REPRINTED FROM
THEORY AND APPLICATIONS OF SPLINE FUNCTIONS

Generalizations of Spline Functions and Applications to Nonlinear Boundary Value and Eigenvalue Problems

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0. Introduction. In the past few years, the subject of spline functions has undergone rapid development. It is the purpose of this paper to sketch some of these developments, as well as to present some new results. In §1, a historical survey is given of the evolution of splines from their introduction by Schoenberg in 1946. The survey is necessarily brief and does not attempt to give an exhaustive catalogue of all results obtained about splines. A very general kind of spline function, called an Lg-spline, is developed in §2, and error estimates are obtained in §3 for approximation by Lg-splines. The new results of the paper begin in §3 where the error estimates are shown to hold for a much wider class of approximating splines than had previously been established. The applications are to be found in §§4 and 5. Solutions of nonlinear boundary value problems are approximated in §4 by variational techniques using the improved results of §3, and in §5, similar techniques are employed to approximate the eigenvalues and eigenfunctions of eigenvalue problems.

1. Historical survey. The class of polynomial spline functions of degree n and global continuity class C^{n-1}, as introduced by Schoenberg [42], has admitted generalizations and extensions in many different directions. At the very outset, one must distinguish between two different types of approximation for which splines are utilized, viz., interpolation and best approximation, the latter usually taken in the uniform norm. It is perhaps a startling fact that, for a class of splines known as natural spline functions, the spline of interpolation coincides with the spline of best approximation in an appropriate

Hilbert space norm. This property, fully described by de Boor and Lynch [10], has become the starting point for many generalizations of interpolating splines.

The purpose of this section is to sketch a historical survey of the development of the subject of interpolating spline functions, where interpolation will be understood in a quite broad sense.

The fundamental result on the existence of splines interpolating given values at points not necessarily coincident with the knots or juncture points of the spline was obtained by Schoenberg and Whitney in [45]. Indeed, given a set of points $x_1 < x_2 < \cdots < x_k$, an integer $n \geq 1$, and abscissae $t_1 < t_2 < \cdots < t_{k+n+1}$, we then have

Theorem 1.1. For every choice of $\{y_i\}_1^{k+n+1}$, there exists a unique spline function $s(x)$ pieced together at $\{x_i\}_1^k$ by polynomials of degree n in the intervals $(-\infty, x_1)$, $(x_1, x_2), \ldots, (x_k, \infty)$ in such a way that

$$(1.1) \qquad s \in C^{n-1}(-\infty, \infty), \text{ and } s(t_i) = y_i, \quad 1 \leq i \leq n+k+1$$

if and only if

$$t_1 < x_1 < t_{n+2}, \; t_2 < x_2 < t_{n+3}, \ldots, t_k < x_k < t_{k+n+1}.$$

Theorem 1.1 was used by de Boor [9] to prove the now celebrated result on the existence of a unique natural spline function of odd degree $2m-1$ interpolating given data at $k \geq m$ points coinciding with the knots. The term natural refers to the fact that the spline reduces to a polynomial of degree $m-1$ in the intervals $(-\infty, x_1)$, (x_k, ∞). Since natural interpolating (polynomial) splines are treated in considerable detail elsewhere in this volume, we will move on now to the notion of generalized spline functions, first treated by Greville in [26].

Let $L = \sum_{j=0}^{m} a_j(x) D^j$, $D \equiv \frac{d}{dx}$, where the $a_j(x) \in C^j[a, b]$ and $a_m(x) \geq \alpha > 0$, and let $L^* f = \sum_{j=0}^{m} (-1)^j D^j(a_j f)$ denote the formal adjoint of L. Let $a < x_1 < \cdots < x_k < b$

be any partition of (a, b) with $k \geq m$. Then we may define the generalized L-spline interpolating given values r_i at the x_i as the unique solution, s, of the problem

$$
(1.2)\quad \begin{cases} \text{(i)} & L^*Ls(x) = 0 \quad \text{if } x_i < x < x_{i+1},\quad 1 \leq i \leq n-1, \\ \text{(ii)} & s(x_i) = r_i, \quad 1 \leq i \leq n, \\ \text{(iii)} & s \in C^{2m-2}[a, b], \text{ and} \\ \text{(iv)} & Ls(x) = 0 \quad \text{if } a < x < x_1 \text{ or if } x_k < x < b, \end{cases}
$$

where we assume that the null space of L is spanned by a Chebyshev system $\{u_i\}_1^m$, i.e., no linear combination of the $u_i(x)$ has more than $m - 1$ zeros, except the identically zero function. We formalize this result in

<u>Theorem 1.2</u>. There exists a unique solution s of (1.2) for each $r = (r_1, \ldots, r_k) \in E^k$ provided $k \geq m$ and the null space of L is spanned by a Chebyshev system.

The terminology of "generalized spline" was introduced by Greville in [26] where he proved Theorem 1.2 and also showed that the unique solution s of (1.2) minimizes $\int_a^b (Lf)^2 dx$ over the class of functions f such that $f \in W^{m,2}[a, b]$ and $f(x_i) = r_i$, $1 \leq i \leq k$, where $W^{m,2}[a, b]$ denotes the Sobolev class of functions f such that $D^{m-1}f$ is absolutely continuous on $[a, b]$, with $D^m f \in L^2[a, b]$. It is interesting that the converse result is true. In fact, we have

<u>Theorem 1.3</u>. The unique solution s of (1.2) uniquely minimizes the integral

$$
(1.3)\qquad \int_a^b (Lf)^2 \, dx,
$$

over all $f \in W^{m,2}[a, b]$ such that $f(x_i) = r_i$, $1 \leq i \leq k$. In particular, equality is taken on in

$$
\int_a^b (Lf)^2 dx \geq \int_a^b (Ls)^2 dx, \quad f \in W^{m,2}[a, b], \; f(x_i) = r_i, \; 1 \leq i \leq k
$$

if and only if $f = s$.

Somewhat earlier, Schoenberg [43] had introduced trigonometric splines of degree m as the class of periodic functions f in $C^{4m}[0,2\pi]$ with $f(0)=f(2\pi)$, pieced together at the knots $0=x_1<x_2<\cdots<x_{k+1}=2\pi$ by null solutions of the operator

$$M^2 = D^2(D^2+1)^2 \cdots (D^2+m^2)^2 ,$$

i.e., by linear combinations of 1, $\cos rx$, $\sin rx$, x, $x\cos rx$, $x\sin rx$ where $1 \le r \le m$. He showed that if the values $\{y_i\}_1^k$ are prescribed with $k \ge 2m+1$, and knots $\{x_i\}_1^k$ are prescribed with $x_1=0$ and $x_k < 2\pi$, then there exists a unique solution s of the problem

$$(1.4)\quad \begin{cases} \text{(i)} & M^2 s(x) = 0 \qquad \text{if } x_i < x < x_{i+1},\ 1 \le i \le k, \\ \text{(ii)} & s(x_i) = y_i, \qquad 1 \le i \le k, \\ \text{(iii)} & s \in C^{4m}[0,2\pi]. \end{cases}$$

He also showed that s minimizes

$$(1.5)\qquad \int_0^{2\pi} (Mf)^2 dx$$

over the class of periodic f in the Sobolev class $W^{2m+1,2}[0,2\pi]$ satisfying $s(x_i)=y_i$, $1 \le i \le k$. We summarize these results in

Theorem 1.4. If $k \ge 2m+1$, then (1.4) possesses a unique solution s which uniquely minimizes (1.5) over the class of periodic $f \in W^{2m+1}[0,2\pi]$ such that $f(x_i) = y_i$, $1 \le i \le k$.

The g-splines were first introduced by Ahlberg and Nilson [1] to generalize the interpolation conditions satisfied by the natural interpolating splines. Their work was later refined by Schoenberg in [44]. Indeed, let $m \ge 1$ be given and let $\{\lambda_i\}_1^k$ be a collection of Hermite-Birkhoff type linear functionals, i.e., to each λ_i there corresponds a pair (j_i, x_i), $0 \le j_i < m$ and $a \le x_i \le b$ such that

$$\lambda_i f \equiv D^{j_i} f(x_i).$$

The set $\{x_i\}$ will be called the specified knots. Under the assumption that any function u with $D^m u \equiv 0$ satisfying $\lambda_i u = 0$ for $1 \leq i \leq k$, must be identically zero, Ahlberg and Nilson [1] showed that there exists a unique solution s to the problem, given $\{y_i\}_1^k$,

$$
(1.6)\quad \begin{cases}
\text{(i)} & D^{2m} s(x) = 0 \quad \text{if } x \text{ is not a knot,} \\
\text{(ii)} & \lambda_i s = y_i , \\
\text{(iii)} & D^m s(x) = 0 \quad \text{if } x < \text{first knot, } x > \text{last knot,} \\
\text{(iv)} & [D^{2m-j-1} s]_x = 0 \quad \text{if } x \text{ is a knot and the } j\text{-th derivative is not specified at } x,
\end{cases}
$$

where $[D^{2m-j-1}s]_x = D^{2m-j-1}s(x+) - D^{2m-j-1}s(x-)$ if x is an interior point of $[a, b]$ while $[D^{2m-j-1} s]_a = D^{2m-j-1}s(a+)$ and $[D^{2m-j-1}s]_b = -D^{2m-j-1}s(b-)$.

A corresponding minimization property is valid. These results are summarized in

Theorem 1.5. If $D^m u \equiv 0$ and $\lambda_i u = 0$ for all $1 \leq i \leq k$, implies that u is identically zero, then (1.6) has a unique solution s which uniquely minimizes

$$\int_a^b (D^m f)^2 dx, \ f \in W^{m,2}[a, b], \quad \lambda_i f = y_i, \quad 1 \leq i \leq k.$$

Polynomial splines have been generalized in other directions. For differential operators L satisfying Pólya's "property W" (see [34]), there exist positive functions $\{w_i\}_0^{m-1}$ of class $C^{m-1}[a, b]$ such that the null space of L may be spanned by functions $\{u_i\}_0^{m-1}$ of the form

$$u_0(t) = w_0(t)$$

$$u_1(t) = w_0(t) \int_a^t w_1(\xi)\, d\xi$$

$$\vdots$$

$$u_{m-1}(t) = w_0(t) \int_a^t w_1(\xi_1) \int_a^{\xi_1} \cdots \int_a^{\xi_{m-2}} w_{m-1}(\xi_{m-1})\, d\xi_{m-1} \cdots d\xi_1 .$$

We say that a function is a u-polynomial if it is a linear combination of the $u_i(t)$, $0 \leq i \leq m-1$.

Functions $\varphi_r(t,x)$ may be defined by replacing the lower limit a in the definition of the $\{u_i\}$ by x if $x \leq t$; $\varphi_r(t,x) = 0$ if $x > t$.

