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Numerical Methods of High-Order Accuracy for Nonlinear Boundary Value Problems

III. Eigenvalue Problems*

P. G. CIARLET, M. H. SCHULTZ, and R. S. VARGA

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§ 1. Introduction

We shall consider here the numerical approximation by the Rayleigh-Ritz method of the eigenvalues and eigenfunctions of the following real self-adjoint elliptic boundary value problem (cf. (2.1)):

$$\mathscr{L}u(x) = \lambda \mathscr{M}u(x), \quad 0 < x < 1, \tag{1.1}$$

subject to the homogeneous boundary conditions (cf. (2.2)):

$$\mathscr{U}u(x) = 0. \tag{1.2}$$

We shall do this by applying the Rayleigh-Ritz method to appropriately selected finite dimensional subspaces. In particular, we consider subspaces of "L-spline" functions, defined in [15], as well as polynomial subspaces. In so doing, we generalize the results of [1, 8, and 17], and obtain new error estimates for the approximate eigenvalues, as well as new error estimates in the *uniform* norm for the approximate eigenfunctions. These results improve upon known results in the literature, both for discrete difference methods applied to (1.1)—(1.2) (cf. [9, 12, 16]), as well as continuous methods applied to (1.1)—(1.2) (cf. [1, 8, 17]).

§ 2. The Rayleigh-Ritz Method

For the eigenvalue problem of (1.1)—(1.2), we assume that $\mathscr{L}$ and $\mathscr{M}$ are formally self-adjoint operators of the form

$$\begin{aligned} \mathscr{L}u(x) &= \sum_{j=0}^{n} (-1)^j [p_j(x)\, u^{(j)}(x)]^{(j)}; \\ \mathscr{M}u(x) &= \sum_{j=0}^{r} (-1)^j [q_j(x)\, (u^{(j)} x)]^{(j)}, \quad 0 < x < 1, \end{aligned} \tag{2.1}$$

where we require that $0 \leqq r < n$, and we assume that the homogeneous boundary conditions of (1.2) consist of $2n$ linearly independent conditions of the form

$$U_j u(x) = \sum_{k=1}^{2n} \{m_{j,k}\, u^{(k-1)}(0) + n_{j,k}\, u^{(k-1)}(1)\} = 0, \quad 1 \leqq j \leqq 2n. \tag{2.2}$$

One special case of interest of (2.2) is the boundary conditions of

$$u^{(j)}(0) = u^{(j)}(1) = 0, \quad 0 \leqq j \leqq n-1. \tag{2.2'}$$

The coefficient functions $p_j(x)$ and $q_k(x)$ are real-valued functions of class $C^j[0, 1]$, $0 \leqq j \leqq n$, and class $C^k[0, 1]$, $0 \leqq k \leqq r$, respectively, and in addition, we require that

$$p_n(x) \text{ and } q_r(x) \text{ do not vanish on } [0, 1]. \tag{2.3}$$

A value of λ for which a nontrivial solution of (1.1)—(1.2) exists is called an *eigenvalue*, and a corresponding nontrivial solution $u(x)$ is called an *eigenfunction*. Letting $\mathscr{D}$ denote the set of real-valued functions of class $C^{2n}[0, 1]$ which satisfy the homogeneous boundary conditions (2.2), and defining the inner product $(u, v)_{L^2} = \int_0^1 u(x)\, v(x)\, dx$ for functions $u(x)$ and $v(x)$ in $L^2[0, 1]$, we assume that the eigenvalue problem (1.1)—(1.2) is self-adjoint in the sense that

$$(\mathscr{L}u, v)_{L^2} = (u, \mathscr{L}v)_{L^2} \quad \text{for all} \quad u, v \in \mathscr{D}, \tag{2.4}$$

and

$$(\mathscr{M}u, v)_{L^2} = (u, \mathscr{M}v)_{L^2} \quad \text{for all} \quad u, v \in \mathscr{D}. \tag{2.5}$$

We also assume that there exist real constants K' and $d > 0$ such that

$$(\mathscr{L}u, u)_{L^2} \geqq K'(u, u)_{L^2} \quad \text{and} \quad (\mathscr{M}u, u)_{L^2} \geqq d(u, u)_{L^2} \quad \text{for all} \quad u \in \mathscr{D}. \tag{2.6}$$

With $\mathscr{L}_1 \equiv \mathscr{L} + (\alpha + \beta)\mathscr{M}$ where α is chosen so that $K' + \alpha d \geqq 0$ and $\beta > 0$, it follows from (2.6) that $(\mathscr{L}_1 u, u)_{L^2} \geqq \beta(\mathscr{M}u, u)_{L^2}$ for all $u \in \mathscr{D}$. Because the eigenvalue problem $\mathscr{L}_1 x = \lambda \mathscr{M} x$, $\mathscr{U} x = 0$, has the same eigenfunctions as $\mathscr{L} x = \lambda \mathscr{M} x$, $\mathscr{U} x = 0$, and eigenvalues simply translated by $\alpha + \beta$, then assuming (2.6) is essentially equivalent to assuming that there exist constants $K_1 > 0$ and $d > 0$ such that

$$(\mathscr{L}u, u)_{L^2} \geqq K_1(\mathscr{M}u, u)_{L^2} \quad \text{and} \quad (\mathscr{M}u, u)_{L^2} \geqq d(u, u)_{L^2} \quad \text{for all} \quad u \in \mathscr{D}. \tag{2.6'}$$

Because we seek error estimates for the approximate eigenfunctions in the *uniform norm*, we shall assume later in §4 that there exists a constant $K_2 > 0$ such that

$$\|u\|_{L^\infty} \equiv \max_{0 \leqq x \leqq 1} |u(x)| \leqq K_2 \{(\mathscr{L}u, u)_{L^2}\}^{\frac{1}{2}} \quad \text{for all} \quad u \in \mathscr{D}. \tag{2.7}$$

To give specific examples of eigenvalue problems (1.1)—(1.2) which satisfy the above assumptions, consider the second-order eigenvalue problem

$$-(p_1(x)\, u^{(1)}(x))^{(1)} + p_0(x)\, u(x) = \lambda\, q_0(x)\, u(x), \quad 0 < x < 1, \tag{2.8}$$

where $p_1(x) > 0$, and $q_0(x) > 0$ on $[0, 1]$, with boundary conditions

$$u^{(1)}(0) = \alpha_0 u(0), \quad u^{(1)}(1) = -\alpha_1 u(1), \quad \text{where} \quad \alpha_0 \geqq 0 \quad \text{and} \quad \alpha_1 \geqq 0, \tag{2.9}$$

which was treated in [1] and [8], for example. In this case, we have

$$\begin{aligned} (\mathscr{L}u, v)_{L^2} &= (u, \mathscr{L}v)_{L^2} = \alpha_0 p_1(0)\, u(0)\, v(0) + \alpha_1 p_1(1)\, u(1)\, v(1) \\ &\quad + \int_0^1 \{p_1(x)\, u^{(1)}(x)\, v^{(1)}(x) + p_0(x)\, u(x)\, v(x)\}\, dx \quad \text{for all} \quad u, v \in \mathscr{D}, \end{aligned} \tag{2.10}$$

and that

$$(2.11)\qquad (\mathcal{M}u, v)_{L^2} = (u, \mathcal{M}v)_{L^2} = \int_0^1 q_0(x)\, u(x)\, v(x)\, dx \quad \text{for all} \quad u, v \in \mathcal{D}.$$

