\leqq k$, in the $(x,y)$-plane, and let $C$ be a regular collection of partitions of $T$. If $f\in S^{p,r}(T)$ where $p\geqq 2m$ and $f_{m,\varrho}$ is the $H^{(m)}(\varrho;R_i)$-interpolate of $f$ on each $R_i$, $1\leqq i\leqq k$, then, setting $v\equiv\max_{1\leqq i\leqq k}\bar{\pi}_i$, there exists a constant $M$, such that

$$(4.33)\qquad \|D^{(h,\ell)}(f-f_{m,\varrho})\|_{L^r(T)}\leqq M(v)^{2m-h-\ell}\quad\text{for all}\quad \varrho\in C,$$

and for all $0\leqq h,\ \ell\leqq m$ with $0\leqq h+\ell\leqq 2m-1$.

## § 5. Best Exponents in Two Variables

It is again natural to ask, as in §3, if the exponent of $\bar{\pi}$ in (4.30) is best possible for the set $S^{p,r}(R)$, where $p \geqq 2m$. The result of this section (Theorem 7) is that this exponent is best possible, in the same sense as that of §3, when $r=+\infty$. In fact, we show that it is best possible for the set of all polynomials in $x$ and $y$. Whether the exponent of $\bar{\pi}$ in (4.30) is best possible for $1 \leqq r < \infty$ remains an open question.

Analogous to the construction of §3, we consider the function $f(x, y)=(x^2+y^2)^m$ defined on $R_u \equiv [0,1] \times [0,1]$.

**Theorem 7.** With $f(x, y) \equiv (x^2+y^2)^m$, let $\{\pi_n \times \pi_n'\}_{n=1}^{\infty}$ be the regular collection of partitions of $R_u$ with

$$\underline{\pi}_n = \bar{\pi}_n = \underline{\pi}_n' = \bar{\pi}_n' = \frac{1}{n}.$$

Then, there exists a positive constant $M$, dependent on $k$ and $\ell$, but independent of $n$, such that

$$\|D^{(k,\ell)}(f-f_n)\|_{L^\infty(R_u)} \geqq M(\bar{\pi}_n)^{2m-k-\ell} \quad \text{for all} \quad n \geqq 1, \tag{5.1}$$

for *any* $f_n \in H^{(m)}(\pi_n \times \pi_n'; R_u)$, and for all $0 \leqq k+\ell \leqq 2m-1$. Thus, the exponent of $\bar{\pi}$ in (4.30) of Theorem 5 cannot in general be improved for the special case $r=+\infty$.

*Proof.* Consider the uniform partition $\pi_n \times \pi_n'$ of $R_u$, $n \geqq 1$, and let $f_n \in H^{(m)}(\pi_n \times \pi_n'; R_u)$. By definition

$$\|D^{(k,\ell)}(f-f_n)\|_{L^\infty(R_u)} \geqq \|D^{(k,\ell)}(f-f_n)\|_{L^\infty([0,1/n]\times[0,1/n])}. \tag{5.2}$$

On the other hand, since $f(x, y)=(x^2+y^2)^m$, we have that

$$\frac{\partial^{k+\ell} f(x,y)}{\partial x^k \partial y^\ell} = \left(\frac{1}{n}\right)^{2m-k-\ell} \frac{\partial^{k+\ell} f(t,s)}{\partial t^k \partial s^\ell} \quad \text{where} \quad x=\frac{t}{n} \quad \text{and} \quad y=\frac{s}{n}.$$

Thus, we can write that

$$\|D^{(k,\ell)}(f-f_n)\|_{L^\infty([0,1/n]\times[0,1/n])} \geqq \left(\frac{1}{n}\right)^{2m-k-\ell} \|D^{(k,\ell)}(f(t,s)-g_n(t,s))\|_{L^\infty(R_u)}, \tag{5.3}$$

where $g_n(t,s) \equiv f_n(t/n, s/n)\, n^{(2m-k-\ell)}$, which is also an element of $H^{(m)}(R_u)$. Just as in §3, we use the fact that $(x^2+y^2)^m \notin H^{(m)}(R_u)$, since the elements of $H^{(m)}(R_u)$ are polynomials of the form $\sum_{i,j=0}^{2m-1} c_{i,j} x^i y^j$. Thus, we form

$$\sigma_\infty(h; k, \ell, m) = \inf_{g \in H^{(m)}(R_u)} \|D^{(k,\ell)}((x^2+y^2)^m - g(hx, hy))\|_{L^\infty(R_u)}. \tag{5.4}$$

As $\sigma_\infty(h; k, \ell, m)$ is positive and continuous in $h$ for $0 \leqq h \leqq 1$, then

$$\min_{0 \leqq h \leqq 1} \sigma_\infty(h; k, \ell, m) = c_\infty(k, \ell, m) > 0. \tag{5.5}$$

With this definition of $c_\infty(k, \ell, m)$, it follows from (5.3) that

$$\|D^{(k,\ell)}(f-f_n)\|_{L^\infty([0,1/n]\times[0,1/n])} \geqq \left(\frac{1}{n}\right)^{2m-k-\ell} c_\infty(k, \ell, m) > 0. \tag{5.6}$$