Karlin and Ziegler [34] defined a "Chebyshevian spline function" $p(t)$ of order m on $[a,b]$ possessing the knots $a < x_1 < \cdots < x_k < b$ with associated multiplicities $\mu_1, \mu_2, \ldots, \mu_k$, $1 \leq \mu_i \leq m$, as a function p satisfying

$$(1.7)\quad \begin{cases} \text{(i)} & p \text{ is a u-polynomial in each of the intervals } [a,x_1), [x_1,x_2), \ldots, [x_k,b]; \\ \text{(ii)} & p \text{ is of continuity class } C^{m-\mu_i-1} \text{ at } x_i, \ 1 \leq i \leq k. \end{cases}$$

They showed that p has the representation

$$p(t) = \sum_{i=1}^{k} \sum_{j=1}^{\mu_i} a_{ij}\varphi_{m-j}(t,x_i) + \sum_{i=0}^{m-1} b_i u_i(t) .$$

The work of Karlin, Schumaker, and Ziegler (see [33] and [34]) gave rise to an extended result for interpolation by Chebyshevian spline functions at nodes $\{t_i\}_{i=1}^{m+k+1}$ which interlace knots $\{x_j\}_1^k$ in the manner

$$(1.8)\qquad t_i < x_i < t_{i+m+1}, \quad 1 \leq i \leq k ,$$

where the $\{t_i\}$ and $\{x_j\}$ are indexed now according to multiplicity. Their work shows that unique interpolation by a Chebyshevian spline function is possible if and only if (1.8) is satisfied, which extends the results of Schoenberg and Whitney [45]. We summarize these results in

Theorem 1.6. For each choice of $\{y_i\}_1^{k+m+1}$, there exists a unique Chebyshevian spline function interpolating the values $\{y_i\}_1^{k+m+1}$ at points $\{t_i\}_1^{k+m+1}$ with knots $\{x_i\}_1^k$ if and only if (1.8) is satisfied. Here it is understood that the $\{t_i\}_1^{k+m+1}$ and the $\{x_i\}_1^k$ are indexed according to multiplicity.

Karlin and Ziegler [34] also obtained a generalization of de Boor's earlier result concerning natural interpolating splines. Specifically, they obtained an existence and uniqueness result for natural Chebyshevian spline functions interpolating Hermite data, i. e., consecutive derivatives at points coincident with the knots.

Schultz and Varga in [46] extended simultaneously the results of Karlin and Ziegler [34] and Greville [26] to L-splines with four basic types of Hermite interpolation at the end points, including the case of periodic splines. Using theorems on the zeros of solutions of differential equations, they obtained the existence of a unique Hermite-interpolating L-spline for all sufficiently fine meshes. They then obtained sharp L^∞ and L^2 error bounds for the interpolation of smooth functions by L-splines and the g-splines described earlier, as will be described in detail in §3.

Several authors have taken an abstract approach to interpolation by splines and have worked in a Hilbert space framework. Perhaps the first to do this were de Boor and Lynch in [10], in which they exploited the properties of Hilbert spaces with reproducing kernels. Their work clearly showed that the spline of best approximation in such spaces is precisely the natural interpolating spline.

Anselone and Laurent [3] considered a very general situation suggested by the minimization property satisfied by splines. If X and Y are Hilbert spaces, T is a bounded linear operator mapping X into Y and $K = \text{span}\{k_i\}_1^k$ where the k_i are the representors of linearly independent, continuous, linear functionals on X , then Anselone and Laurent proved the following, where $\eta(T)$ is the null space of T:

<u>Theorem 1.7</u>. If T is a bounded linear mapping of X <u>onto</u> Y , $\eta(T)$ is finite-dimensional, and $\eta(T) \cap K^{\perp} = (0)$, then the minimization problem

$$\min_{f \in K_r^{\perp}} \|Tf\|_Y , \quad \text{where}$$

$$(1.9) \qquad K_r^{\perp} \equiv \{f \in X: (f, k_i) = r_i , \ 1 \leq i \leq k\}$$

has a unique solution s characterized by

$$(1.10) \qquad Ts \in (TK^{\perp})^{\perp} .$$

None of the authors previously mentioned obtained results that distinguish clearly between the properties of existence and uniqueness of interpolating splines. Hypotheses guaranteeing uniqueness are assumed a priori, and then existence is shown to follow. The first to recognize that these are not equivalent properties and that, indeed, existence can hold without uniqueness was Golomb in [25]. His insight was later used as the starting point for a general development of splines by Jerome and Schumaker [31], which is described in §2 of this paper.

2. On Lg-splines. In this section, we will simultaneously generalize results on L-splines and g-splines by Ahlberg and Nilson [1], Schoenberg [44], and Schultz and Varga [46] to obtain splines associated with a differential operator L which interpolate very general conditions expressed in the form of linear functionals. These results, due to Jerome and Schumaker [31], have appeared elsewhere in more complete form.

Let L be a linear differential operator of the form

$$(2.1) \qquad L = \sum_{j=0}^{n} a_j(x) D^j , \quad D \equiv \frac{d}{dx} ,$$

where $a_n(x) \neq 0$ on $[a, b]$ and $a_j(x) \in C^j[a, b]$, $0 \leq j \leq n$, and as in §1, let $W^{n,2}[a, b]$ denote the Sobolev space of real-valued functions $f(x)$ defined on $[a, b]$ such that $D^{n-1}f(x)$ is absolutely continuous on $[a, b]$ with $D^n f \in L^2[a, b]$. It is well known that $W^{n,2}[a, b]$ is a real Hilbert space under the inner product

$$(2.2) \qquad (f, g)_n \equiv \int_a^b \{ \sum_{j=0}^{n} D^j f(x) \cdot D^j g(x) \} dx, \quad f, g \in W^{n,2}[a, b].$$

Now let $\Lambda = \{\lambda_j\}_1^k$, be a set of continuous linear functionals linearly independent over $W^{n,2}[a, b]$, and suppose that $r = (r_1, r_2, \ldots, r_k) \in E^k$, where E^k is real Euclidean k-space.

Definition 2.1. A function $s \in W^{n,2}[a,b]$ is called an Lg-spline interpolating r with respect to Λ, provided it solves the following minimization problem:

$$(2.3)\begin{cases} \|Ls\|_{L^2[a,b]} = \inf_{f \in U(r)} \|Lf\|_{L^2[a,b]}, & \text{where} \\ U(r) \equiv \{f \in W^{n,2}[a,b] : \lambda_j f = r_j, \quad 1 \le j \le k\}. \end{cases}$$

The assumption that the $\{\lambda_j\}_1^k$ are linearly independent over $W^{n,2}[a,b]$ insures that $U(r)$ is not empty for each $r \in E^k$. Also, the differential operator L is easily seen to define a bounded linear operator from $W^{n,2}[a,b]$ onto $L^2[a,b]$. Its null space $\mathfrak{n} = \mathfrak{n}_L$ is of dimension n, and is spanned by the functions $\{u_i(x)\}_1^k$ in $C^n[a,b]$. We now formulate a fundamental result on the existence and uniqueness of Lg-splines in

Theorem 2.1. There exists an $s(x) \in W^{n,2}[a,b]$ satisfying (2.3). A function $s(x) \in U(r)$ solves (2.3) if and only if

$$(2.4) \quad \int_a^b Ls \cdot Lg \, dx = 0 \qquad \text{for every } g \in U(0).$$

Moreover, any two solutions of (2.3), corresponding to a prescribed $r \in E^k$, differ by a function in $\mathfrak{n}$, and (2.3) possesses a unique solution if and only if $\mathfrak{n} \cap U(0) = (0)$.

We remark that this result is an essential improvement of the result of Anselone and Laurent [3] in that it clearly distinguishes the questions of existence and uniqueness for interpolating splines. Existence always holds, and uniqueness holds if and only if $\mathfrak{n} \cap U(0) = (0)$. This differs sharply from [3] in which the assumption $\mathfrak{n} \cap U(0) = (0)$ is made in order to obtain both existence and uniqueness.

Proof. Note that $U(r)$ is a simple translate of $U(0)$, i.e.,

$$U(r) = x_r + U(0)$$

for an arbitrary $x_r \in U(r)$. The facts that $U(0)$ is closed and $\mathfrak{n}$ is finite-dimensional imply that $U(0) + \mathfrak{n}$ is closed.

Since L is a bounded linear operator from $W^{n,2}[a, b]$ onto $L^2[a, b]$, it follows from Lemma 2.1 of [25] that $LU(0)$ and thus $LU(r)$ is closed in $L^2[a, b]$, and hence the minimization problem (2.3) possesses a solution. Viewing (2.3) as a projection problem in L^2, the orthogonality relation (2.4) is immediate.

Conversely, if (2.4) holds for some $s \in U(r)$, then it follows easily that s is a solution of (2.3). Indeed,

$$\int_a^b (Lf)^2 dx = \int_a^b (Ls)^2 dx + 2\int_a^b (Ls)\cdot(Lf-Ls)\,dx + \int_a^b (Lf-Ls)^2 dx$$

$$= \int_a^b (Ls)^2 dx + \int_a^b (Lf-Ls)^2 dx$$

for every $f \in U(r)$, and thus $\int (Ls)^2 dx \leq \int (Lf)^2 dx$ for all $f \in U(r)$. Clearly, (2.4) implies that any two solutions of (2.3) differ by a function in $\mathfrak{n}$, and hence (2.3) possesses a unique solution if and only if $\mathfrak{n} \cap U(0) = (0)$, which completes the proof.

As immediate corollaries, we have

Corollary 2.1. The class $Sp(L,\Lambda)$ of Lg-splines s such that s satisfies (2.3) for some $r \in E^k$ is a linear space of dimension $k + \dim\{\mathfrak{n} \cap U(0)\}$ in $W^{n,2}[a, b]$ and $\mathfrak{n} \subset Sp(L, \Lambda)$.

Corollary 2.2. $L\{Sp(L, \Lambda)\}$ and $L\{U(0)\}$ form an orthogonal decomposition of L^2. In particular, for every $f \in W^{n,2}[a, b]$, we have

$$\int_a^b (Lf)^2 dx = \int_a^b (Ls)^2 dx + \int_a^b (Lf-Ls)^2 dx \tag{2.5}$$

where s interpolates $\{\lambda_j f\}_1^k$, with respect to Λ.

We remark that relation (2.5) is known in the literature as the first integral relation (cf. [2, p. 193]). Also, it can be easily demonstrated that a basis for $Sp(L, \Lambda)$ may be chosen to consist of the so-called cardinal splines $\{s_j(x)\}_1^k$, i.e., splines $s_j(x)$ such that $\lambda_i s_j = \delta_{ij}$, together with a basis for $\mathfrak{n} \cap U(0)$.