Next, since $q_0(x) > 0$ on $[0, 1]$, then

$$(2.12)\qquad (\mathcal{M}u, u)_{L^2} \geqq d(u, u)_{L^2} \quad \text{for all} \quad u \in \mathcal{D}, \qquad \text{where} \quad d \equiv \min_{0 \leqq x \leqq 1} q_0(x) > 0.$$

With $\alpha_0 \geqq 0$, $\alpha_1 \geqq 0$, and $p_1(x) > 0$ on $[0, 1]$, the boundary terms of $(\mathcal{L}u, u)_{L^2}$ are nonnegative, and hence

$$(\mathcal{L}u, u)_{L^2} \geqq \int_0^1 \{p_1(x)\,(u^{(1)}(x))^2 + p_0(x)\,(u(x))^2\}\, dx.$$

By virtue of the remarks following (2.6), we may also assume without loss of generality that $p_0(x) > 0$ on $[0, 1]$. It then follows from the above inequality that

$$(2.12')\quad (\mathcal{L}u, u)_{L^2} \geqq K_1 (\mathcal{M}u, u)_{L^2} \quad \text{for all} \quad u \in \mathcal{D} \quad \text{where} \quad K_1 \equiv \min_{0 \leqq x \leqq 1} \left\{\frac{p_0(x)}{q_0(x)}\right\} > 0.$$

Moreover, the inequality above (2.12′) also gives that

$$(\mathcal{L}u, u)_{L^2} \geqq K_2 \int_1^0 \{(u^{(1)}(x))^2 + (u(x))^2\}\, dx \equiv K_2 \|u\|_{1,2}^2$$

where

$$K_2 \equiv \min_{0 \leqq x \leqq 1} \{p_0(x), p_1(x)\} > 0,$$

and by Sobolev's inequality [18, p. 174] in one dimension, there exists a constant $K_3 > 0$ such that $K_3 \|u\|_{1,2} \geqq \|u\|_{L^\infty}$. Thus,

$$(2.13)\qquad \{(\mathcal{L}u, u)_{L^2}\}^{\frac{1}{2}} \geqq K_2^{\frac{1}{2}} K_3 \|u\|_{L^\infty} \quad \text{for all} \quad u \in \mathcal{D}.$$

As another example of an eigenvalue problem (1.1)—(1.2) which satisfies the assumptions of (2.4)—(2.7), consider the problem of transverse vibrations of a cantilever beam of variable cross-section but possessing constant flexural rigidity (cf. [5, p. 253]):

$$(2.14)\qquad u^{(4)}(x) = \lambda(1 + x)\, u(x), \qquad 0 < x < 1,$$

with boundary conditions

$$(2.15)\qquad u(0) = u^{(1)}(0) = u^{(2)}(1) = u^{(3)}(1) = 0.$$

In this case,

$$(2.16)\qquad (\mathcal{L}u, v)_{L^2} = (u, \mathcal{L}v)_{L^2} = \int_0^1 u^{(2)}(x)\, v^{(2)}(x)\, dx \quad \text{for all} \quad u, v \in \mathcal{D},$$

and

$$(2.17)\qquad (\mathcal{M}u, v)_{L^2} = (u, \mathcal{M}v)_{L^2} = \int_0^1 (1 + x)\, u(x)\, v(x)\, dx \quad \text{for all} \quad u, v \in \mathcal{D}.$$

From (2.17), we see that $(\mathcal{M}u, u)_{L^2} \geqq (u, u)_{L^2}$, giving the second inequality of 2.6′).(Because of the boundary conditions of (2.15), the Rayleigh-Ritz inequality

[11, p. 184] gives

$$\int_0^1 (u^{(2)}(x))^2\,dx \geqq \frac{4}{\pi^2}\int_0^1 (u^{(1)}(x))^2\,dx \geqq \frac{16}{\pi^4}\int_0^1 (u(x))^2\,dx,$$

and as $|u(x)| = |\int_0^x u^{(1)}(t)\,dt| \leqq \{\int_0^1 (u^{(1)}(t))^2\,dt\}^{\frac{1}{2}}$ by Schwarz's inequality, we thus deduce that

$$(2.18)\qquad (\mathscr{L}u, u)_{L^2} \geqq \frac{8}{\pi^4}(\mathscr{M}u, u)_{L^2} \quad \text{for all} \quad u \in \mathscr{D},$$

and

$$(2.18')\qquad \{(\mathscr{L}u, u)_{L^2}\}^{\frac{1}{2}} \geqq \frac{2}{\pi}\|u\|_{L^\infty} \quad \text{for all} \quad u \in \mathscr{D}.$$

To describe the relevant theory for the eigenvalue problem (1.1)—(1.2) under the assumptions of (2.4)—(2.6'), we define the following inner products on $\mathscr{D}$:

$$(2.19)\qquad (u, v)_D \equiv (\mathscr{M}u, v)_{L^2} \quad \text{for all} \quad u, v \in \mathscr{D},$$

and

$$(2.20)\qquad (u, v)_N \equiv (\mathscr{L}u, v)_{L^2} \quad \text{for all} \quad u, v \in \mathscr{D}.$$

Hence, from (2.6'), we have, with $\|u\|_D^2 \equiv (u, u)_D$ and $\|u\|_N^2 \equiv (u, u)_N$, that

$$(2.21)\qquad \|u\|_D^2 \geqq d(u, u)_{L^2} \quad \text{for all} \quad u \in \mathscr{D}, \quad \text{where} \quad d > 0,$$

and

$$(2.22)\qquad \|u\|_N \geqq \sqrt{K_1}\|u\|_D \quad \text{for all} \quad u \in \mathscr{D}, \quad \text{where} \quad K_1 > 0.$$

Thus, $\|\cdot\|_D$ and $\|\cdot\|_N$ are norms on $\mathscr{D}$, and we denote by H_D and H_N the Hilbert space completions of $\mathscr{D}$ with respect to the norms $\|\cdot\|_D$ and $\|\cdot\|_N$, respectively. As a consequence of the inequality of (2.22), we note that

$$(2.22')\qquad H_N \subset H_D.$$

It is well-known [5, p. 230], [6], [7, p. 406], [10, p. 108] that solving the eigenvalue problem (1.1)—(1.2) under the assumptions of (2.4)—(2.6') is equivalent to finding the extreme values and critical points of the *Rayleigh quotient:*

$$(2.23)\qquad R[w] \equiv \frac{\|w\|_N^2}{\|w\|_D^2}, \quad w \in H_N, \quad w \not\equiv 0.$$

We remark that, because of the inequality of (2.22), $R[w] \geqq K_1 > 0$ for any $w \in H_N$ with $w \not\equiv 0$. We now state the results of Brauer [2] and Kamke [13], [14] for the eigenvalue problem (1.1)—(1.2).