Thus, as $\max_{0\leq j\leq 2m}(\|D^{(j,2m-j)}f(x,y)\|_{L^\infty(R_u)})$ is a constant $K$, then from (5.2) and (5.6),

$$\frac{\|D^{(k,\ell)}(f-f_n)\|_{L^\infty(R_u)}}{\max_{0\leq j\leq 2m}(\|D^{(j,2m-j)}f\|_{L^\infty(R_u)})}\geq(\overline{\pi})^{2m-k-\ell}\cdot M, \tag{5.7}$$

where $\overline{\pi}_n=1/n$, and $M$ is a positive constant independent of $n$ and $f_n$, which establishes (5.1). Q.E.D.

## § 6. Piecewise Bivariate Hermite Interpolation: Method of Stancu and Simonsen

We have found it convenient, especially for the applications of §4 to boundary value problems for differential equations in rectangular domains, to have started with the Taylor series development of Lemma 1 which restricts the indices $i$ and $j$ in (4.4) through $i+j<2m$. This results in (4.11) in a representation of the functional $F$ of (4.6) in terms of integrals of the derivatives $D^{(i,2m-i)}f$. It is also possible to make a similar analysis based on a Taylor series development in which the indices $i$ and $j$ in (4.4) vary over the rectangular grid $0\leq i,j\leq 2m-1$. This is the approach of SIMONSEN [20] and STANCU [22], and this approach results in a different representation of the functional $F$ of (4.6).

To make matters precise, let $K^{p,r}\times K^{p,r}(R_u)$ be the collection of all real-valued functions $f(x,y)$ defined in $R_u=[0,1]\times[0,1]$ such that $D^{(i,j)}f\in C^0(R_u)$ for all $0\leq i,j\leq p-1$, and $D^{(i,j)}f\in L^r(R_u)$ for all $0\leq i,j\leq p$.

**Lemma 3.** Given any $f\in K^{p,r}\times K^{p,r}(R_u)$, then for all $(x_0,y_0)\in R_u$

$$\begin{aligned} f(x,y)&=\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}\frac{(x-x_0)^i}{i!}\frac{(y-y_0)^j}{j!}D^{(i,j)}f(x_0,y_0)+\int_{x_0}^{x}\frac{(x-t)^{p-1}}{(p-1)!}D^{(p,0)}f(t,y)\,dt\\ &+\int_{y_0}^{y}\frac{(y-z)^{p-1}}{(p-1)!}D^{(0,p)}f(x,z)\,dz-\int_{x_0}^{x}\int_{y_0}^{y}\frac{(x-t)^{p-1}(y-z)^{p-1}}{(p-1)!\,(p-1)!}D^{(p,p)}f(t,z)\,dt\,dz.\end{aligned}\tag{6.1}$$

*Proof.* The proofs of SIMONSEN [20] and STANCU [22] for the case $f\in C^{p,p}(R_u)$, extend to the case $f\in K^{p,r}\times K^{p,r}(R_u)$.

We consider again the functionals depending on $x_0$, $y_0$, $h$, and $\ell$,

$$F(f)=D^{(h,\ell)}f(x_0,y_0)-D^{(h,\ell)}f_m(x_0,y_0), \tag{6.2}$$

$(x_0,y_0)$ fixed in $R_u$, $f\in K^{p,r}\times K^{p,r}(R_u)$ where $p\geq m$, and the integers $h$ and $\ell$ are now restricted by

$$0\leq h,\ell\leq p-1. \tag{6.3}$$

Note that when $p\geq m$, any $f\in K^{p,r}\times K^{p,r}(R_u)$ possesses a unique $H^{(m)}(R_u)$-interpolate $f_m(x,y)$. If $m\leq p\leq 2m$, then, as in the one-dimensional case, $F(g)=0$ for any polynomial $g(x,y)$ of degree less than or equal to $2m-1$ in each variable $x$ and $y$.

Based on the representation of (6.1), SIMONSEN [20] and STANCU [22] have given the following integral representation for $F(f)$ for $f\in C^{p,p}(I)$, which also holds for $F\in K^{p,r}\times K^{p,r}(R_u)$.

**Theorem 8.** Let $f \in K^{p,r} \times K^{p,r}(R_u)$ where $p \geqq m$, $f_m$ be the $H^{(m)}(R_u)$-interpolate of $f(x,y)$, and $R_u = [0,1] \times [0,1]$. Then,

$$F(f) = D^{(h,\ell)}(f - f_m)(x_0, y_0) = \int_0^1 k_{h,m,s}(t; x_0)\, D^{(s,\ell)} f(t, y_0)\, dt$$

$$+ \int_0^1 k_{\ell,m,s}(z; y_0)\, D^{(h,s)} f(x_0, z)\, dz \tag{6.4}$$

$$- \int_0^1 \int_0^1 k_{h,m,s}(t, x_0)\, k_{\ell,m,s}(z; y_0)\, D^{(s,s)} f(t,z)\, dt\, dz$$

for all $0 \leqq h, \ell \leqq s-1$ where $s \equiv \min(p, 2m)$ and where the kernels $k_{h,m,s}(t, x_0)$ and $k_{\ell,m,s}(t, y_0)$ are defined in (2.9).