We wish now to derive analytical characterizations for members of $Sp(L,\Lambda)$ when the $\{\lambda_j\}_1^k$ are chosen to be Hermite-Birkhoff-type linear functionals. This specializes the results of Jerome and Schumaker [31], where the characterizations are carried out for more general linear functionals of so-called extended Hermite-Birkhoff-type. We will also characterize $Sp(L, \Lambda)$ when periodic conditions are added and will obtain generalizations of the well-known periodic splines. We now formalize the notion of an Hermite-Birkhoff interpolation problem, described earlier in §1 .

Definition 2. 2. We say that $\Lambda = \{\lambda_j\}_1^k$ generates an Hermite-Birkhoff (or H-B) interpolation problem if to each $\lambda_i \in \Lambda$, there corresponds a pair (x_i, j_i) such that $\lambda_i f = D^{j_i} f(x_i)$ where $a \leq x_i \leq b$ and $0 \leq j_i \leq n-1$.

The points $\{x_i\}$ are called the knots of s . The formal adjoint L^* of the differential operator L is given by

$$L^* f(x) = \sum_{j=0}^{n} (-1)^j D^j (a_j(x) f(x)) .$$

We have the following important characterization theorem for $Sp(L, \Lambda)$ in the case of an H-B interpolation problem.

Theorem 2. 2. Suppose Λ generates an H-B interpolation problem. Then $s(x) \in Sp(L, \Lambda)$ if and only if $s(x) \in W^{n,2}[a, b]$ and

$$(2.6) \begin{cases} \text{(i)} \quad L^* Ls(x) = 0 , & x \in (a, b) - \{x_i\} \\ \text{(ii)} \quad Ls(x) = 0 & \text{if } a < x < \min\{x_i\} \text{ or } \max\{x_i\} < x < b \\ \text{(iii)} \quad [O_j Ls]_x = 0 & \text{if } x \in \{x_i\} \text{ and } (D^j)_x \notin \Lambda , \ 0 \leq j \leq n-1 . \end{cases}$$

Here, $(D^j)_x$ denotes the evaluation of the j-th derivative of a function at x , and the operators O_j are those that arise naturally in the integration by parts formula:

$$(2.7) \qquad \int_\alpha^\beta (vLu - uL^* v)\, dx = - \sum_{j=0}^{n-1} D^j u(x) \cdot O_j v(x) \Big|_\alpha^\beta$$

and are defined by

$$O_j v = \sum_{i=0}^{n-j-1} (-1)^{i+1} D^i (a_{i+j+1}(x) v(x)) .$$

The bracket notation $[f]_x$ is defined by

$$[f]_x = f(x+) - f(x-) \quad \text{if} \quad a < x < b,$$

while $[f]_a = f(a+)$ and $[f]_b = -f(b-)$.

<u>Proof</u> (of Theorem 2.2). For any $s(x) \in Sp(L,\Lambda)$, let J be any open subinterval of (a, b) such that $J \cap \{x_i\}$ is empty. If $C_c^{2n}(J)$ denotes the set of all functions in $C^{2n}(J)$ which vanish outside a compact interval contained in J, then every $\varphi \in C_c^{2n}(J)$ is also in $U(0)$, and by (2.4) and (2.7),

$$(2.8) \qquad 0 = \int_a^b Ls \cdot L\varphi \, dx = \int_J Ls \cdot L\varphi \, dx = \int_J sL^* L\varphi \, dx.$$

It follows from (2.8), by the argument used in Gelfond and Shilov [24] to prove that every distribution solution is a classical solution for operators L considered here, that the restriction of s to J is in $C^{2n}(J)$ and satisfies $L^*Ls(x) = 0$ for all $x \in J$. This proves (2.6)(i).

To prove (2.6)(ii), suppose $\xi = \min(x_i)$ and $a < x < \xi$. If $Ls(x) \neq 0$ then, by the continuity of Ls in a neighborhood of x, it follows that $\int_a^b (Ls)^2 dx > \int_\xi^b (Ls)^2 \, dx$. Now, define a new function $\tilde{s}(x) \in W^{n,2}[a, b]$ by:

$$\tilde{s}(y) = \begin{cases} s(y) & \text{if} \quad \xi \leq y \leq b \\ u(y) & \text{if} \quad a \leq y \leq \xi \end{cases}$$

where $u \in \mathfrak{n}$ is determined by the n conditions $D^j u(\xi) = D^j s(\xi)$, $0 \leq j \leq n-1$. Clearly,

$$\int_a^b (L\tilde{s})^2 dx = \int_\xi^b (L\tilde{s})^2 dx = \int_\xi^b (Ls)^2 dx < \int_a^b (Ls)^2 dx.$$

Since $\lambda_j \tilde{s} = \lambda_j s$ for $j = 1, 2, \ldots, k$, this contradicts the fact that $s \in Sp(L,\Lambda)$. A similar proof holds if $\max\{x_i\} < x < b$.

To prove (2.6)(iii), let $x \in \{x_i\}$ be an interior point of $[a, b]$ and suppose that the evaluation of the j-th derivative at x is not in Λ where $0 \leq j \leq n-1$. With j held fixed, let $J = (x - \epsilon, x + \epsilon)$ be an interval such that $J \cap \{x_i\} = x$ and choose $g \in C_c^\infty(J)$ such that $D^i g(x) = \delta_{ij}$, $0 \leq i \leq n-1$. Then, $g \in U(0)$ and, by (2.4), (2.6)(i), and (2.7),

$$0 = \int_a^b Ls \cdot Lg \, dx = \int_{x-\epsilon}^{x} Ls \cdot Lg \, dx + \int_x^{x+\epsilon} Ls \cdot Lg \, dx$$

$$= -\sum_{i=0}^{n-1} g^i(y) \, O_i Ls(y) \Big|_{x-\epsilon}^{x} - \sum_{i=0}^{n-1} g^i(y) \, O_i Ls(y) \Big|_{x}^{x+\epsilon}$$

$$= \sum_{i=0}^{n-1} g^{(i)}(x) \, [O_i Ls]_x = [O_j Ls]_x ,$$

since $g^{(i)}(x) = \delta_{ij}$. The proof in the case $x = a$ or $x = b$ is similar.

Conversely, suppose $s \in W^{n,2}$ and satisfies (2.6). We will show that $s \in Sp(L, \Lambda)$ by showing that $\int_a^b Ls \cdot Lg \, dx = 0$ for all $g \in U(0)$. Indeed, if $g \in U(0)$ and $\xi = \min\{x_i\}$ and $\eta = \max\{x_i\}$, then by (2.6)(ii),

$$\int_a^b Ls \cdot Lg \, dx = \int_\xi^\eta Ls \cdot Lg \, dx$$

and by (2.6)(i) and (2.7), we conclude that

$$\int_a^b Ls \cdot Lg \, dx = \sum_{\{x_i\}} \sum_{j=0}^{n-1} D^j g(x_i) \, [O_j Ls]_{x_i}$$

and, for each x_i, the inner sum is zero by (2.6)(iii) and the fact that $g \in U(0)$. This completes the proof of Theorem 2.2.

A number of corollaries follow immediately from Theorem 2.2.

<u>Corollary 2.3</u>. If $L = D^n$, then (2.6)(iii) becomes

$$D^{2n-j-1} s(x+) = D^{2n-j-1} s(x-)$$

if $x \in (a, b)$ and the j-th derivative evaluated at x is not in Λ; if $x = a$ or $x = b$, then

$$D^{2n-j-1} s(a+) = 0, \quad D^{2n-j-1} s(b-) = 0 ,$$

respectively.

<u>Corollary 2.4</u>. Suppose $s \in Sp(L, \Lambda)$ where Λ defines an H-B interpolation problem and let $x \in \{x_i\}$. If v denotes the order of the highest derivative specified at x,

then s is locally of class C^{2n-2-v} about x.

Let $W_p^{n,2}[a,b]$ denote the closed subspace of $W^{n,2}[a,b]$ consisting of those periodic functions f on $[a,b]$ satisfying $D^j f(a+) = D^j f(b-)$, $j = 0, 1, \ldots, n-1$. $W_p^{n,2}[a,b]$ is the null space of the set $\{\mu_j\}_0^{n-1}$ of continuous linear functionals on $W^{n,2}[a,b]$ defined by

$$\mu_j f = D^j f(a+) - D^j f(b-).$$

Definition 2.3. $Sp_p(L,\Lambda)$ denotes the class of solutions s of (2.3) where $U(r)$ is replaced by $U(r) \cap W_p^{n,2}[a,b]$ and $\{\lambda_i\}_1^N$ are linearly independent over $W_p^{n,2}[a,b]$. The elements of $Sp_p(L,\Lambda)$ are called periodic spline functions.

It is easy to see that Theorem 2.1 carries over to the class $Sp_p(L,\Lambda)$. $U(0)$ is replaced by $U(0) \cap W_p^{n,2}[a,b]$ and the uniqueness characterization becomes in this case:

$$s(x) \text{ is unique if and only if } \eta \cap W_p^{n,2}[a,b] \cap U(0) = (0).$$

We have as a final corollary to Theorem 2.2,

Corollary 2.5. Let Λ generate an H-B interpolation problem. Then $s \in Sp_p(L,\Lambda)$ if and only if $s \in W_p^{n,2}[a,b]$ and s satisfies (2.6) where (2.6)(iii) is modified in the following sense: a is identified with b and, if $a \in \{x_i\}$ and if $D^j s(a+) \notin \Lambda$, then

$$[O_j Ls]_a = O_j Ls(a+) - O_j Ls(b-).$$

We remark in closing this section that the first integral relation (2.5) is also valid for $f \in W_p^{n,2}$ in terms of the periodic Lg-spline s interpolating f. Finally, if $f \in W^{n,2}[a,b]$ and $s_f \in Sp(L,\Lambda)$ interpolates f then, for any $s \in Sp(L,\Lambda)$, we have by (2.5)

$$(2.9) \quad \int_a^b (Lf-Ls)^2 dx = \int_a^b (Ls_f - Ls)^2 dx + \int_a^b (Lf - Ls_f)^2 dx.$$

A corresponding property holds for $f \in W_p^{n,2}[a,b]$. We remark that if $f \in W^{2n,2}[a,b]$ and s is chosen to be equal to s_f, then the left side of (2.9) has the alternate expression

$$(2.10)\qquad \int_a^b (Lf - Ls)^2 dx = \int_a^b (f - s)\cdot L^*Lf\, dx ,$$

which is known in the literature as the second integral relation (cf. [2, p. 205]).