Theorem 1. With the assumptions of (2.4)—(2.6'), the eigenvalue problem of (1.1)—(1.2) has countably many eigenvalues $\{\lambda_j\}_{j=1}^\infty$ which are real and have no finite limit point, and can be arranged as

$$(2.24)\qquad 0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k \leqq \lambda_{k+1} \leqq \cdots.$$

There is a corresponding sequence of eigenfunctions $\{\varphi_j(x)\}_{j=1}^\infty$ where $\varphi_j(x) \in C^{2n}[0, 1]$ and $\mathscr{L}\varphi_j(x) = \lambda_j \mathscr{M}\varphi_j(x)$, $j \geqq 1$, and these eigenfunctions can be chosen to be orthonormal in the D-inner product of (2.19), i.e.,

$$(\varphi_i, \varphi_j)_D = \delta_{i,j} \quad \text{for all} \quad i, j = 1, 2, \ldots, \tag{2.25}$$

and also

$$(\varphi_i, \varphi_j)_N = \lambda_i \, \delta_{i,j} \quad \text{for all} \quad i, j = 1, 2, \ldots, \tag{2.26}$$

and the sequence of eigenfunctions $\{\varphi_j(x)\}_{j=1}^{\infty}$ is complete in H_D. Each eigenvalue λ_k, $k \geqq 1$, has the following characterization in terms of the Rayleigh quotient of (2.23):

$$\lambda_k = \inf\{R[w] : w \in H_N \text{ such that } (w, \varphi_\ell)_D = 0 \text{ for } 1 \leqq \ell \leqq k-1\} = R[\varphi_k], \tag{2.27}$$

$$\lambda_k = \max_{c_1, c_2, \ldots, c_k} \left\{ R\left[\sum_{i=1}^{k} c_i \varphi_i\right] \right\}, \tag{2.28}$$

$$\lambda_k = \max_{\substack{v_1(x), v_2(x), \ldots, v_{k-1}(x) \\ \text{linearly independent}}} (\inf\{R[w];\ w \in H_N,\ (w, v_\ell)_D = 0,\ \ell = 1, 2, \ldots, k-1\}) \tag{2.29}$$

and

$$\lambda_k = \min_{\substack{v_1(x), v_2(x), \ldots, v_k(x) \\ \text{linearly independent}}} \left(\max_{c_1, c_2, \ldots, c_k} \left\{ R\left[\sum_{i=1}^{k} c_i v_i\right] \right\} \right). \tag{2.30}$$

Now, let S_M be any *finite* dimensional subspace of H_N, of dimension M, and let $\{w_i(x)\}_{i=1}^{M}$ be M linearly independent functions from the subspace. Thus, any function $w(x)$ in S_M can be written as

$$w(x) = \sum_{i=1}^{M} u_i \, w_i(x). \tag{2.31}$$

Instead of looking for the extremal points of the Rayleigh quotient $R[w]$ over the whole space H_N, as in Theorem 1, the Rayleigh-Ritz procedure consists in looking for the extremal points of $R[w]$ over the subspace S_M. Equivalently, we now can view $R[w]$ as a Rayleigh quotient of a symmetric matrix defined over R^M. More precisely, let

$$\mathscr{R}[\boldsymbol{u}] = \mathscr{R}[u_1, u_2, \ldots, u_M] = R\left[w = \sum_{i=1}^{M} u_i w_i\right] = \frac{\left\|\sum_{i=1}^{M} u_i w_i\right\|_N^2}{\left\|\sum_{i=1}^{M} u_i w_i\right\|_D^2} = \frac{\mathscr{A}[\boldsymbol{u}]}{\mathscr{B}[\boldsymbol{u}]}. \tag{2.32}$$

To find the stationary values of $\mathscr{R}[\boldsymbol{u}]$, we write

$$\frac{\partial \mathscr{A}[\boldsymbol{u}]}{\partial u_i} = \lambda \frac{\partial \mathscr{B}[\boldsymbol{u}]}{\partial u_i}, \tag{2.33}$$

which yields the matrix eigenvalue problem,

$$A_M \, \boldsymbol{u} = \lambda B_M \, \boldsymbol{u}, \tag{2.34}$$

where the $M \times M$ matrices $A_M = (\alpha_{i,j}^{(M)})$ and $B_M = (\beta_{i,j}^{(M)})$ have their entries given by

$$\alpha_{i,j}^{(M)} = (w_i, w_j)_N, \quad \beta_{i,j}^{(M)} = (w_i, w_j)_D, \quad 1 \leqq i, j \leqq M. \tag{2.35}$$

It is clear, from the assumptions made in (2.4)—(2.6′), that the matrices A_M and B_M are real, symmetric, and positive definite. Thus, the matrix eigenvalue problem (2.34) has M positive eigenvalues $0 < \hat{\lambda}_1 \leqq \hat{\lambda}_2 \leqq \cdots \leqq \hat{\lambda}_M$ and M corresponding linearly independent eigenvectors $\hat{\boldsymbol{u}}_1, \hat{\boldsymbol{u}}_2, \ldots, \hat{\boldsymbol{u}}_M$. To each eigenvector $\hat{\boldsymbol{u}}_k$, $1 \leqq k \leqq M$, we associate the function

$$\hat{\varphi}_k(x) = \sum_{i=1}^{k} \hat{u}_{k,i}\, w_i(x), \tag{2.36}$$

where $\hat{u}_{k,i}$ is the i-th component of the vector $\hat{\boldsymbol{u}}_k$, and henceforth we will call $\hat{\lambda}_k$ an *approximate eigenvalue* and $\hat{\varphi}_k(x)$ an *approximate eigenfunction* for (1.1)—(1.2). Clearly, we have the following characterizations, which are analogues of (2.27), (2.28), (2.29), and (2.30):

$$\begin{aligned} \hat{\lambda}_k = R[\hat{\varphi}_k] &= \max_{c_1, c_2, \dots, c_k} \left\{ R\left[\sum_{i=1}^{k} c_i \hat{\varphi}_i \right] \right\} \\ &= \inf\{R[w];\ w \in S_M,\ (w_i \hat{\varphi}_i)_D = 0,\quad 1 \leqq i \leqq k-1\} \\ &= \min_{\substack{v_1(x), \dots, v_k(x) \in S_M \\ \text{linearly independent}}} \left(\max_{c_1, c_2, \dots, c_k} \left\{ R\left[\sum_{i=1}^{k} c_i v_i \right] \right\} \right), \quad 1 \leqq k \leqq M. \end{aligned} \tag{2.37}$$

Lemma 1. Let S_M be any finite dimensional subspace of H_N with $\dim S_M = M$, and let $\hat{\lambda}_k$, $1 \leqq k \leqq M$, be the corresponding approximate eigenvalues of (1.1) to (1.2). Then,

$$\lambda_k \leqq \hat{\lambda}_k, \qquad 1 \leqq k \leqq M. \tag{2.38}$$

Proof. The result follows immediately from (2.30) and (2.37). Q.E.D.