Using the definition of the constants $c_{j,m,s,r,q}$ of (2.10), we now apply Hölder's inequality to each term of (6.4), which results in the

**Corollary.** Given any $f \in K^{p,r} \times K^{p,r}(R_u)$ with $p \geqq m$ then

$$\|D^{(h,\ell)}(f - f_m)\|_{L^q(R_u)} \leqq c_{h,m,s,r,q} \|D^{(s,\ell)} f\|_{L^r(R_u)} + c_{\ell,m,s,r,q} \|D^{(h,s)} f\|_{L^r(R_u)} \tag{6.5}$$
$$+ c_{h,m,s,r,q}\, c_{\ell,m,s,r,q} \|D^{(s,s)} f\|_{L^r(R_u)}$$

for all $0 \leqq h, \ell \leqq s-1$, for all $1 \leqq q \leqq +\infty$, and for all $0 \leqq h, \ell \leqq s$, if $q \leqq r$, where $s \equiv \min(p, 2m)$.

Before we extend the results of Theorem 8 and its Corollary to full partitions, we briefly discuss the case of equality in (6.5). Using further results of Simonsen [20] and Stancu [22], based on the constancy of sign of the kernels $k_{0,m,2m}$, it is possible to show that (6.5) is indeed sharp for the case when $q = \infty$, $h = \ell = 0$, and $m$ is even. We have not investigated this question for other values of the parameters.

The extensions of the results of Theorem 8 and its Corollary to the case of partitions of an arbitrary rectangle again follow quite closely the method of Sard. Thus, we can omit obvious details, and we simply state

**Theorem 9.** Let $\varrho \equiv \pi \times \pi'$ be any partition of $R \equiv [a,b] \times [c,d]$, $f \in K^{p,r} \times K^{p,r}(R)$, where $p \geqq m$, and $f_{m,\varrho}$ be the $H^{(m)}(\varrho; R)$-interpolate of $f$. Then with $s \equiv \min(p, 2m)$

$$\|D^{(h,\ell)}(f - f_{m,\varrho})\|_{L^q(R)} \leqq c_{h,m,s,r,q} (\bar{\pi})^{s-h-\frac{1}{r}+\frac{1}{q}} \|D^{(s,\ell)} f\|_{L^r(R)}$$

$$+ c_{\ell,m,s,r,q} (\bar{\pi}')^{s-\ell-\frac{1}{r}+\frac{1}{q}} \|D^{(h,s)} f\|_{L^r(R)} \tag{6.6}$$

$$+ c_{h,m,s,r,q}\, c_{\ell,m,s,r,q} (\bar{\pi})^{s-h-\frac{1}{r}+\frac{1}{q}} (\bar{\pi}')^{s-\ell-\frac{1}{r}+\frac{1}{q}} \|D^{(s,s)} f\|_{L^r(R)}$$

for any $q \geqq r$, for any $0 \leqq h, \ell \leqq m-1$ and also for $0 \leqq h, \ell \leqq m$ if $p > m$, and

$$\|D^{(h,\ell)}(f - f_{m,\varrho})\|_{L^q(R)} \leqq c_{h,m,s,r,q} (\bar{\pi})^{s-h} (b-a)^{\frac{r-q}{rq}} \|D^{(s,\ell)} f\|_{L^r(R)}$$

$$+ c_{\ell,m,s,r,q} (\bar{\pi}')^{s-\ell} (d-c)^{\frac{r-q}{rq}} \|D^{(h,s)} f\|_{L^r(R)} \tag{6.6'}$$

$$+ c_{h,m,s,r,q}\, c_{\ell,m,s,r,q} (\bar{\pi})^{s-h} (\bar{\pi}')^{s-\ell} (b-a)^{\frac{r-q}{rq}} (d-c)^{\frac{r-q}{rq}} \|D^{(s,s)} f\|_{L^r(R)}$$

for any $q \leqq r$, for any $0 \leqq h, \ell \leqq m-1$ and also for $0 \leqq h, \ell \leqq m$ if $p > m$.

Note that the inherently less complicated bounds of (6.6) and (6.6′) require *no* mesh uniformity. This, as we shall see, is at the expense of assuming higher order smoothness in $f$, in which detracts from its applicability to differential equations.

Finally, the analogue of Theorem 6 which follows is less complicated, again because we need no assumption of mesh uniformity.