3. Error Estimates. In this section, we will obtain new error estimates for the approximation of smooth functions by Lg-splines or periodic Lg-splines interpolating H-B-type data. We will obtain six distinct approximation theorems while considering L^∞ and L^2 convergence of Lg-splines interpolating $W^{n,2}$ and $W^{2n,2}$ functions (see Schultz and Varga [46] where these results were obtained for Hermite L-splines and g-splines). For the special case of piecewise Hermite interpolation, convergence results of Birkhoff, Schultz and Varga [8] will be stated for functions in the class $W^{m,p}$ where $n \leq m \leq 2n$ and $1 \leq p \leq \infty$.

Definition 3.1. Let $\Lambda = \{\lambda_j\}_1^k$ define an H-B problem, and let $\{x_j\}$ be the corresponding knots. The subset Δ consisting of all $x \in \{x_j\}$ such that there exists $\lambda \in \{\lambda_j\}$ satisfying $\lambda f = f(x)$ will be called the partition of $[a, b]$ induced by Λ.

If Δ is not empty, we define $\bar{\Delta}$ as the maximum length of the subintervals into which $[a, b]$ is decomposed by the points of Δ, and we define $\underline{\Delta}$ as the minimum such length. Next, if Δ is not empty and $x \in \Delta$, let $i(x)$ be defined as the maximum positive integer such that there exists a $\lambda_k \in \Lambda$ for which

$$\lambda_k f = D^k f(x)$$

for each $0 \leq k \leq i(x) - 1$. In other words, $i(x)$ is the number of consecutive derivative point functionals associated with the point $x \in \Delta$. With this notation, we then define $\gamma(\Delta)$ by

$$\gamma(\Delta) \equiv \sum_{x \in \Delta} i(x)$$

if Δ is not empty, and we set $\gamma(\Delta) = 0$ if Δ is empty.

We now state

Theorem 3.1. Let $f \in W^{n,2}[a,b]$, let Λ generate an H-B problem, and assume that $\gamma(\Delta) \geq n$. If $s \in Sp(L,\Lambda)$ interpolates f, then for $\bar{\Delta}$ sufficiently small,

$$\|D^j(f-s)\|_{L^\infty[a,b]} \leq M_j(\bar{\Delta})^{n-j-(1/2)} \|L(f-s)\|_{L^2[a,b]} \tag{3.1}$$
$$\leq M_j(\bar{\Delta})^{n-j-(1/2)} \|Lf\|_{L^2[a,b]}$$

for all $0 \leq j < n$, where M_j is independent of Λ and f.

Proof. The argument here is an extension of that used in Schultz and Varga [46]. Let $\Delta = \{\xi_0 < \xi_1 < \cdots < \xi_N\}$, and let $s(x)$ be any $Sp(L,\Delta)$-interpolate of $f(x)$. Note that $s(x)$ need not be uniquely determined. Since $f - s \in C^{n-1}[a,b]$ and $\gamma(\Delta) \geq n$, we can apply a generalized Rolle's Theorem, i.e., setting $\xi_i^{(0)} = \xi_i$ for $0 \leq i \leq N$, there exist points $\Delta^{(j)} = \{\xi_\ell^{(j)}\}_0^{N_j}$ in $[a,b]$ such that

$$D^j f(\xi_\ell^{(j)}) - D^j s(\xi_\ell^{(j)}) = 0, \quad 0 \leq \ell \leq N_j, \quad 0 \leq j \leq n-1, \tag{3.2}$$

where $N = N_0 \geq N_1 \geq N_2 \geq \cdots \geq N_{n-1} \geq 0$, where the points of $\Delta^{(j)}$ satisfy

$$a \leq \xi_0^{(j)} < \xi_1^{(j)} < \cdots < \xi_{N_j}^{(j)} \leq b$$

and $\Delta^{(j+1)}$ is related to $\Delta^{(j)}$ by

$$\xi_\ell^{(j)} \leq \xi_\ell^{(j+1)} < \xi_{\ell+1}^{(j)} \quad \text{for all } 0 \leq \ell \leq N_j \quad \text{and } 0 \leq j \leq n-2.$$

It follows immediately that $|\xi_{\ell+1}^{(j)} - \xi_\ell^{(j)}| \leq (j+1)\bar{\Delta}$, $|a - \xi_0^{(j)}| \leq (j+1)\bar{\Delta}$ and $|b - \xi_{N_j}^{(j)}| \leq (j+1)\bar{\Delta}$ for any $0 \leq j \leq n-1$. Now, for each such j, let $x_j \in [a,b]$ be such that

$$|D^j(f(x_j) - s(x_j))| = \|D^j(f-s)\|_{L^\infty[a,b]}, \quad 0 \leq j \leq n-1. \tag{3.3}$$

Since $\overline{\Delta^{(j)}} \equiv \max_i |\xi_{i+1}^{(j)} - \xi_i^{(j)}| \le (j+1)\bar{\Delta}$, there is a point $\xi_k^{(j)}$ such that $|x_j - \xi_k^{(j)}| \le (j+1)\bar{\Delta}$. By (3.2) and (3.3),

$$\|D^j(f-s)\|_{L^\infty[a,b]} = \left|\int_{\xi_k^{(j)}}^{x_j} D^{j+1}(f(t) - s(t))dt\right|, \quad 0 \le j \le n-1,$$

(3.4)

$$\le (j+1)\bar{\Delta}\, \|D^{j+1}(f-s)\|_{L^\infty[a,b]}, \quad 0 \le j < n-1.$$

Arguing inductively, we conclude that for $0 \le j \le n-1$,

$$(3.5)\quad \|D^j(f-s)\|_{L^\infty[a,b]} \le \frac{(n-1)!}{j!}(\bar{\Delta})^{n-j-1}\|D^{n-1}(f-s)\|_{L^\infty[a,b]}.$$

Applying the Schwarz inequality to the integral of (3.4) for $j = n-1$, we obtain

$$(3.6)\quad \|D^{n-1}(f-s)\|_{L^\infty[a,b]} \le \sqrt{n\bar{\Delta}}\, \|D^n(f-s)\|_{L^2[a,b]}.$$

By (3.5) and (3.6),

$$(3.7)\quad \|D^j(f-s)\|_{L^\infty[a,b]} \le \frac{n!}{\sqrt{n}\, j!}(\bar{\Delta})^{n-j-(1/2)}\|D^n(f-s)\|_{L^2[a,b]},$$

$$0 \le j \le n-1.$$

Now in order to put the right-hand side of (3.7) into the form of (3.1) we write,

$$a_n(x) D^n(f(x) - s(x)) = L[f(x) - s(x)] - \sum_{j=0}^{n-1} a_j(x) D^j(f(x) - s(x))$$

and since $|a_n(x)| \ge \omega > 0$ on $[a, b]$, we have by the triangle inequality

$$(3.8)\qquad \| D^n(f-s)\|_{L^2[a,b]} \le \frac{1}{\omega}\Big\{\|L(f-s)\|_{L^2[a,b]} + \sum_{j=0}^{n-1} \|a_j\|_{L^\infty[a,b]} \|D^j(f-s)\|_{L^2[a,b]}\Big\}.$$

But, since $\|D^j(f-s)\|_{L^2[a,b]} \le (b-a)^{1/2}\|D^j(f-s)\|_{L^\infty[a,b]}$, (3.7) and (3.8) yield

$$\Big\{1 - \frac{1}{\omega}\sum_{j=0}^{n-1}\|a_j\|_{L^\infty[a,b]} \frac{(b-a)^{1/2}}{\sqrt{n}}\frac{n!}{j!}(\bar{\Delta})^{n-j-(1/2)}\Big\}$$

$$\times \|D^n(f-s)\|_{L^2[a,b]} \le \frac{1}{\omega}\|L(f-s)\|_{L^2[a,b]}.$$

Clearly, if $\bar{\Delta}$ is sufficiently small, the coefficient $c = c(n, L)$ of $\|D^n(f-s)\|$ is greater than $1/2$, and hence

$$(3.9)\qquad \|D^n(f-s)\|_{L^2[a,b]} \le \frac{2}{\omega}\|L(f-s)\|_{L^2[a,b]}$$

for all such $\bar{\Delta}$. Inequalities (3.7) and (3.9) then imply the first inequality of (3.1) with $M_j = (\frac{2}{\omega})\frac{n!}{\sqrt{n}\,j!}$. The last inequality of (3.1) follows from the first integral relation (2.3). This completes the proof.

If we are interested in L^2-type rather than L^∞-type error bounds, the result of Theorem 3.1 can be improved in

<u>Theorem 3.2.</u> Let $f \in W^{n,2}[a,b]$, let Λ generate an H-B problem, and assume that $\gamma(\Delta) \ge n$. If $s \in Sp(L, \Lambda)$ interpolates f, then for $\bar{\Delta}$ sufficiently small,

$$(3.10)\qquad \begin{aligned}\|D^j(f-s)\|_{L^2[a,b]} &\le M_j^{(1)}(\bar{\Delta})^{n-j}\|L(f-s)\|_{L^2[a,b]}\\ &\le M_j^{(1)}(\bar{\Delta})^{n-j}\|Lf\|_{L^2[a,b]}\end{aligned}$$

for all $0 \le j \le n$, where $M_j^{(1)}$ is independent of f and Λ.