§ 3. Eigenvalue Bounds

We are now interested in the following problem. Let λ_k be the k-th eigenvalue of (1.1)—(1.2), where we keep k fixed. Whenever the subspace S_M has dimension $M \geqq k$, we obtain an approximate eigenvalue $\hat{\lambda}_k$ which is always an upper bound for λ_k from Lemma 1. Suppose now that we have a (not necessarily nested) sequence of subspaces $\{S_{M_i}\}_{i=1}^{\infty}$ where, if $M_i = \dim S_{M_i}$, then $\lim_{i \to \infty} M_i = +\infty$. What is a sufficient condition on the asymptotic properties of S_{M_i} which will insure that $\hat{\lambda}_{k,i}$ the approximate eigenvalue in S_{M_i} corresponding to λ_k, tends to λ_k as $i \to \infty$? We will give an answer to this in Theorem 3. But first, we will develop some machinery which, in essence, amounts to finding an upper bound for $\hat{\lambda}_{k,i}$. For this purpose, we state the following result [1, Theorem 1], which, is valid for our problem as well.

Theorem 2. With the assumptions of (2.4)—(2.6′), let $\{\varphi_i(x)\}_{i=1}^{k}$ be the first k eigenfunctions of (1.1)—(1.2), orthonormalized in the sense that

$$(\varphi_i, \varphi_j)_D = \delta_{i,j}, \qquad 1 \leqq i, j \leqq k. \tag{3.1}$$

Let $\{\tilde{\varphi}_i(x)\}_{i=1}^{k}$ be any "globally approximating set of functions" to $\{\varphi_i(x)\}_{i=1}^{k}$ in H_N, in the sense that

$$\sum_{i=1}^{k} \|\tilde{\varphi}_i - \varphi_i\|_D^2 < 1. \tag{3.2}$$

Then, the functions $\{\tilde{\varphi}_i(x)\}_{i=1}^{k}$ are linearly independent, and if we define

$$\varepsilon_i(x) \equiv \tilde{\varphi}_i(x) - \varphi_i(x), \qquad 1 \leqq i \leqq k, \tag{3.3}$$

then

$$\lambda_j \leqq \tilde{\lambda}_j \leqq \lambda_j + \frac{\left(\sum_{i=1}^{j} \|\varepsilon_i\|_N^2\right)}{\left(1 - \sqrt{\sum_{i=1}^{j} \|\varepsilon_i\|_D^2}\right)^2} \quad \text{for all} \quad 1 \leqq j \leqq k, \tag{3.4}$$

where

$$\tilde{\lambda}_j \equiv \max_{c_1, c_2, \ldots, c_j} \frac{\left\| \sum_{i=1}^{j} c_i \tilde{\varphi}_i \right\|_N^2}{\left\| \sum_{i=1}^{j} c_i \tilde{\varphi}_i \right\|_D^2}, \quad 1 \leqq j \leqq k, \tag{3.5}$$

is the j-th-approximate eigenvalue for the finite dimensional subspace of H_N spanned by $\{\tilde{\varphi}_i(x)\}_{i=1}^{j}$.

We now give our general convergence criterion for the eigenvalues.

Theorem 3. With the assumptions of (2.4)—(2.6′), let k be a fixed positive integer, let $\{S_{M_n}\}_{n=1}^{\infty}$ be a given (not necessarily nested) sequence of subspaces of H_N with $\dim S_{M_n} = M_n \geqq k$ for all $n \geqq 1$, and let $\hat{\lambda}_{k,n}$ be the k-th approximate eigenvalue, obtained from applying the Rayleigh-Ritz method (2.37) to the subspace S_{M_n}, $n \geqq 1$. If the first k eigenfunctions of (1.1)—(1.2) are $\varphi_1(x), \ldots, \varphi_k(x)$, and $\lim_{n\to\infty} \left\{ \inf_{w \in S_{M_n}} \|w - \varphi_i\|_N \right\} = 0$ for each $1 \leqq i \leqq k$, then the sequence $\{\hat{\lambda}_{k,n}\}_{n=1}^{\infty}$ converges to λ_k (from above). Moreover, there exists a positive integer n_0 such that for each $n \geqq n_0$, there exist k functions $\{\tilde{\varphi}_{i,n}\}_{i=1}^{k}$ in S_{M_n} which are globally approximating functions to $\{\varphi_i\}_{i=1}^{k}$, and consequently

$$\lambda_k \leqq \hat{\lambda}_{k,n} \leqq \lambda_k + \frac{\sum_{i=1}^{k} \|\tilde{\varphi}_{i,n} - \varphi_i\|_N^2}{\left(1 - \sqrt{\sum_{i=1}^{k} \|\tilde{\varphi}_{i,n} - \varphi_i\|_D^2}\right)^2} \quad \text{for all} \quad n \geqq n_0. \tag{3.6}$$

Proof. If $\{\varphi_i(x)\}_{i=1}^{k}$ are the first k eigenfunctions of (1.1)—(1.2), then for each $n \geqq 1$, let $\{\tilde{\varphi}_{i,n}(x)\}_{i=1}^{k}$ be defined in S_{M_n} such that $\|\tilde{\varphi}_{i,n} - \varphi_i\|_N = \inf_{w \in S_{M_n}} \|w - \varphi_i\|_N$, $1 \leqq i \leqq k$. By hypothesis, $\lim_{n\to\infty} \|\tilde{\varphi}_{i,n} - \varphi_i\|_N = 0$ for each $1 \leqq i \leqq k$, and thus there exists an $n_0 \geqq k$ such that $\sum_{i=1}^{k} \|\tilde{\varphi}_{i,n} - \varphi_i\|_N^2 \leqq \lambda_1/2$ for all $n \geqq n_0$, where λ_1 is the first (positive) eigenvalue of (1.1)—(1.2). From the definition of λ_1 in (2.27), it follows that $\|\tilde{\varphi}_{i,n} - \varphi_i\|_D^2 \leqq \frac{1}{\lambda_1} \|\tilde{\varphi}_{i,n} - \varphi_i\|_N^2$ for all $1 \leqq i \leqq k$ and all $n \geqq 1$. Summing this inequality over i then yields

$$\sum_{i=1}^{k} \|\tilde{\varphi}_{i,n} - \varphi_i\|_D^2 \leqq \frac{1}{\lambda_1} \sum_{i=1}^{k} \|\tilde{\varphi}_{i,n} - \varphi_i\|_N^2 \leqq \frac{1}{2} \text{ for all } \quad n \geqq n_0. \tag{3.7}$$