**Theorem 10.** Let $T$ be any rectangular polygon, composed of the rectangles $R_i=[a_i, b_i]\times[c_i, d_i]$, $1\leqq i\leqq k$, in the $(x, y)$-plane, $\varrho\equiv\pi\times\pi'$ be a partition of $T$, i.e., the restriction of $\pi\times\pi'$ to each rectangle, $R_i$, is a partition $\pi_i\times\pi_i'$, and $f\in K^{p,r}\times K^{p,r}(T)$, where $p\geqq m$. Then with $s\equiv\min(p, 2m)$ and $v\equiv\max\limits_{1\leqq i\leqq k}(\bar{\pi}_i, \bar{\pi}_i')$, there exist constants $M$ and $M'$, independent of $\varrho$, such that

$$\|D^{(h,\ell)}(f-f_{m,\varrho})\|_{L^q(T)}\leqq M(v)^{s-\max(h,\ell)-\frac{1}{r}+\frac{1}{q}}, \tag{6.7}$$

for any $q\geqq r$, for any $0\leqq h, \ell\leqq m-1$, and also for $0\leqq h, \ell\leqq m$ if $p>m$, and

$$\|D^{(h,\ell)}(f-f_{m,\varrho})\|_{L^q(T)}\leqq M'(v)^{s-\max(h,\ell)}, \tag{6.8}$$

for any $q\leqq r$, for any $0\leqq h, \ell\leqq m-1$, and also for $0\leqq h, \ell\leqq m$ if $p>m$.

## § 7. Sobolev Norms

Starting with the one-dimensional results of §2, let $\pi$ denote a partition of $I=[a, b]$, $f\in K^{p,r}(I)$ where $p\geqq m$, and $f_{m,\pi}$ be its $H^{(m)}(\pi; I)$-interpolate. The Sobolev norm [26, p. 55] of $f$ is

$$\|f\|_{\ell,2}\equiv\left\{\sum_{0\leqq j\leqq\ell}\int_a^b|D^j f(x)|^2\,dx\right\}^{\frac{1}{2}} \quad\text{for any}\quad 0\leqq\ell\leqq p-1. \tag{7.1}$$

Since $f-f_{m,\pi}$ is an element of $K^{m,r}(I)$ if $p=m$ and $r<2$ and an element of $K^{m,2}(I)$ otherwise, we can interpret the result of Theorem 2 in the norm of (7.1). Specifically, we have

**Theorem 11.** Let $\pi$ be any partition of $I$. If $f(x)\in K^{p,r}(I)$ where $p\geqq m$ and $f_{m,\pi}$ denotes the $H^{(m)}(\pi; I)$-interpolate of $f(x)$, then, with $s\equiv\min(p, 2m)$, there exists a constant $M$, independent of $\pi$, such that

$$\|f-f_{m,\pi}\|_{\ell,2}\leqq M(\bar{\pi})^{s-\ell-\frac{1}{r}+\frac{1}{2}}, \quad\text{for any}\quad 1\leqq r\leqq 2 \tag{7.2}$$

for any $0\leqq\ell\leqq m-1$, and also for $\ell=m$ if $p>m$ or $r=2$, and

$$\|f-f_{m,\pi}\|_{\ell,2}\leqq M(b-a)^{\frac{r-2}{2r}}(\bar{\pi})^{s-\ell} \tag{7.2′}$$

for any $r\geqq 2$ for any $0\leqq\ell\leqq m$.

The interpretation of the two-dimensional results of §4 in terms of Sobolev norms follows similarly. Let us now consider any partition $\varrho\equiv\pi\times\pi'$ of a rectangular polygon $T$, and any element $f\in S^{p,r}(T)$, $r\geqq 2$, $p\geqq 2m$. Let $f_{m,\varrho}$ be the unique $H^{(m)}(\varrho; T)$-interpolate of $f$. The Sobolev norm of $f$ is

$$\|f\|_{\ell,2}=\left\{\sum_{0\leqq\alpha_1+\alpha_2\leqq\ell}\iint_T|D^{(\alpha_1,\alpha_2)}f(x,y)|^2\,dx\,dy\right\}^{\frac{1}{2}} \tag{7.3}$$

for any $0 \leqq \ell \leqq p$. Since $f - f_{m,\varrho}$ is an element of $S^{m,2}(T)$, we can interpret the results of Theorem 6 in the norm of (7.3). Specifically, we have

**Theorem 12.** Let $C$ be any regular collection of partitions $\varrho \equiv \pi \times \pi'$ of $T$, $f \in S^{p,r}(T)$, $p \geqq 2m$, $r \geqq 2$, and $f_{m,\varrho}$ be the unique $H^{(m)}(\varrho; T)$-interpolate of $f$. Then, with $v \equiv \max(\bar{\pi}, \bar{\pi}')$ there exists a constant $M$, independent of the particular partition in $C$, such that

$$\|f - f_{m,\varrho}\|_{\ell,2} \leqq M(v)^{2m-\ell} \tag{7.4}$$

for any $0 \leqq \ell \leqq m$, and any $\varrho \in C$.

## § 8. An Application to Elliptic Differential Equations

To give some applications of these preceding error bounds, let us consider the Galerkin method for approximating the solution of the Dirichlet problem for a class of linear elliptic differential equations in the interval $I$ and the rectangular polygon, $T$, in the plane.