Proof. For any $0 \le j \le n-1$ we have from (3.2) that $D^j(f-s)$ vanishes at $\xi_\ell^{(j)}$ for $0 \le \ell \le N_j$. Hence, applying the Rayleigh-Ritz inequality (cf. [27, p. 184]), we have for $0 \le \ell \le N_j$,

$$(3.11)\quad \int_{\xi_\ell^{(j)}}^{\xi_{\ell+1}^{(j)}} \{D^j(f(t)-s(t))\}^2 dt \le \left[\frac{(j+1)\bar{\Delta}}{\pi}\right]^2 \int_{\xi_\ell^{(j)}}^{\xi_{\ell+1}^{(j)}} \{D^{j+1}(f(t)-s(t))\}^2 dt$$

since $|\xi_{\ell+1}^{(j)} - \xi_\ell^{(j)}| \le (j+1)\bar{\Delta}$. Summing both sides of (3.11) with respect to ℓ, we have for $0 \le j \le n-1$,

$$(3.12)\quad \int_{\xi_0^{(j)}}^{\xi_{N_j}^{(j)}} \{D^j(f(t)-s(t))\}^2 dt \le \left[\frac{(j+1)\bar{\Delta}}{\pi}\right]^2 \|D^{j+1}(f-s)\|^2_{L^2[a,b]}.$$

Setting $j = n-1$ in this inequality we have from (3.9),

$$(3.13)\quad \int_{\xi_0^{(n-1)}}^{\xi_{N_{n-1}}^{(n-1)}} \{D^{n-1}(f(t)-s(t))\}^2 dt \le \frac{4}{\omega^2}\left(\frac{n\bar{\Delta}}{\pi}\right)^2 \|L(f-s)\|^2_{L^2[a,b]}$$

for all $\bar{\Delta}$ sufficiently small. Next, (3.1) gives us that

$$(3.14)\quad \begin{aligned}\int_a^{\xi_0^{(j)}} \{D^j(f(t)-s(t))\}^2 dt &\le |\xi_0^{(j)} - a|\, \|D^j(f-s)\|^2_{L^\infty[a,b]} \\ &\le M_j''(\bar{\Delta})^{2n-2j} \|L(f-s)\|^2_{L^2[a,b]}\end{aligned}$$

for all $0 \le j \le n-1$ since $|\xi^{(j)} - a| \le (j+1)\bar{\Delta}$, as well as a similar inequality

$$(3.15)\quad \int_{\xi_{N_j}^{(j)}}^{b} \{D^j(f(t)-s(t))\}^2 dt \le M_j''(\bar{\Delta})^{2n-2j} \|L(f-s)\|^2_{L^2[a,b]}.$$

Setting $j = n - 1$ in (3.14) and (3.15) and summing the left-hand sides of (3.13), (3.14), and (3.15), we obtain, upon taking square roots,

$$(3.16) \qquad \| D^{n-1}(f - s) \|_{L^2[a,b]} \leq M'' \bar{\Delta} \| L(f-s) \|_{L^2[a,b]}$$

for all $\bar{\Delta}$ sufficiently small. Now, arguing by induction on $v = n - j$ and using (3.12), (3.14), and (3.15) we can easily establish (3.10). This completes the proof.

We remark that Theorems 3.1 and 3.2 are also valid if $f \in W_p^{n,2}[a, b]$ and $s \in Sp_p(L, \Lambda)$, so long as the other hypotheses are satisfied.

Now, suppose $f \in W^{2n,2}[a, b]$. As we know from (2.10), the second integral relation is

$$(3.17) \qquad \int_a^b (Lf - Ls)^2 dx = \int_a^b (f - s) L^* Lf \, dx .$$

We are now ready to state

Theorem 3.3. Let $f \in W^{2n,2}[a, b]$, let Λ generate an H-B problem, and assume that $\gamma(\Delta) \geq n$. If $s \in Sp(L, \Lambda)$ interpolates f, then for $\bar{\Delta}$ sufficiently small,

$$(3.18) \qquad \| D^j(f-s) \|_{L^\infty[a,b]} \leq M_j^{(2)} (\bar{\Delta})^{2n-j-(1/2)} \| L^* Lf \|_{L^2[a,b]}$$

for all $0 \leq j \leq n-1$.

Proof. Schwarz's inequality applied to (3.17) yields

$$\| L(f-s) \|^2_{L^2[a,b]} \leq \| f - s \|_{L^2[a,b]} \| L^* Lf \|_{L^2[a,b]} .$$

Applying (3.10) with $j = 0$ to the right side of this inequality and dividing through by $\| L(f - s) \|_{L^2[a,b]}$, we obtain

$$(3.19) \qquad \| L(f-s) \|_{L^2[a,b]} \leq M_0^{(1)} (\bar{\Delta})^n \| L^* Lf \|_{L^2[a,b]} .$$

It is clear that (3.1) and (3.19) then imply (3.18) and the proof is concluded.

If we are again interested in L^2-type error bounds the result of Theorem 3.3 can be improved in

Theorem 3.4. Let f, Λ, s, and Δ satisfy the hypotheses of Theorem 3.3. Then,

$$(3.20) \qquad \| D^j(f-s)\|_{L^2[a,b]} \leq M_j^{(3)}(\bar{\Delta})^{2n-j} \| L^* Lf\|_{L^2[a,b]}$$

for all $0 \leq j \leq n$.

Proof. This follows directly from (3.10) and (3.19).

If we extend the usual definition of the L^∞-norm on $[a, b]$ by defining

$$\| D^j(f-s) \|_{L^\infty[a,b]} = \max_{0 \leq k \leq N-1} \{ \| D^j(f-s)\|_{L^\infty[\xi_k, \xi_{k+1}]} \}$$

for $n \leq j \leq 2n-1$, then an analogous extension of the results of [46] yields

Theorem 3.5. Let $f \in W^{2n,2}[a, b]$, let $\{\Lambda_i\}_1^\infty$ generate a sequence of H-B problems, with $\gamma(\Delta_i) \geq n$ for all $i \geq 1$, let $s_i \in Sp(L, \Lambda_i)$ interpolate f, and let $\bar{\Delta}_i$ tend to zero in such a way that $\bar{\Delta}_i/\underline{\Delta}_i \leq \sigma$ for all $i \geq 1$. Then, there exists an i_0 such that for all $i \geq i_0$,

$$(3.21) \qquad \| D^j(f-s_i)\|_{L^\infty[a,b]} \leq M_j^{(4)}(\bar{\Delta}_i)^{2n-j-(1/2)} \| L^* Lf\|_{L^2[a,b]}$$

for all $0 \leq j \leq 2n-1$.

An L^2 estimate for higher order derivatives is also available and has been obtained by Perrin [37]. We state this as

Theorem 3.6. Let the hypotheses of Theorem 3.5 be satisfied. Then there exists i_0 such that for all $i \geq i_0$,

$$(3.22) \qquad \| D^j(f-s_i) \|_{L^2[a,b]} \leq M_j^{(5)}(\bar{\Delta}_i)^{2n-j} \| L^* Lf\|_{L^2[a,b]}$$

for all $0 \leq j \leq 2n-1$.

Error estimates also can be derived for different assumptions upon f. For example, in the often-cited paper of Birkhoff and de Boor [5], convergence of cubic spline interpolates to functions f in $C^4[a, b]$ is shown to be of the order $(\bar{\Delta}_i)^4$ for nearly uniform meshes Δ_i, with convergence of the j-th derivatives of order $(\bar{\Delta}_i)^{4-j}$, for $0 \leq j \leq 3$. Certain limited generalizations of this result have been made but we will not pursue them here. (See for example Birkhoff and de Boor [5], and Sharma and Meir [47].)

We turn now to the topic of piecewise Hermite interpolation, which will conclude this section.

Let $W^{n,r}[a, b]$ for $n \geq 1$ and $1 \leq r \leq \infty$ denote the set of functions f such that $D^{n-1}f$ is absolutely continuous on $[a, b]$, and $D^n f \in L^r[a, b]$, and let $m \geq 1$ be such that $m \leq n \leq 2m$. It is clear that, given an arbitrary partition

$$\Delta: \quad a = x_0 < x_1 < \cdots < x_{N+1} = b$$

of $[a, b]$ and $f \in W^{n,r}[a, b]$, there exists a unique function f_m which is a piecewise polynomial of degree $2m - 1$ on each subinterval $[x_i, x_{i+1}]$ of $[a, b]$ satisfying

$$D^j f_m(x_i) = D^j f(x_i), \quad i = 0, 1, \ldots, N+1, \quad 0 \leq j \leq m-1.$$

The class of all such f_m is designated by $H_\Delta^{(m)}(a, b)$. The following theorem is taken from Birkhoff, Schultz, and Varga [8].

Theorem 3.7. Let Δ be any partition of $[a, b]$, let $f \in W^{n,r}[a, b]$, where $n \geq 1$ and $1 \leq r \leq \infty$. Let m satisfy $m \leq n \leq 2m$, and let f_m be the unique $H_\Delta^{(m)}[a, b]$ interpolant of f. Then

$$\text{(3.23)} \qquad \| D^j(f - f_m) \|_{L^q[a, b]} \leq c_{j,m,n,r,q} (\bar{\Delta})^{n-j-(1/r)+(1/q)} \| D^n f \|_{L^r[a, b]}$$

for any $q \geq r$ and any $0 \leq j \leq m-1$, and also for $j = m$ if $n > m$ or $q = r$, and

$$(3.24) \qquad \| D^j(f - f_m) \|_{L^q(a,b]} \leq c_{j,m,n,r,q} (\bar{\Delta})^{n-j} (b-a)^{(r-q)/(rq)} \| D^m f \|_{L^r[a,b]}$$

for any $1 \leq q \leq r$, and any $0 \leq j \leq m$.

Several remarks are now in order. First, based on the results of Schultz and Varga [46] and Birkhoff, Schultz, and Varga [8], we can assert that the exponents of $\bar{\Delta}$ in Theorems 3.1-3.7 cannot be improved for the function spaces considered in these theorems. Second, the results of Theorems 3.1-3.6 constitute a generalization of the error bounds given by Schultz and Varga [46], because the error bounds of [46] were derived for Hermite-Birkhoff problems for which the $Sp(L, \Delta)$ -interpolates were always unique. But, the proofs of this section show moreover that the error bounds of Theorems 3.1 and 3.2 are in fact valid for the more general case in which $\Lambda = \{\lambda_j\}_1^k$ consists of at least point functionals for which $\gamma(\Delta) \geq n$, the remaining functionals of Λ being arbitrary, provided that $\Lambda = \{\lambda_j\}_1^k$ is a linearly independent set. Moreover, the error bounds in Theorems 3.1-3.6 are valid for extended Hermite-Birkhoff problems of Jerome and Schumaker [31].