Thus, the functions $\{\tilde{\varphi}_{i,n}\}_{i=1}^{k}$ are globally approximating to $\{\varphi_i\}_{i=1}^{k}$ in the sense of (3.2) for $n \geqq n_0$, and (3.4) of Theorem 2 can be applied. Thus, as $\hat{\lambda}_{k,n} \leqq \tilde{\lambda}_{k,n}$ from the Min-Max Principle applied to S_{M_n}, then (3.6) follows from (3.4). For the particular choice of functions $\{\tilde{\varphi}_{i,n}(x)\}_{i=1}^{k}$ in S_{M_n}, it is clear by hypothesis that both $\sum_{i=1}^{k} \|\tilde{\varphi}_{i,n} - \varphi_i\|_N^2$ and $\sum_{i=1}^{k} \|\tilde{\varphi}_{i,n} - \varphi_i\|_N^2$ tend to zero as $n \to \infty$, and we conclude from (3.6) that the sequence $\{\hat{\lambda}_{k,n}\}_{n=1}^{\infty}$ converges to λ_k. Q.E.D.

In place of $\lim_{n\to\infty}\left\{\inf_{w\in S_{M_n}}\|w-\varphi_i\|_N\right\}=0$ for each $1\leqq i\leqq k$, we could of course make the stronger hypothesis in Theorem 3 that $\lim_{n\to\infty}\{\inf_{w\in S_{M_n}}\|w-g\|_N\}=0$ for *every* $g(x)$ in H_N. In practice, where the eigenfunctions $\varphi_i(x)$ are in general not known a priori, this latter hypothesis is more readily checked, and is in fact valid for the subspaces considered in §5. The same comment applies to Theorem 5 of §4.

As an immediate consequence of Theorem 3, we have

Corollary 1. If the hypotheses of Theorem 3 are satisfied, then for n sufficiently large, we have

(3.8) $$\hat{\lambda}_{k,n}=\lambda_k+\varrho_n\sum_{i=1}^{k}\|\tilde{\varphi}_{i,n}-\varphi_i\|_N^2,$$

where

(3.9) $$\varrho_n\to 1+\quad\text{as}\quad n\to\infty.$$

§ 4. Eigenfunction Error Bounds

With the added assumption of (2.7), our next theorem, which again makes use of results of [1], gives an error bound in the *uniform norm* for the approximate eigenfunction $\tilde{\varphi}_k$ obtained on a finite dimensional subspace S_M of H_N.

Theorem 4. With the assumptions of (2.4)—(2.7), let $\varphi_1,\varphi_2,\ldots,\varphi_k$ be the first k eigenfunctions of (1.1)—(1.2), where it is assumed that the corresponding eigenvalues λ_j satisfy $0<\lambda_1<\lambda_2<\cdots<\lambda_k$, let S_M be any finite dimensional subspace of H_N with $\dim S_M=M\geqq k$, and let $\hat{\lambda}_1,\hat{\lambda}_2,\ldots,\hat{\lambda}_k$, and $\hat{\varphi}_1,\ldots,\hat{\varphi}_k$ be the first k approximate eigenvalues and approximate eigenvectors obtained from applying the Rayleigh-Ritz method for (1.1)—(1.2) to S_M. Then, there exists a constant C, dependent on k but independent of S_M, such that

(4.1) $$\|\hat{\varphi}_k-\varphi_k\|_{L^\infty}\leqq K_2\|\hat{\varphi}_k-\varphi_k\|_N\leqq C\left\{\sum_{i=1}^{k}(\hat{\lambda}_j-\lambda_j)\right\}^{\frac{1}{2}}.$$

Proof. With the normalizations $\|\varphi_j\|_D=\|\hat{\varphi}_j\|_D=1$ for $1\leqq j\leqq k$, we define

(4.2) $$\sigma_j^2=1-[(\varphi_j,\hat{\varphi}_j)_D]^2,\quad 1\leqq j\leqq k,$$

where the sign of $\hat{\varphi}_j$ is chosen so that $(\varphi_j,\hat{\varphi}_j)_D\geqq 0$. Hence, we can write $\sigma_j=\sin\theta_j$, where θ_j is the angle between φ_j and $\hat{\varphi}_j$ in the Hilbert space H_D determined from the D-inner product. Thus, from (4.2), we have that

(4.3) $$\|\hat{\varphi}_j-\varphi_j\|_D^2=4\sin^2(\theta_j/2)=2\{1-\sqrt{1-\sigma_j^2}\},\quad 1\leqq j\leqq n.$$

Now, the result of Corollary 1 of Theorem 2 of [1], valid also for our problem, directly gives us that

(4.4) $$(\lambda_{j+1}-\lambda_j)\,\sigma_j^2\leqq(\hat{\lambda}_j-\lambda_j)+\sum_{i=1}^{j-1}(\lambda_{j+1}-\lambda_i)\,\sigma_i^2,\quad 1\leqq j\leqq k.$$

Next, if we define $\|\cdot\|_{N_j}^2\equiv\|\cdot\|_N^2-\lambda_j\|\cdot\|_D^2$ for $1\leqq j\leqq k$, then

$$\begin{aligned}\hat{\lambda}_j-\lambda_j&=\|\hat{\varphi}_j\|_{N_j}^2=\|\hat{\varphi}_j\|_{N_j}^2-\|\varphi_j\|_{N_j}^2=\|\varphi_j+(\hat{\varphi}_j-\varphi_j)\|_{N_j}^2-\|\varphi_j\|_{N_j}^2\\&=2(\varphi_j,\hat{\varphi}_j-\varphi_j)_N-2\lambda_j(\varphi_j,\hat{\varphi}_j-\varphi_j)_D+\|\hat{\varphi}_j-\varphi_j\|_{N_j}^2=\|\hat{\varphi}_j-\varphi_j\|_{N_j}^2\\&=\|\hat{\varphi}_j-\varphi_j\|_N^2-\lambda_j\|\hat{\varphi}_j-\varphi_j\|_D^2,\end{aligned}$$

so that

$$\hat{\lambda}_j - \lambda_j = \|\hat{\varphi}_j - \varphi_j\|_N^2 - \lambda_j \|\hat{\varphi}_j - \varphi_j\|_D^2. \tag{4.5}$$

Thus, using the expression of (4.3), this becomes

$$0 \leqq \|\hat{\varphi}_j - \varphi_j\|_N^2 = (\hat{\lambda}_j - \lambda_j) + 2\lambda_j\{1 - \sqrt{1-\sigma_j^2}\}, \qquad 1 \leqq j \leqq n. \tag{4.6}$$