Let the Hilbert space $W_0^{\ell,2}(I)$, $\ell \geqq 1$, be the completion of $C_0^\infty(I)$, the space of infinitely differentiable real-valued functions with compact support in the interior of $I$, with respect to the Sobolev $\|\cdot\|_{\ell,2}$ norm of (7.1). By Sobolev's inequality [26, p. 174], there exists a constant $Q$ such that $\|D^j f\|_{L^\infty(I)} \leqq Q\|f\|_{\ell,2}$, $0 \leqq j \leqq \ell - 1$, for all $f \in W_0^{\ell,2}(I)$. We consider the class of linear ordinary differential operators in divergence form

$$L[u] = \sum_{j=0}^{\ell} (-1)^j D^j[p_j(x) D^j u(x)], \quad \ell \geqq 1, \tag{8.1}$$

such that the coefficients $p_j(x)$ are real-valued uniformly bounded functions in $I$ and for some constant $K > 0$

$$K\|u\|_{\ell,2}^2 \leqq \left| \int_a^b \left\{ \sum_{j=0}^{\ell} p_j(x) [D^j u(x)]^2 \right\} dx \right| \tag{8.2}$$

for all $u \in W_0^{\ell,2}(I)$.

If these conditions are satisfied, we say, following CEÀ [6], that the bilinear Dirichlet form (cf. [26, p. 92])

$$a(u, v) = \int_a^b \left\{ \sum_{j=0}^{\ell} p_j(x) D^j u(x) D^j v(x) \right\} dx \tag{8.3}$$

is $W_0^{\ell,2}(I)$-*elliptic*. Moreover, by our assumption, there exists a constant $C > 0$ such that

$$|a(u, v)| \leqq C\|u\|_{\ell,2}\|v\|_{\ell,2} \tag{8.3'}$$

for all

$$u, v \in W_0^{\ell,2}(I), \quad \left(\text{e.g., } C \equiv \max_{0 \leqq j \leqq \ell} \|p_j(x)\|_{L^\infty(I)}\right).$$

The function $u(x)$ is said to be a *weak solution* of the homogeneous Dirichlet problem

$$L[u](x) = f(x), \quad x \in I, \quad f(x) \in L^2(I), \tag{8.4}$$

$$D^p u(a) = D^p u(b) = 0, \quad 0 \leqq p \leqq \ell - 1 \tag{8.5}$$

if and only if

$$a(u,v)=\int_a^b f(x)\,v(x)\,dx \quad \text{for all} \quad v\in W_0^{\ell,2}(I). \tag{8.6}$$

By the Riesz Representation Theorem in Hilbert spaces, (8.6) is equivalent to

$$a(u,v)=(g,v)_{\ell,2} \tag{8.7}$$

for some unique $g\in W_0^{\ell,2}(I)$, where $(\cdot,\cdot)_{\ell,2}$ denotes the inner product in $W_0^{\ell,2}(I)$.

Similarly, let the Hilbert space, $W_0^{\ell,2}(T)$, be the completion of $C_0^\infty(T)$, the space of infinitely differentiable real-valued functions with compact support in the interior of $T$, with respect to the Sobolev $\|\cdot\|_{\ell,2}$ norm of (7.3). We consider the class of linear partial differential operators

$$L=\sum_{\substack{s+t\leqq \ell\\ q+r\leqq \ell}}(-1)^{s+t}D^{(s,t)}\left(c_{s,t,q,r}(x,y)\,D^{(q,r)}\right) \tag{8.8}$$

in divergence form, having coefficients $c_{s,t,q,r}(x,y)$ which are real-valued uniformly bounded functions in $T$ and there exists a constant $K>0$ such that

$$K\|u\|_{\ell,2}^2\leqq\left|\iint_T\left\{\sum_{\substack{s+t\leqq \ell\\ q+r\leqq \ell}}c_{s,t,q,r}(x,y)\,D^{(s,t)}u\,D^{(q,r)}u\right\}dx\,dy\right| \tag{8.9}$$

for all $u\in W_0^{\ell,2}(T)$.

If these conditions are satisfied, we again say, following CEÀ [6], that the bilinear Dirichlet form (cf. [26, p. 92])

$$a(u,v)=\iint_T\left\{\sum_{\substack{s+t\leqq \ell\\ q+r\leqq \ell}}c_{s,t,q,r}(x,y)\,D^{(s,t)}u(x,y)\,D^{(q,r)}v(x,y)\right\}dx\,dy \tag{8.10}$$

is $W_0^{\ell,2}(T)$-*elliptic*. Moreover, by our assumption, there exists a constant $C>0$ such that $|a(u,v)|\leqq C\|u\|_{\ell,2}\|v\|_{\ell,2}$ for all $u,v\in W_0^{\ell,2}(T)$.