4. Applications of Lg-splines to nonlinear differential equations. As a particular application of the theory of Lg-splines given in the previous sections, we consider the approximate solution of the following two-point nonlinear boundary value problem:

$$(4.1) \qquad M[u(x)] + f(x, u(x)) = 0, \qquad a < x < b,$$

where

$$(4.1') \qquad M[u(x)] \equiv \sum_{0 \leq i,j \leq n} (-1)^j D^j(\sigma_{i,j}(x) D^i u(x)), \qquad n \geq 1, \quad D \equiv \frac{d}{dx},$$

subject to the homogeneous boundary conditions of

$$(4.2) \qquad D^j u(a) = D^j u(b) = 0 , \qquad 0 \le j \le n-1 .$$

For the coefficient functions $\sigma_{i,j}(x)$ of (4.1'), we assume that

(4.3)
(i) the coefficient functions $\sigma_{i,j}(x)$, $0 \le i, j \le n$, are bounded, real-valued, and measurable in x in $[a, b]$, and

(ii) there exists a positive constant c such that
$$\int_a^b \{ \sum_{0 \le i, j \le n} \sigma_{i,j}(x) D^i w(x) D^j w(x) \} dx \ge c \|w\|_n^2$$
for all $w(x) \in W_0^{n,2}[a, b]$,

where $W_0^{n,2}[a, b]$ denotes the linear subspace of $W^{n,2}[a, b]$ of all real-valued functions $w(x)$ defined on $[a, b]$ which satisfy (4.2). As is known (cf. (2.2)), $W_0^{n,2}[a, b]$ is a Hilbert space with a norm defined by

$$\|u\|_n^2 \equiv \int_a^b \{ \sum_{j=0}^{n} (D^j u(x))^2 \} dx .$$

It follows from (4.3)(ii) that

$$(4.4) \quad \Lambda \equiv \inf_{\substack{w \in W_0^{n,2}[a, b] \\ w \neq 0}} \frac{\int_a^b \{ \sum_{0 \le i, j \le n} \sigma_{i,j}(x) D^i w(x) D^j w(x) \} dx}{\int_a^b (w(x))^2 dx}$$

is <u>positive</u>. With respect to the function $f(x, u)$ of (4.1), we assume that

(4.5)
(i) $f(x, u)$ is a real-valued function on $[a, b] \times R$ such that $f(x, u(x)) \in L^2[a, b]$ for any $u(x) \in W_0^{n,2}[a, b]$,

(ii) there exists a real constant γ such that $\frac{f(x, u) - f(x, v)}{u - v} \ge \gamma > -\Lambda$ for almost all $x \in [a, b]$ and all $-\infty < u, v < +\infty$ with $u \neq v$,

(iii) for each positive real number c, there exists a positive constant $M(c)$ such that $\frac{f(x, u) - f(x, v)}{u - v} \le M(c)$ for almost all $x \in [a, b]$, and all $-\infty < u, v < \infty$ with $u \neq v$ and $|u| < c$, $|v| \le c$.

Consider now the quasi-bilinear form

$$(4.6)\quad a(u,v) \equiv \int_a^b \Big\{ \sum_{0 \le i,j \le n} \sigma_{i,j}(x) D^i u(x) \cdot D^j v(x) + f(x,u(x)) \cdot v(x) \Big\} dx$$

for any $u, v \in W_0^{n,2}[a,b]$. This is obtained formally by multiplying the left side of (4.1) by $v(x)$, and then integrating by parts. Following Aubin [4], Browder [12], and Céa [13], an element $u(x) \in W_0^{n,2}[a,b]$ is called a generalized solution of (4.1)-(4.2) in $W_0^{n,2}[a,b]$ if

$$(4.7)\qquad a(u,v) = 0 \qquad \text{for all } v(x) \in W_0^{n,2}[a,b] .$$

Similarly, if B^k is any finite-dimensional subspace of $W_0^{n,2}[a,b]$, then $u_k \in B^k$ is called a Galerkin approximation in B^k of the solution $u(x)$ of (4.1)-(4.2) if

$$(4.8)\qquad a(u_k, v) = 0 \qquad \text{for all } v \in B^k .$$

We now state a result which is proved using results of Browder [12], Minty [35], and Zarantonello [54] on monotone operators (cf. Ciarlet, Schultz and Varga [18] and Varga [49]).

Theorem 4.1. With the assumptions of (4.3) and (4.5), the two-point nonlinear boundary value problem of (4.1)-(4.2) has a unique generalized solution (cf. (4.7)), $u(x)$, in $W_0^{n,2}[a,b]$. Moreover, if B^k is any finite-dimensional subspace of $W_0^{n,2}[a,b]$, then there exists a unique Galerkin approximation (cf. (4.8)), $u_k(x)$, and there exist positive constants K_1 and K_2, independent of the choice of B^k, such that

$$(4.9)\quad \| D^i(u_k - u)\|_{L^\infty[a,b]} \le K_1 \| u_k - u \|_n \le K_2 \inf\{ \|w_k - u\|_n ; \ w_k \in B^k \}$$

for all $0 \le i \le n-1$.

We remark that the first inequality in (4.9) is a consequence of the Sobolev imbedding theorem in one dimension (cf. [36, p. 72] and [53, p. 174]).

Consider now the algebraic problem of determining the Galerkin approximation from (4.8). If $\{w_i(x)\}_{i=1}^M$ is a basis for the finite-dimensional subspace B^k of $W_0^{n,2}[a,b]$, we can express the unique Galerkin approximation $u_k(x)$ in B^k as $u_k(x) = \sum_{i=1}^M c_i w_i(x)$. Using (4.6), then (4.8) takes the matrix form

$$(4.10) \qquad Ac + g(c) = 0,$$

where $A = (\alpha_{k,\ell})$ is an $M \times M$ real matrix, and c and $g(c)$ are column vectors with M components, where

$$(4.11) \qquad \alpha_{\ell,k} \equiv \int_a^b \{ \sum_{0 \le i,j \le n} \sigma_{i,j}(x) D^i w_k(x) \cdot D^j w_\ell(x) \} dx, \qquad 1 \le k, \ell \le M,$$

and

$$(4.11') \qquad g_\ell(c) \equiv \int_a^b f(x, \sum_{k=1}^M c_k w_k(x)) \cdot w_\ell(x)\, dx, \quad 1 \le \ell \le M.$$

From Theorem 4.1, the nonlinear matrix problem of (4.10) admits a unique solution c, from which the Galerkin approximation $u_k(x)$ can be constructed.

To give a concrete illustration of the foregoing material in this section, suppose we consider the numerical approximation of the two-point boundary value problem

$$(4.12) \qquad D^2 u(x) = e^{u(x)}, \qquad 0 < x < 1,$$

subject to

$$(4.13) \qquad u(0) = u(1) = 0.$$

This corresponds to the case in (4.1') where $\sigma_{1,1}(x) \equiv 1$, and where $\sigma_{i,j}(x) \equiv 0$ for all $0 \le i, j \le 1$ with $0 \le i+j < 2$. Thus, to verify that (4.3) is satisfied in this case, we need only show that there exists a positive constant c such that

$$\int_0^1 (Dw(x))^2 dx \geq c\|w\|_1^2 \quad \text{for all } w(x) \in W_0^{1,2}[0,1] .$$

Using the Rayleigh-Ritz inequality (cf. [27, p. 184]), we have for any $w(x) \in W_0^{1,2}[0,1]$ that

$$(\pi^2+1)\int_0^1 (Dw(x))^2 dx = \pi^2\int_0^1 (Dw(x))^2 dx + \int_0^1 (Dw(x))^2 dx$$
$$\geq \pi^2 \int_0^1 (Dw(x))^2 dx + \pi^2 \int_0^1 (w(x))^2 dx = \pi^2 \|w\|_1^2 ,$$

i.e., c can be chosen to be $\pi^2(1+\pi^2)^{-1}$. Use of the Rayleigh-Ritz inequality again shows that Λ of (4.4) is π^2, and as $f(x,u) = e^u$ in this case, it also follows that the assumptions of (4.5) are fulfilled with $\gamma = 0$. Thus, Theorem 4.1 is applicable.

Continuing with the particular example of (4.12)-(4.13), consider a uniform mesh Δ: $0=x_0<x_1<\cdots<x_{N+1}=1$ on $[0,1]$ where $x_i = ih$, $0 \leq i \leq N+1$, and where $h = 1/(N+1)$. Let $\Lambda = \{\lambda_j\}_1^N$ be the class of continuous linear functionals over $W_0^{1,2}[a,b]$, defined by

$$\lambda_j(f) = f(x_j) , \qquad 1 \leq j \leq N .$$

These functionals are obviously linearly independent, and in this case, with $L \equiv D$, the Lg-spline space $Sp(D, \underline{\Lambda})$ of $W_0^{1,2}[0,1]$ is well-defined. To show the dependence on the partition Δ of $[0,1]$, we now write $Sp(D,\Delta)$ for $Sp(D,\Lambda)$. A convenient computational basis $\{t_i(x)\}_1^N$ for $Sp(D,\Delta)$ is defined by

$$(4.14) \qquad t_i(x) = \begin{cases} 1 - \dfrac{x-x_i}{h} , & x_i \leq x \leq x_{i+1} , \\ 1 + \dfrac{x-x_i}{h} , & x_{i-1} \leq x \leq x_i , \\ 0 , & x \notin [x_{i-1}, x_{i+1}] . \end{cases}$$

With this choice of basis for $Sp(D,\Delta)$, the $N \times N$ matrix A of (4.10) takes the familiar tridiagonal form

$$(4.15) \qquad A = \frac{1}{h}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix},$$

and the components of the vector $g(c)$ of (4.11') in this case are given by

$$(4.16) \qquad g_\ell(c) = \int_{x_{\ell-1}}^{x_{\ell+1}} e^{\sum_{i=1}^{N} c_i t_i(x)} t_\ell(x)\,dx, \quad 1 \leq \ell \leq N.$$

From Theorem 4.1, we know that there is a unique Galerkin solution $u_N(x) = \sum_{k=1}^{N} c_k t_k(x)$ of the matrix problem (4.10). Since $u_N(ih) = c_i$, it is interesting to note that if $g_\ell(c)$ of (4.16) is approximated by the trapezoidal rule on the intervals $[x_{\ell-1}, x_\ell]$ and $[x_\ell, x_{\ell+1}]$, i.e.,

$$g_\ell(c) \doteq he^{c_\ell} = he^{u_N(\ell h)},$$

then the associated nonlinear matrix problem is precisely the one which arises from the usual three-point central-difference approximations of (4.12)-(4.13). For further computational details, we refer to Herbold [28] and Herbold, Schultz, and Varga [29].

To show how the error bound of (4.9) can be utilized in this particular case, the solution u(x) of (4.12)-(4.13) is given explicitly (cf. [15]) by

$$(4.17) \qquad u(x) = -\ln 2 + 2 \ln \{c \sec[c(x - \tfrac{1}{2})/2]\}, \quad c \doteq 1.3360557,$$

which shows that $u(x) \in C^\infty[0,1]$, and, in particular, $u(x)$ is a classical solution of (4.12)-(4.13). Hence, if $\tilde{u}_N(x)$ is the unique $Sp(D, \Delta)$-interpolation of $u(x)$, then from Theorem 3.4, we know that there exist positive constants C_1 and C_2, independent of h, such that

$$(4.18)\qquad \begin{cases} \|u - \tilde{u}_N\|_{L^2[0,1]} \le C_1 h^2 , \\ \|D(u - \tilde{u}_N)\|_{L^2[0,1]} \le C_2 h , \end{cases}$$

and consequently,

$$(4.18')\qquad \|u - \tilde{u}_N\|_1 \le C'h ,$$

where C' is independent of h. Thus, by choosing $w_k(x) = \tilde{u}_N(x)$ in (4.9), we obtain

$$(4.19)\qquad \|u_N - u\|_{L^\infty[0,1]} \le K_1 \|u_N - u\|_1 \le C''h ,$$

where C'' is independent of h. In other words, the convergence of $u_N(x)$ to $u(x)$ as h tends to zero is assured. We further remark that the analysis above is valid also for nonuniform partitions Δ of $[0,1]$. The only change is that h is replaced by $\bar{\Delta}$ in (4.19).