Now, using the hypothesis that $\lambda_2 > \lambda_1$, we see from (4.4) that $\sigma_1^2 \leqq \left(\frac{\hat{\lambda}_1 - \lambda_1}{\lambda_2 - \lambda_1}\right)$, and it follows from (4.4) by induction that there is a constant M, dependent on k only, such that

$$\sigma_j^2 \leqq M \sum_{i=1}^{j} (\hat{\lambda}_i - \lambda_i), \qquad 1 \leqq j \leqq k. \tag{4.7}$$

Hence, since $|1 - \sqrt{1-x}| \leqq x$ for all $x \geqq 0$, we see from (4.3) and (4.7) that

$$\|\hat{\varphi}_j - \varphi_j\|_D^2 \leqq M \sum_{i=1}^{j} (\hat{\lambda}_i - \lambda_i), \qquad 1 \leqq j \leqq k, \tag{4.8}$$

and from (4.6) and (4.7), we similarly have

$$\|\hat{\varphi}_j - \varphi_j\|_N^2 \leqq (\hat{\lambda}_j - \lambda_j) + 2M\lambda_j \sum_{i=1}^{j} (\hat{\lambda}_i - \lambda_i) \leqq M' \sum_{i=1}^{j} (\hat{\lambda}_i - \lambda_i), \qquad 1 \leqq j \leqq k, \tag{4.9}$$

where again M' is dependent only on k. Thus, by virtue of our basic assumption of (2.7), we have

$$\|\hat{\varphi}_j - \varphi_j\|_{L^\infty}^2 \leqq K_2^2 \|\hat{\varphi}_j - \varphi_j\|_N^2, \qquad 1 \leqq j \leqq k, \tag{4.10}$$

and thus, with the inequality of (4.9), we have

$$\|\hat{\varphi}_j - \varphi_j\|_{L^\infty[0,1]}^2 \leqq K_2^2 \|\hat{\varphi}_j - \varphi_j\|_N^2 \leqq K_2^2 M' \sum_{i=1}^{j} (\hat{\lambda}_i - \lambda_i), \qquad 1 \leqq j \leqq k, \tag{4.11}$$

which is the desired result. Q.E.D.

If we couple this last result with Theorem 3, we obtain

Theorem 5. With the assumptions of (2.4)—(2.7), let $\varphi_1, \varphi_2, \ldots, \varphi_k$ be the first k eigenfunctions of (1.1)—(1.2), where it is assumed that the corresponding eigenvalues λ_j satisfy $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$. Then, given any sequence of (not necessarily nested) subspaces $\{S_{M_n}\}_{n=1}^{\infty}$ of H_N with $\dim S_{M_n} = M_n \geqq k$ for all $n \geqq 1$ such that

$$\lim_{n\to\infty} \left\{ \inf_{w \in S_{M_n}} \|w - \varphi_j\|_N \right\} = 0 \quad \text{for all} \quad 1 \leqq j \leqq k, \tag{4.12}$$

then the approximate eigenfunctions $\{\tilde{\varphi}_{j,n}\}_{n=1}^{\infty}$, obtained from the Rayleigh-Ritz method for (1.1)—(1.2) applied to the subspaces S_{M_n}, converge *uniformly* to φ_j for each $1 \leqq j \leqq k$.

§ 5. High-Order Methods

Assuming that we have at our disposal an *actual* sequence of subspaces, $\{S_{M_n}\}_{n=1}^{\infty}$, of H_N satisfying the hypothesis of Theorems 3 and 5, our interest lies in finding the asymptotic order of accuracy which will be naturally derived from (3.6) of Theorem 3 and (4.1) of Theorem 4, i.e., we want an estimate of the rate at which $\sum_{i=1}^{k} \|\tilde{\varphi}_{i,n} - \varphi_i\|_N^2$ converges to zero for a fixed k as $n \to \infty$. Since the positive integer k is always supposed to be fixed, it suffices to have a general estimate for $\|\varphi - \tilde{\varphi}_n\|_N$, where $\tilde{\varphi}_n$ is any approximation in S_{M_n} for which we may compute the above quantity.

Because of the rather great generality of the boundary conditions of (2.2) for n large, we restrict ourselves now for reasons of brevity to the special homogeneous boundary conditions of (2.2'), which have been used in [3]. This is not to say that other boundary conditions cannot be similarly considered. In fact, boundary conditions of type (2.9) have been dealt with in [4], and, for the eigenvalue problem of (2.14)—(2.15), since only the boundary conditions $u(0) = u^{(1)}(0) = 0$ are *essential* boundary conditions, the rest being *suppressible* boundary conditions (cf. [5, p. 4]), the results to follow apply equally well to the eigenvalue problem of (2.14)—(2.15).

As our first example, we consider $P_0^{(m)}$, the $(m+1-2n)$-dimensional subspace of H_N consisting of all real polynomials of degree m which satisfy the boundary conditions of (2.2'). The following result is obtained from Theorems 5 and 7 of [3] and Corollary 1 above, where we make use of the fact that the eigenfunctions $\varphi_j(x)$ of (1.1)—(1.2) are by Theorem 1 necessarily of class $C^{2n}[0, 1]$.

Theorem 6. With the assumptions of (2.4)—(2.6'), let $\hat{\lambda}_{k,m}$ be the k-th approximate eigenvalue of (1.1)—(2.2'), obtained by applying the Rayleigh-Ritz method to the subspace $S_M \equiv P_0^{(m)}$ of H_N where $m+1 \geqq k+2n$. If the eigenfunctions $\{\varphi_i(x)\}_{i=1}^{\infty}$ of (1.1)—(2.2') are of class $C^t[0, 1]$, with $t \geqq 2n$, then there exist constants M_1 and M_2 dependent on k and n but independent of m, such that

$$\lambda_k \leqq \hat{\lambda}_{k,m} \leqq \lambda_k + M_1 \left\{ \frac{1}{(m-n)^{t-n}} \max_{1 \leqq i \leqq k} \omega \left(D^t \varphi_i, \frac{1}{m-n} \right) \right\}^2 \tag{5.1}$$

for all $m \geqq M_2$ where ω is the modulus of continuity[1]. If in addition, the eigenfunctions $\{\varphi_i(x)\}_{i=1}^{\infty}$ are analytic in some open set of the complex plane containing the interval $[0, 1]$, then there exists a constant μ_1 with $0 \leqq \mu_1 < 1$ such that

$$\lambda_k \leqq \hat{\lambda}_{k,m}, \tag{5.2}$$

and

$$\overline{\lim_{m \to \infty}} \, (\hat{\lambda}_{k,m} - \lambda_k)^{1/m} = \mu_1. \tag{5.3}$$

The following result, giving the asymptotic order of accuracy for the approximate eigenfunctions, is obtained from Theorems 4 and 5.