The function $u(x,y)$ is said to be a *weak solution* of the homogeneous Dirichlet problem

$$L[u](x,y)=f(x,y),\qquad (x,y)\in T,\qquad f(x,y)\in L^2(T), \tag{8.11}$$

$$D^{(p,r)}u(x,y)=0,\qquad (x,y)\in\partial T,\qquad 0\leqq p+r\leqq \ell-1, \tag{8.12}$$

where $\partial T$ is the boundary of $T$, if and only if

$$a(u,v)=\iint_T f(x,y)\,v(x,y)\,dx\,dy \quad \text{for all} \quad v\in W_0^{\ell,2}(T). \tag{8.13}$$

By the Riesz Representation Theorem in Hilbert spaces, (8.13) is equivalent to

$$a(u,v)=(g,v)_{\ell,2} \tag{8.14}$$

for some unique $g\in W_0^{\ell,2}(T)$, where $(\cdot,\cdot)_{\ell,2}$ denotes the inner product in $W_0^{\ell,2}(T)$. Under the above hypotheses, the Dirichlet problems (8.4), (8.5), and (8.11), (8.12) has a *unique* weak solution, $u$, by the Lax-Milgram Lemma [26, p. 92].

We consider Galerkin's procedure for obtaining an approximate solution of (8.7) or (8.14). More precisely, let $S$ be any finite dimensional subspace of $W_0^{\ell,2}(I)$

(Resp. $W_0^{\ell,2}(T)$). Consider the approximate problem of finding a $w \in S$ such that

$$a(w, v) = (g, v)_{\ell,2}, \quad \text{for all} \quad v \in S. \tag{8.15}$$

Céa in [6] has proved the following

**Lemma 4.** If the Dirichlet form $a(u, v)$ is $W_0^{\ell,2}(I)$-elliptic (Resp. $W_0^{\ell,2}(T)$-elliptic) and $S$ is a finite dimensional subspace of $W_0^{\ell,2}(I)$ (Resp. $W_0^{\ell,2}(T)$), then (8.15) has a unique solution $w \in S$.

Moreover, the following error estimate holds.

**Theorem 13.** Under the hypotheses of Lemma 4, there exists a constant, namely $CK^{-1} > 0$, (cf. (8.2) and (8.3')) independent of $S$, such that

$$CK^{-1}\left(\inf_{v \in S} \|u - v\|_{\ell,2}\right) \geqq \|u - w\|_{\ell,2}. \tag{8.16}$$

*Proof.* From (8.14) and (8.15), we have $a(u, w - v) = (g, w - v)_{\ell,2}$ and $a(w, w - v) = (g, w - v)_{\ell,2}$ for all $v \in S$. Thus, $a(u - w, w - v) = 0$ and

$$\begin{aligned} C\|u - w\|_{\ell,2}\|u - v\|_{\ell,2} &\geqq |a(u - w, u - v)| \\ &\geqq |a(u - w, u - w) + a(u - w, w - v)| \\ &= |a(u - w, u - w)| \geqq K\|u - w\|_{\ell,2}^2. \end{aligned} \tag{8.17}$$

The result then follows from (8.17). Q.E.D.

Theorem 13 was obtained in the one-dimensional case for second order problems in [24].

Combining Lemma 4 and Theorems 11 and 13 with Sobolev's inequality, we obtain the following results which sharpen the results of [24, 7, 19].

**Theorem 14.** Let the Dirichlet form $a(u, v)$ be $W_0^{\ell,2}(I)$-elliptic, $\pi$ be any partition of $I$ and the solution, $u$, of (8.7) belong to $K^{p,r}(I)$, where $r \leqq 2$. Then, setting $S \equiv H^{(m)}(\pi; I)$ in the Galerkin procedure, where either $p > m \geqq \ell$ or $p \geqq m > \ell$, there exists a constant $M$, independent of $\pi$, such that if $w$ is the solution of (8.15),

$$\|D^j(u - w)\|_{L^\infty(I)} \leqq Q\,\|u - w\|_{\ell,2} \leqq M(\bar{\pi})^{s-\ell-\frac{1}{r}+\frac{1}{2}}, \tag{8.18}$$

$0 \leqq j \leqq \ell - 1$, where $s \equiv \min(p, 2m)$.

**Theorem 15.** Let the Dirichlet form $a(u, v)$ be $W_0^{\ell,2}(I)$-elliptic, $\pi$ be any partition of $I$, and let the solution, $u$, of (8.7) belong to $K^{p,r}(I)$, where $r \geqq 2$. Then, setting $S \equiv H^{(m)}(\pi; I)$ in the Galerkin procedure, where $p \geqq m \geqq \ell$, there exists a constant $M_1$, independent of $\pi$, such that if $w$ is the solution of (8.15),

$$\|D^j(u - w)\|_{L^\infty(I)} \leqq K\|u - w\|_{\ell,2} \leqq M_1(\bar{\pi})^{s-\ell}, \quad 0 \leqq j \leqq \ell - 1, \tag{8.19}$$

where $s \equiv \min(p, 2m)$.