The argument above connecting the error bounds of (4.9) with interpolation errors for Lg-spline interpolation can be easily extended to the general case. Specifically, for the boundary value problem (4.1)-(4.2), finite-dimensional subspaces B^k of $W_0^{n,2}[a,b]$ of the following form can be considered. Let $\Delta: a = x_0 < x_1 < \cdots < x_{N+1} = b$ be any partition of $[a,b]$, and let $z = (z_1, z_2, \ldots, z_N)$ be any vector with positive integer components z_i where $1 \le z_i \le m$. Let $\Lambda = \{\lambda_j^{(k)}\}$ now be the class of continuous linear functionals over $W_0^{n,2}[a,b]$, defined by

$$\lambda_j^{(k)}(f) = D^k f(x_j), \quad 0 \le k \le z_j - 1, \quad 1 \le j \le N .$$

Again, these functionals are linearly independent, and the Lg-spline space $Sp(L,\Lambda)$ of $W_0^{n,2}[a,b]$ is expressed as $Sp(L,\Delta,z)$, to show the dependence on Δ and z.

We now apply the L^2-interpolation results of Theorems 3.2 and 3.4 of §3. As in the example above, this gives rise to interpolation errors in the norm $\|\cdot\|_n$, and combining with the error bounds of (4.9) of Theorem 4.1 gives

Theorem 4.2. With the assumptions of (4.3) and (4.5), let $u(x)$ be the unique generalized solution of (4.1)-(4.2) in $W_0^{n,2}[a,b]$. For any partition Δ of $[a,b]$, and any incidence vector z, let $\hat{u}$ be the unique Galerkin approximation of $u(x)$ in $Sp(L,\Delta,z)$, where the order of L satisfies $m \geq n$. Then, there exist positive constants K_1 and K_2, independent of Δ and z, such that if $u(x) \in W^{t,2}[a,b]$ with $t \geq m$, then

$$\|D^i(\hat{u}-u)\|_{L^\infty[a,b]} \leq K_1\|\hat{u}-u\|_n \leq K_2(\Delta)^{m-n}\|Lu\|_{L^2[a,b]},$$

(4.20)

$$0 \leq i \leq n-1 .$$

Similarly, if $u(x) \in W^{t,2}[a,b]$ with $t \geq 2m$, there exist positive constants K_1 and K_2', independent of Δ and z, such that

$$(4.21) \quad \|D^i(\hat{u}-u)\|_{L^\infty[a,b]} \leq K_1\|\hat{u}-u\|_n \leq K_2'(\bar{\Delta})^{2m-n}\|L^*Lu\|_{L^2[a,b]}$$

for $0 \leq i \leq n-1$.

Because the error bounds of Theorems 3.2 and 3.4 hold for more general Lg-spline subspaces of $W^{n,2}[a,b]$, the results of Theorem 4.2 then improve the recent results of Varga [49].

Returning to the example of (4.12)-(4.13), we know that its solution $u(x)$ is of class $C^\infty[0,1]$, so that the error bound of (4.21) is always applicable. Consider then any Lg-spline space of the form $Sp(D^2,\Delta,z)$, where z is any incidence vector with positive integer components satisfying $1 \leq z_i \leq 2$, $1 \leq i \leq N$. If $\hat{z}_1 = \hat{z}_2 = \cdots = \hat{z}_N = 1$, then $Sp(D^2,\Delta,\hat{z})$ consists of natural cubic splines, while if $\bar{z}_1 = \bar{z}_2 = \cdots = \bar{z}_N = 2$, then $Sp(D^2,\Delta,\bar{z})$ consists of piecewise cubic polynomials of class $C^1[a,b]$, which is also known as the smooth cubic Hermite space $H^{(2)}(\Delta)$ (cf. [15] and [18]). For $Sp(D^2,\Delta,z)$, the error bound for the boundary value problem of (4.12)-(4.13) from (4.21) is

(4.22) $\|\hat{u}-u\|_{L^\infty[a,b]} \le K_1\|\hat{u}-u\|_1 \le K_2'(\bar{\Delta})^3\|L^*Lu\|_{L^2[a,b]}$.

Numerical results for (4.12)-(4.13) for the subspaces $Sp(D^2, \Delta, \bar{z})$ and $Sp(D^2, \Delta, \hat{z})$ are given respectively in Tables I and II (cf. [15], [28], and [49]). In all cases, $\Delta = \Delta(h)$ was chosen to be a uniform partition of $[0,1]$ with mesh size h.

h	dim	$\|u - \hat{u}_h\|_{L^\infty[0,1]}$
1/3	6	$3.13 \cdot 10^{-5}$
1/4	8	$1.03 \cdot 10^{-5}$
1/5	10	$4.40 \cdot 10^{-6}$
1/6	12	$2.17 \cdot 10^{-6}$
1/7	14	$1.19 \cdot 10^{-6}$
1/8	16	$7.15 \cdot 10^{-7}$

Table I - $Sp(D^2, \Delta(h), \bar{z})$, Smooth Cubic Hermite

The error bounds of (4.21) of Theorem 4.2 can be improved if

(i) the generalized solution $u(x)$ of (4.1)-(4.2) is smoother, say of class $W^{2m,2}[a,b]$ where $m = n + q$ and q is a nonnegative integer, and

(ii) appropriate Lg-spline subspaces of $W_0^{n,2}[a,b]$ are selected.

Specifically, suppose that we can express the differential operator M of (4.1) as

(4.23) $M[v(x)] = \ell^*\ell\,[v(x)] + \sum_{0\le i,j\le k} (-1)^j D^j(\tilde{\sigma}_{i,j}(x)\, D^i v(x))$

h	dim	$\|u - \hat{u}_h\|_{L^\infty[0,1]}$
1/4	5	$5.70 \cdot 10^{-6}$
1/5	6	$2.39 \cdot 10^{-6}$
1/6	7	$1.19 \cdot 10^{-6}$
1/7	8	$6.44 \cdot 10^{-7}$
1/8	9	$3.43 \cdot 10^{-7}$
1/9	10	$2.50 \cdot 10^{-7}$

Table II-$Sp(D^2, \Delta(h), \hat{z})$, Natural Cubic Splines

where $\tilde{\sigma}_{i,j}(x) \in C^i[a,b]$ for all $0 \leq i, j \leq k$, where $0 \leq k \leq n$ and

$$(4.24) \qquad \ell[v(x)] \equiv \sum_{j=0}^{n} \beta_j(x) D^j v(x) ,$$

where we assume that $\beta_j(x) \in C^j[a,b]$ for all $0 \leq j \leq n$, and that

$$\beta_n(x) \geq \omega > 0 \quad \text{for all} \quad x \in [a,b]$$

for some positive constant. In this case, we select the finite-dimensional subspaces $H_q(\ell, \Delta, z)$ of $W_0^{n,2}[a,b]$, which are defined as follows. The construction for this is similar to the recent work of Hulme [30].

Given the positive integers m and n with $m = n + q$, $q \geq 0$, let $\Delta: a = x_0 < x_1 < \cdots < x_{N+1} = b$ be any partition of $[a,b]$, and let $z = (z_0, z_1, \ldots, z_{N+1})$ be any associated incidence vector with integer components satisfying $1 \leq z_i \leq m+q$ for all $0 \leq i \leq N+1$. With $L \equiv \ell \cdot D^{2q}$, consider the Lg-spline space $Sp(L, \Delta, z)$. Then, let $\hat{S}p(L, \Delta, z)$ be defined as the subspace of $Sp(L, \Delta, z)$ of elements $s(x)$ which satisfy the particular boundary behavior

$$D^{\ell} s(a) = D^{\ell} s(b) = 0 \quad \text{for all } 0 \le \ell \le q-1 \text{ if } q \ge 1 \tag{4.25}$$
$$\text{and for all } 2q \le \ell \le n-1+2q .$$

Then, we define $H_q(\ell, \Delta, z)$ as the set of all real-valued functions $w(x)$ defined on $[a, b]$ such that

$$w(x) = D^{2q} s(x), \quad a \le x \le b, \quad \text{where } s(x) \in \hat{S}p(L, \Delta, z). \tag{4.26}$$

Because of the boundary conditions of (4.25) for $0 \le \ell \le q-1$ if $q \ge 1$, there is a one to one correspondence between elements of $H_q(\ell, \Delta, z)$ and $\hat{S}p(L, \Delta, z)$. Next, because of the boundary conditions of (4.25) for $2q \le \ell \le n-1+2q$, it follows from (4.26) that each element $w(x)$ of $H_q(\ell, \Delta, z)$ satisfies the boundary conditions of (4.2). Because each element $s(x)$ of $Sp(L, \Delta, z)$ is necessarily of class $W^{m+q, 2}[a, b]$, it follows that each $w(x)$ of $H_q(\ell, \Delta, z)$ is of class $W^{n, 2}[a,b]$, and from this, we deduce that $H_q(\ell, \Delta, z)$ is a finite-dimensional subspace of $W_0^{n, 2}[a, b]$. To give examples, let $\ell = D^n$, so that $L = D^{m+q}$. If we choose the components of the incidence vector z to satisfy $z_i = m + q$ for all $0 \le i \le N+1$, then each element $s(x)$ of $Sp(L, \Delta, z)$ is a piecewise polynomial of degree $2m + 2q - 1$, of continuity class $C^{n+2q-1}[a, b]$. Thus, each element $w(x)$ of $H_q(\ell, \Delta, z)$ is a piecewise polynomial of degree $2m - 1$, of continuity class $C^{n-1}[a, b]$. These subspaces of $W_0^{n, 2}[a, b]$ have been called the <u>nonsmooth Hermite spaces</u> $H(\Delta; n; 2m)$ (cf. [15, p. 413]). Similarly, if we choose $z_i = 1$, $1 \le i \le N$, then $H_q(D^n, \Delta, z)$ can be verified to be the <u>natural spline subspace</u> of piecewise polynomials of degree $2m - 1$, of continuity class $C^{2m-2}[a, b]$. We remark that the unusual interpolation property of splines at the knots x_i is <u>not</u> in general valid for the spaces $H_q(D^n, \Delta, z)$.

With these finite-dimensional subspaces $H_q(\ell, \Delta, z)$ of $W_0^{n, 2}[a, b]$, we state the following improved form of Theorem 4.2, which generalizes results of Perrin, Price, and Varga [38], and Varga [49].