Theorem 7. With the assumptions of (2.4)—(2.7), let $\hat{\varphi}_{k,m}$ be the k-th approximate eigenfunction of (1.1)—(2.2'), obtained by applying the Rayleigh-Ritz method to the subspace $S_M \equiv P_0^{(m)}$ of H_N where $m+1 \geqq k+2n$. If $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$ and the eigenfunctions, $\{\varphi_i(x)\}_{i=1}^{\infty}$, are of class $C^t[0, 1]$, with $t \geqq 2n$ then there exist constants M_3 and M_4, dependent on k and n but independent of m, such that

$$\|\varphi_k - \hat{\varphi}_{k,m}\|_{L^\infty} \leqq K_2 \|\varphi_k - \hat{\varphi}_{k,m}\|_N \leqq M_3 \left(\frac{1}{(m-n)^{t-n}} \max_{1 \leqq k \leqq i} \omega \left(D^t \varphi_i, \frac{1}{m-n} \right) \right) \tag{5.4}$$

---

[1] As usual, $\omega(f, \delta) \equiv \sup_{\substack{|x-y| \leqq \delta \\ x, y \in [0,1]}} |f(x) - f(y)|$.

for all $m \geqq M_4$. If, in addition, the eigenfunctions $\{\varphi_i(x)\}_{i=1}^{\infty}$ are analytic in some open set of the complex plane containing the interval $[0, 1]$, then there exists a constant μ_2 with $0 \leqq \mu_2 < 1$ such that

$$(5.5) \qquad \overline{\lim_{m\to\infty}} \left(\|\varphi_k - \hat{\varphi}_{k,m}\|_{L^\infty}\right)^{1/m} = \mu_2 .$$

We remark, as in [4], that if the eigenfunctions $\{\varphi_i(x)\}_{i=1}^{\infty}$ are *entire* functions, then the constants μ_1 and μ_2 of Theorems 6 and 7 are zero, suggesting very rapid convergence. This assumption of analyticity is true for the example of §6, and the numerical results of Table 1 corroborate this rapid convergence.

As our second example, we consider subspaces of L-spline functions introduced in [15]. These subspaces include, as special cases, the cubic spline functions and piecewise cubic Hermite functions considered in [1].

Let L be the m-th order linear differential operator defined by

$$(5.6) \qquad L[u] = \sum_{j=0}^{m} a_j(x)\, D^j u(x), \qquad x \in [0, 1],$$

for all $u \in K_2^m[0, 1]$, where $K_2^m[0, 1]$ is defined as the class of all real-valued functions $u(x)$ on $[0, 1]$ such that $u \in C^{m-1}[0, 1]$ and $D^{m-1}u(x)$ is absolutely continuous with $D^m u(x) \in L^2[0, 1]$. We assume that $a_j(x) \in K_2^m[0, 1]$, $0 \leqq j \leqq m$, and $a_m(x) \geqq \omega > 0$ for all $x \in [0, 1]$. Let $\pi\colon 0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$ denote a partition of the interval $[0, 1]$, and let $\mathfrak{z} = (z_0, z_1, \ldots, z_N, z_{N+1})$, the *incidence vector*, be an $(N+2)$-vector with positive integer components each less than or equal to m, i.e., $1 \leqq z_i \leqq m$, $j = 0, \ldots, N+1$. If the formal adjoint, L^*, of L is defined by $L^*[v(x)] = \sum_{j=0}^{n} (-1)^j D^j [a_j(x)\, v(x)]$, we have

Definition 1. The real-valued function $s(x)$, $x \in [0, 1]$, is said to be an *L-spline* for π and $\mathfrak{z}$ if and only if

$$(5.7) \qquad \begin{aligned} &\text{(i)} \quad L^*L[s(x)] = 0 \quad \text{for all} \quad x \in (x_i, x_{i+1}), \quad \text{and each} \quad i = 0, \ldots, N, \\ &\text{and} \\ &\text{(ii)} \quad D^k s(x_j-) = D^k s(x_j+), \qquad k = 0, 1, \ldots, 2m-1-z_j, \quad j = 1, \ldots, N. \end{aligned}$$

The class of all L-splines for fixed π and $\mathfrak{z}$ with $z_0 = z_{N+1} = m$ is denoted by $Sp_{\mathrm{I}}(L, \pi, \mathfrak{z})$.

We remark that other choices for the incidence vector components z_0 and z_{N+1} can similarly be considered (cf. [15]), but for purposes of brevity, we shall consider here only the class $Sp_{\mathrm{I}}(L, \pi, \mathfrak{z})$, which corresponds to boundary interpolation of Type I in [15]. We do remark however, that the other types of boundary interpolation (Types II, III, IV) considered in [15] can be especially useful for the more general boundary conditions of (2.2).

We remark that if $m \geqq n$, then $Sp_{\mathrm{I},0}(L, \pi, \mathfrak{z})$, the subset of elements of $Sp_{\mathrm{I}}(L, \pi, \mathfrak{z})$ which satisfy the boundary conditions of (2.2′), is a finite-dimensional subspace of H_N. Moreover, given any function $f(x)$ in $C^{m-1}[0, 1]$, there exists a *unique* $Sp_{\mathrm{I}}(L, \pi, \mathfrak{z})$-interpolate of $f(x)$, i.e., there exists a unique $s(x)$ in $Sp_{\mathrm{I}}(L, \pi, \mathfrak{z})$ such that

$$(5.8) \qquad D^k s(x_i) = D^k f(x_i), \qquad 0 \leqq k \leqq z_i - 1, \qquad 0 \leqq i \leqq N+1 .$$

This interpolation allows us to obtain error bounds for our eigenvalue problem in the following way. With the notation $\bar{\pi} = \max_{0 \leq i \leq N} (x_{i+1} - x_i)$, we combine Theorem 9 of [15] and Corollary 1, again making use of the fact that the eigenfunctions $\varphi_j(x)$ of (1.1)—(1.2) are by Theorem 1 necessarily of class $C^{2n}[0, 1]$, to obtain

Theorem 8. Let $\{\pi_j\}_{j=1}^{\infty}$ be a sequence of partitions of $[0, 1]$ such that $\lim_{j \to \infty} \bar{\pi}_j = 0$, and let $\{z_j\}_{j=1}^{\infty}$ be a corresponding sequence of incidence vectors associated with $\{\pi_j\}_{j=1}^{\infty}$. With the assumptions of (2.4)—(2.6'), let $\hat{\lambda}_{k,j}$ be the k-th approximate eigenvalue of (1.1)—(2.2'), obtained by applying the Rayleigh-Ritz method to the subspace $S_{M_j} \equiv Sp_{\mathrm{I},0}(L, \pi_j, z_j)$ of H_N. If the eigenfunctions $\{\varphi_i(x)\}_{i=1}^{\infty}$ of (1.1)—(2.2') are of class $K_2^t[0, 1]$, with $t \geq 2m \geq 2n$, then there exists a constant M_7, dependent on k and n but independent of j, and a positive integer j_0 such that

$$\lambda_k \leq \hat{\lambda}_{k,j} \leq \lambda_k + M_7 (\bar{\pi}_j)^{2(2m-n)} \quad \text{for all} \quad j \geq j_0. \tag{5.9}$$

We remark that, since subspaces of cubic splines and cubic Hermite piecewise polynomial functions are special cases of $m=2$ of $Sp_{\mathrm{I}}(L, \pi, z)$, the above result generalizes the result of [1, Corollary 2 of Theorem 1].