**Corollary 1.** Let the Dirichlet form $a(u, v)$, corresponding to a second order operator, i.e., $\ell = 1$, be $W_0^{1,2}(I)$-elliptic, $\pi$ be any partition of $I$, and the solution, $u$, of (8.7) belong to $K^{2,r}(I)$. Then, setting $S \equiv H^{(1)}(\pi; I)$ in the Galerkin procedure, there exists constants $M_2$ and $M_3$ independent of $\pi$, such that if $w$ is the solution of (8.15)

$$\|u-w\|_{L^\infty(I)} \leqq K\|u-w\|_{1,2} \leqq M_2(\bar{\pi})^{\frac{3}{2}-\frac{1}{r}}, \quad \text{if} \quad r \leqq 2 \tag{8.20}$$

and

$$\|u-w\|_{L^\infty(I)} \leqq K\|u-w\|_{1,2} \leqq M_3\bar{\pi}_n, \qquad \text{if} \quad r \geqq 2. \tag{8.21}$$

Setting $S \equiv H^{(2)}(\pi; I)$ in the Galerkin procedure, there exist constants $M_4$ and $M_5$ such that

$$\|u-w\|_{L^\infty(I)} \leqq \|u-w\|_{1,2} \leqq M_4(\bar{\pi})^{\frac{3}{2}-\frac{1}{r}}, \quad \text{if} \quad r \leqq 2 \tag{8.22}$$

and

$$\|u-w\|_{L^\infty(I)} \leqq K\|u-w\|_{1,2} \leqq M_5\bar{\pi}, \qquad \text{if} \quad r \geqq 2. \tag{8.23}$$

We remark that the problem $D^2u(x)=f(x)$, where $f(x)\in L^r(I)$ but $f$ is not continuous satisfies all the hypotheses of the above Corollary. It appears, at least theoretically, from (8.20), (8.21), (8.22), and (8.23), that piecewise linear Hermite subspaces yield as accurate results in the Galerkin procedure as piecewise cubic Hermite subspaces.

**Corollary 2.** Let the Dirichlet form $a(u, v)$, corresponding to a fourth-order operator, i.e., $\ell=2$, be $W_0^{2,2}(I)$-elliptic, $\pi$ be a partition of $I$, and the solution, $u$, of (8.7) belong to $K^{4,r}(I)$. Then, setting $S \equiv H^{(2)}(\pi; I)$ in the Galerkin procedure, there exists constants $M_6$ and $M_6'$, independent of $\pi$, such that if $w$ is the solution of (8.15),

$$\|D^j(u-w)\|_{L^\infty(I)} \leqq K\|u-w\|_{2,2} \leqq M_6(\bar{\pi})^2, \qquad j=0,1, \quad \text{if} \quad r \geqq 2, \tag{8.24}$$

and

$$\|D^j(u-w)\|_{L^\infty(I)} \leqq K\|u-w\|_{2,2} \leqq M_6'(\bar{\pi})^{\frac{5}{2}-\frac{1}{r}}, \quad j=0,1, \quad \text{if} \quad r \leqq 2. \tag{8.24'}$$

We now move on to two-dimensional results. Combining Lemma 4 and Theorems 12 and 13, we obtain

**Theorem 16.** Let $T$ be a rectangular polygon, composed of the rectangles $R_i$, $1 \leqq i \leqq k$, the Dirichlet form $a(u, v)$ be $W_0^{\ell,2}(T)$-elliptic, $C$ be a regular collection of partitions of $T$, and the solution, $u$, of (8.14) belong to $S^{p,r}(T)$, where $r \geqq 2$ and $p \geqq 2\ell$. Then setting $S \equiv H^{(m)}(\varrho; T)$ for any $\varrho \in C$ in the Galerkin procedure, where $\ell \leqq m \leqq \frac{1}{2}p$, there exists a constant $M_7$, independent of $\varrho$, such that if $w$ is the solution of (8.15) and $v \equiv \max_{1\leqq i\leqq k}(\bar{\pi}_i)$,

$$\|u-w\|_{\ell,2} \leqq M_7(v)^{2m-\ell}. \tag{8.25}$$

In [13], Nitsche and Nitsche discussed the application of the discrete five-point difference scheme on a uniform mesh of side $h$ to the second order elliptic problem, in the unit square $R_u$, i.e., the case $\ell=1$ of (8.8):

$$L[u](x,y) = a(x,y)u_{xx} + 2b(x,y)u_{x,y} + c(x,y)u_{yy} = f(x,y), \tag{8.26}$$
$$(x,y)\in R_u = [0,1]\times[0,1]$$

with

$$u(x,y)=0, \qquad (x,y)\in\partial R_u, \tag{8.27}$$

where the coefficients $a(x, y)$, $b(x, y)$, and $c(x, y)$ are continuous in $R_u$, $f \in L^2(R_u)$, and the operator $L$ is uniformly elliptic on $R_u$. They showed that if $u \in S^{2,2}(T)$, then the discrete solutions $w_{i,j}(h)$ of the corresponding standard five-point difference problem satisfy

$$(8.28) \qquad \max_{0 \leqq i, j \leqq N} |u(ih, jh) - w_{i,j}(h)| \leqq C_1 h^{\frac{2}{5}}, \qquad h \equiv \frac{1}{N},$$

where $C_1$ is independent of $h$. Using Theorem 16, we obtain an analogous result (cf. Corollary 1 below).

In [11], KELLOGG obtained on error bound for a discrete solution of the form $\|u - w_{i,j}\|_1 \leqq C_2 h$, where $\|\cdot\|_1$ is a discrete $L^2$-norm and its first divided differences[2].

The following is an analogue of the results of [11] and [13].

**Corollary 1.** Let $T$ be a rectangular polygon, composed of the rectangles $R_i$, $1 \leqq i \leqq k$, the Dirichlet form $a(u, v)$, corresponding to a second order operator, i.e., $\ell = 1$, be $W_0^{1,2}(T)$-elliptic, $C$ be a regular collection of partitions of $T$, and the solution, $u$, belong to $S^{2,r}(T)$, where $r \geqq 2$. Then, setting $S \equiv H^{(1)}(\varrho; T)$ for any $\varrho \in C$ in the Galerkin procedure, there exists a constant $M_8$, independent of $\varrho$, such that if $w$ is the solution of (8.15) and $v \equiv \max_{1 \leqq i \leqq k} (\bar{\pi}_i)$

$$(8.29) \qquad \|u - u_n\|_{1,2} \leqq M_8 v_n.$$

In the above Corollary, the approximations $w$ are in $H^{(1)}(\varrho; T)$, and thus the $w$ are piecewise *bilinear* functions. The determination of the $w$ leads, interestingly enough, to a system of linear equations which corresponds to a *nonstandard* nine-point difference approximation of (8.26). In particular, when $a = c = 1$ and $b = 0$ in (8.26), this nine-point difference approximation is

$$8w_{i,j} - (w_{i+1,j} + w_{i-1,j} + w_{i+1,j+1} + w_{i,j+1} + w_{i-1,j+1} + w_{i+1,j-1} + w_{i,j-1} + w_{i-1,j-1}) = g_{i,j}.$$

This answers a question raised in [3, Sec. 16].

As a final application of the above results, consider any *fourth*-order operator, i.e., $\ell = 2$, which we assume to be $W_0^{2,2}(T)$-elliptic. An example of such an operator is given by the *biharmonic operator*

$$(8.30) \qquad L[u](x, y) = u_{xxxx}(x, y) + 2u_{xxyy}(x, y) + u_{yyyy}(x, y) = f(x, y), \qquad (x, y) \in T$$

with

$$(8.31) \qquad u(x, y) = u_x(x, y) = u_y(x, y) = 0, \qquad (x, y) \in \partial R.$$

Using piecewise bicubic ($m = 2$) Hermite interpolation, we have, with the aid of Sobolev's inequality [26, p. 174], the

**Corollary 2.** Let $T$ be a rectangular polygon, composed of the rectangles $R_i$, $1 \leqq i \leqq k$, the Dirichlet form $a(u, v)$, corresponding to a fourth-order operator, i.e., $\ell = 2$, be $W_0^{2,2}(T)$-elliptic, $C$ be a regular collection of partitions of $T$, and let the solution of $L[u] = f$ be in $S^{4,r}(T)$, $\infty \geqq r \geqq 2$. Then, setting $S \equiv H^{(2)}(\varrho; T)$

[2] In [12], he obtained a bound of the form $\|u - w_{i,j}\|_0 \leqq M_2 h^2$, where $\|\cdot\|_0$ is a discrete $L^2$-norm.

for any $\varrho \in C$ in the Galerkin procedure, there exist constants $M_9$ and $M_{10}$, independent of $\varrho$, such that if $w$ is the solution of (8.15) and $v \equiv \max_{1\leqq i\leqq k} (\overline{\pi}_i)$,

$$\|u-w\|_\infty \leqq M_9 \|u-w\|_{2,2} \leqq M_{10} v^2. \tag{8.32}$$

We should remark about the fact that Corollaries 1 and 2 are both based on the Sard extension rather than the Stancu-Simonsen extension of the Peano theorem to rectangular polygons. Of course, results can also be obtained from the Stancu-Simonsen form of the Peano theorem, but the differentiability assumptions are different. To illustrate this, let us combine the results for $\ell=1$ of (8.16) of Theorem 13 and for $r=2$ of Theorem 10 and apply them to the problem (8.14). To obtain a positive exponent of $v$ in (6.7), it is necessary to assume now that $p \geqq 2$, and thus $u \in K^{2,2} \times K^{2,2}(T)$. Then setting $S \equiv H^{(1)}(\varrho; T)$, where $\varrho$ is any partition of $T$, in the Galerkin procedure, there exists a constant $M_{11}$, independent of $\varrho$, such that if $w$ is the solution of (8.15) and $v \equiv \max_{1\leqq i\leqq k} (\overline{\pi}_i)$,

$$\|u-w\|_{1,2} \leqq M_{11} v. \tag{8.33}$$

The assumption that $u \in K^{2,2} \times K^{2,2}(T)$ implies that $u_{xxyy}$ exists almost everywhere in $T$, an assumption which is considerably stronger than $u \in S^{2,2}(T)$.

The results we have obtained *can* be extended to higher dimensions, but the smoothness assumptions necessary for applications to the approximate solution of elliptic differential equations make these results far less interesting.

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