From Theorems 4 and 8, we similarly obtain

Theorem 9. Let $\{\pi_j\}_{j=1}^{\infty}$ be a sequence of partitions of $[0, 1]$ such that $\lim_{j \to \infty} \bar{\pi}_j = 0$, and let $\{z_j\}_{j=1}^{\infty}$ be a corresponding sequence of incidence vectors associated with $\{\pi_j\}_{j=1}^{\infty}$. With the assumptions of (2.4)—(2.7), let $\hat{\varphi}_{k,j}$ be the k-th approximate eigenfunction of (1.1)—(2.2'), obtained by applying the Rayleigh-Ritz method to the subspace $S_{M_j} \equiv Sp_{\mathrm{I},0}(L, \pi_j, z_j)$ of H_N. If $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$ and the eigenfunctions $\{\varphi_i(x)\}_{i=1}^{\infty}$ are of class $K_2^t[0, 1]$, with $t \geq 2m \geq 2n$, then there exists a constant M_8, dependent on k and n but independent of j, and a positive integer j_0 such that

$$\|\hat{\varphi}_{k,j} - \varphi_k\|_{L^\infty} \leq K \|\hat{\varphi}_{k,j} - \varphi_j\|_N \leq M_j (\bar{\pi}_j)^{2m-n} \quad \text{for all} \quad j \geq j_0. \tag{5.10}$$

We remark again that the case $m=2$ of cubic splines and cubic Hermite piecewise polynomial functions of [1, Corollary 2 of Theorem 2] is generalized by the above result.

§ 6. Numerical Results

Since the numerical results presented in [1] are very extensive for cubic spline and cubic Hermite piecewise polynomial function subspaces of H_N, we have decided to give here the complementary results of some numerical experiments for polynomial subspaces $P_0^{(m)}$ and quintic Hermite subspaces $H_0^{(3)}(\pi)$, the latter being the special case of $Sp_1(L, \pi, z)$ with $L[u] = D^3 u(x)$ and $z_i = 3$ for all $0 \leq i \leq N+1$ (cf. (5.6) and Definition 1).

The particular eigenvalue problem

$$u^{(2)}(x) + \lambda u(x) = 0, \quad 0 < x < 1, \tag{6.1}$$

subject to the boundary conditions

$$u(0) = u(1) = 0, \tag{6.2}$$

which is a special case of (2.1)—(2.2'), satisfies the assumptions of (2.4)—(2.7) and has the known eigenfunctions $\varphi_j(x)$ and eigenvalues λ_j:

$$(6.3) \qquad \varphi_j(x) = \sin j\pi x, \quad \lambda_j = j^2\pi^2, \quad j = 1, 2, \ldots.$$

In this case, the eigenvalues are positive and distinct and the eigenfunctions $\varphi_j(x)$ are all entire, so that the constants μ_1 and μ_2 of Theorems 6 and 7 for the polynomial subspaces $P_0^{(m)}$ are both zero. In Table 1 below, we give the accuracies of the approximate eigenvalues $\hat{\lambda}_{j,m}$ as they compare with $j^2\pi^2$, $j = 1, 2, 3, 4$.

Table 1. *Polynomial subspaces* $P_0^{(m)}$

m in $P_0^{(m)}$	$\dim(P_0^{(m)})$	$\hat{\lambda}_{1,m} - \pi^2$	$\hat{\lambda}_{2,m} - 4\pi^2$	$\hat{\lambda}_{3,m} - 9\pi^2$	$\hat{\lambda}_{4,m} - 16\pi^2$
3	2	$1.30 \cdot 10^{-1}$	2.52	—	—
4	3	$1.45 \cdot 10^{-4}$	2.52	13.3	—
5	4	$1.45 \cdot 10^{-4}$	$2.31 \cdot 10^{-2}$	13.3	42.6
6	5	$8.66 \cdot 10^{-8}$	$2.31 \cdot 10^{-2}$	$3.47 \cdot 10^{-1}$	42.6
7	6	$8.66 \cdot 10^{-8}$	$5.56 \cdot 10^{-5}$	$3.47 \cdot 10^{-1}$	2.08
8	7	$2.60 \cdot 10^{-12}$	$5.56 \cdot 10^{-5}$	$3.03 \cdot 10^{-3}$	2.08

For the quintic Hermite subspaces $H_0^{(3)}(\pi(h))$, a uniform partition $\pi(h)$ of mesh size h was used. In Table 2 below, we similarly give the accuracies of the approximate eigenvalues $\hat{\lambda}_j(h)$ as they compare with $j^2\pi^2$, $j = 1, 2, 3, 4$. Note that Theorem 8 in this case gives us that $\hat{\lambda}_k(h) - \lambda_k = \mathcal{O}(h^{10})$.

Table 2. *Quintic Hermite subspaces* $H_0^{(3)}(\pi(h))$

h	$\dim(H_0^{(3)}(\pi(h)))$	$\hat{\lambda}_1(h) - \pi^2$	$\hat{\lambda}_2(h) - 4\pi^2$	$\hat{\lambda}_3(h) - 9\pi^2$	$\hat{\lambda}_4(h) - 16\pi^2$
1/2	7	$1.27 \cdot 10^{-7}$	$1.65 \cdot 10^{-3}$	$3.51 \cdot 10^{-2}$	$3.83 \cdot 10^{-1}$
1/3	10	$3.66 \cdot 10^{-9}$	$1.18 \cdot 10^{-5}$	$5.98 \cdot 10^{-3}$	$3.59 \cdot 10^{-2}$
1/4	13	$2.42 \cdot 10^{-10}$	$9.96 \cdot 10^{-7}$	$1.18 \cdot 10^{-4}$	$1.32 \cdot 10^{-2}$
1/5	16	$7.41 \cdot 10^{-11}$	$9.53 \cdot 10^{-8}$	$1.62 \cdot 10^{-5}$	$5.06 \cdot 10^{-4}$

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Prof. R. S. Varga
Case Western Reserve University
Department of Mathematics
University Circle
Cleveland, Ohio, 44106, U.S.A.

Universitätsdruckerei H. Stürtz AG Würzburg