<u>Theorem 4.3</u>. With the assumptions of (4.3), (4.5), and (4.23), assume that $u(x)$, the unique generalized solution of (4.1)-(4.2) in $W_0^{n, 2}[a, b]$, is of class $W^{2m, 2}[a, b]$

where $m = n + q$, $q \geq 0$. For any partition $\Delta: a = x_0 < x_1 < \cdots < x_{N+1} = b$ of $[a, b]$ and any associated incidence vector $z = (z_0, z_1, \ldots, z_{N+1})$ with $1 \leq z_i \leq m + q$ for $1 \leq i \leq N$, let $\hat{u}(x)$ denote the unique Galerkin approximation of $u(x)$ in $H_q(\ell, \Delta, z)$. Then, there exist positive constants K_1 and K_2, independent of Δ and z, such that

$$(4.27)\qquad \| D^i(\hat{u} - u) \|_{L^2[a,b]} \leq K_1(\bar{\Delta})^{2m-\max(\delta, i)}, \quad 0 \leq i \leq n$$

where $\delta \equiv \max\{2k-n; 0\}$, and

$$(4.28)\qquad \| D^i(\hat{u} - u) \|_{L^\infty[a,b]} \leq K_2(\bar{\Delta})^{2m-\max(\delta, i)-1/2}, \quad 0 \leq i \leq n-1 .$$

We remark that similar improved results can be proved for suitable subspaces, such as natural cubic and quintic splines, which improve the error bound in the uniform norm by <u>increasing</u> the exponent of $\bar{\Delta}$ in (4.28) by 1/2 (cf. [38, 49]).

To give an example of the previous result, consider the numerical approximation of the solution of

$$(4.29)\qquad D^4u(x) + u(x) + g(x) = 0, \qquad 0 < x < 1,$$

with boundary conditions

$$(4.30)\qquad u(0) = Du(0) = u(1) = Du(1) = 0,$$

where

$$(4.31)\qquad g(x) = \begin{cases} 6x^4 - 5x^3 + 144, & 0 \leq x \leq 1/2 \\ -945(2x-1)^{1/2} - (2x-1)^{9/2} - 5x^3 + 6x^4 + 144, & 1/2 \leq x \leq 1, \end{cases}$$

which has been considered by Perrin, Price, and Varga [38]. For this example, the quantity Λ of (4.4) is bounded below by π^4, and as $f(x, u)$ is linear in u, we see that the assumption of (4.5) is fulfilled, the inequalities of (4.5)(ii) being satisfied with $\gamma = 1$. It is also clear that the assumptions of (4.3) are fulfilled in this example. We remark that the unique solution $u(x)$ of (4.29)-(4.30) is of class

$C^4[0,1]$, but not of class $C^5[0,1]$.

For approximations of the solution $u(x)$ of (4.29)-(4.30), the Galerkin approximation $\hat{u}_h(x)$ of $u(x)$ was determined for the particular cubic Hermite subspace $Sp(D^2, \Delta(h), z)$ of $W_0^{2,2}[0,1]$, where $\Delta(h)$ denotes a uniform partition of $[0,1]$ with mesh size $h = 1/(N+1)$, and the components of the incidence vector z satisfy $z_1 = z_2 = \cdots = z_N = 2$. The numerical results are given in Table III. For purposes of comparison, the solution $u(x)$ of (4.29)-(4.30) was also approximated by finite differences on a uniform mesh. Here, a standard five-point central-difference approximation to (4.29) was used, and its discrete solution is denoted by w_i, $0 \leq i \leq N+1$. The numerical results are given in Table IV. In Tables III and IV, the computed exponent of $\bar{\Delta} = h$, generated from

$$\ln\left(\frac{\|\hat{u}_{h_2} - u\|_{L^\infty[0,1]}}{\|\hat{u}_{h_1} - u\|_{L^\infty[0,1]}}\right) / \ln(h_2/h_1),$$

is also given. Note the erratic behavior of the computed exponents in Table IV.

Thus far, we have considered here only nonlinear boundary value problems in one dimension with linear homogeneous boundary conditions (cf. (4.2)). The extension to nonlinear boundary conditions in one dimension has been treated in Ciarlet, Schultz, and Varga [16] where numerical results are also described. The extension to nonlinear boundary value problems in higher dimension can also be made, and and this is theoretically considered in Ciarlet, Schultz, and Varga [18], where the theory of monotone operators is used in conjunction with results from interpolation theory of Birkhoff, Schultz, and Varga [8] for piecewise Hermite polynomials. Also, numerical results based on Galerkin's methods for piecewise bicubic Hermite and spline functions in two dimensions for nonlinear second-order differential equations for rectangular domains are given in Herbold [28]. These extensions are basically in the spirit of the Galerkin method described here for one-dimensional problems, and will not be described further.

N	dim	$\lVert \hat{u}_h - u \rVert_{L^\infty}$	α	$\lVert D(\hat{u}_h - u) \rVert_{L^\infty}$	α'
3	6	$6.84 \cdot 10^{-3}$	----	$8.53 \cdot 10^{-2}$	----
4	8	$2.92 \cdot 10^{-3}$	3.81	$4.53 \cdot 10^{-2}$	2.84
6	12	$7.93 \cdot 10^{-4}$	3.88	$1.72 \cdot 10^{-2}$	2.88
8	16	$2.97 \cdot 10^{-4}$	3.86	$8.26 \cdot 10^{-3}$	2.91
10	20	$1.35 \cdot 10^{-4}$	3.95	$4.58 \cdot 10^{-3}$	2.94
16	32	$2.43 \cdot 10^{-5}$	3.94	$1.27 \cdot 10^{-3}$	2.95

Table III - $Sp_0(D^2, \Delta(h), z)$, Cubic Hermite

N	$\max_i \lvert w_i - u(ih) \rvert$	α
9	$3.25 \cdot 10^{-2}$	----
10	$4.10 \cdot 10^{-2}$	-2.30
16	$1.81 \cdot 10^{-2}$	1.88
19	$1.18 \cdot 10^{-2}$	2.60
30	$5.91 \cdot 10^{-3}$	1.58
39	$4.53 \cdot 10^{-3}$	1.06

Table IV - Finite Differences

5. Applications of Lg-splines to eigenvalue problems. As a second distinct application of Lg-splines, we consider next the eigenvalue problem

$$(5.1) \qquad \mathfrak{L}[u(x)] = \lambda \mathfrak{M}[u(x)], \qquad 0 < x < 1,$$

where

$$(5.1') \qquad \begin{cases} \mathfrak{L}[u(x)] \equiv \sum_{j=0}^{n} (-1)^j D^j (p_j(x) D^j u(x)), \\ \mathfrak{M}[u(x)] \equiv \sum_{j=0}^{r} (-1)^j D^j (q_j(x) D^j u(x)), \end{cases}$$

subject to the $2n$ linearly independent homogeneous boundary conditions of

$$(5.2) \qquad U_j[u(x)] = \sum_{k=1}^{2n} \{m_{j,k} D^{k-1} u(0) + n_{j,k} D^{k-1} u(1)\}, \qquad 1 \leq j \leq 2n,$$

or, as a special case,

$$(5.2') \qquad D^j u(0) = D^j u(1) = 0, \qquad 0 \leq j \leq n-1.$$

We assume that $0 \leq r < n$, and that the coefficient functions $p_j(x)$ and $q_k(x)$ are real-valued functions of class $C^j[0,1]$, $0 \leq j \leq n$, and class $C^k[0,1]$, $0 \leq k \leq r$, respectively, and in addition, we require that

$$(5.3) \qquad p_n(x) \text{ and } q_r(x) \text{ do not vanish on } [0,1].$$

Letting $\mathfrak{D}$ denote the set of real-valued functions in $C^{2n}[0,1]$ which satisfy (5.2), we assume that

$$(5.4)\quad \begin{cases} (\mathfrak{L}[u], v)_{L^2[0,1]} = (u, \mathfrak{L}[v])_{L^2[0,1]} & \text{for all } u, v \in \mathfrak{D}, \\ (\mathfrak{M}[u], v)_{L^2[0,1]} = (u, \mathfrak{M}[v])_{L^2[0,1]} & \text{for all } u, v \in \mathfrak{D}, \end{cases}$$

and that there exist positive constants K and d such that

$$(5.5)\quad (\mathfrak{L}[u], u)_{L^2[0,1]} \geq K(\mathfrak{M}[u], u)_{L^2[0,1]} \geq d(u, u)_{L^2[0,1]} \quad \text{for all } u \in \mathfrak{D}.$$

Defining the following inner products on $\mathfrak{D}$,

$$(5.6)\quad \begin{cases} (u, v)_D \equiv (\mathfrak{M}[u], v)_{L^2[0,1]} & \text{for all } u, v \in \mathfrak{D} \\ (u, v)_N \equiv (\mathfrak{L}[u], v)_{L^2[0,1]} & \text{for all } u, v \in \mathfrak{D}, \end{cases}$$

denote by H_D and H_N the Hilbert space completions of $\mathfrak{D}$ with respect to the norms $\|\cdot\|_D$ and $\|\cdot\|_N$, respectively. It is then well known (cf. [19, 20, 21]) that solving the eigenvalue problems (5.1)-(5.2) is equivalent to finding the extreme values and critical points of the Rayleigh quotient:

$$(5.7)\qquad R[w] \equiv \frac{\|w\|_N^2}{\|w\|_D^2}, \qquad w(x) \in H_N.$$

With the above assumptions, it is well known (cf. [11, 32]) that the eigenvalue problem of (5.1)-(5.2) has countably many eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ which are real, have no finite limit point, and can be arranged as

(5.8) $$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots .$$

Moreover, there is a corresponding sequence of eigenfunctions $\{\varphi_j(x)\}_{j=0}^{\infty}$ of (5.1)-(5.2) with $\varphi_j(x) \in \mathfrak{D}$, for which $\mathfrak{L}[\varphi_j] = \lambda_j \mathfrak{M}[\varphi_j]$. These eigenfunctions are orthonormal in the sense that

(5.9) $$(\varphi_i, \varphi_j)_D = \delta_{i,j} \quad \text{for all } i, j = 1, 2, \ldots ,$$

and the sequence $\{\varphi_j(x)\}_{j=1}^{\infty}$ is complete in H_D.

Now, let S_M be any finite-dimensional subspace of H_N, of dimension M, and let $\{w_i(x)\}_{i=1}^{M}$ be M linearly independent functions from the subspace. Thus, any function $w(x)$ in S_M can be written as

(5.10) $$w(x) = \sum_{i=1}^{M} u_i w_i(x) .$$

Instead of looking for the extremal points of the Rayleigh quotient $R[w]$ over the whole space H_N, the Rayleigh-Ritz procedure consists in looking for the extremal points of $R[w]$ over the subspace S_M. Equivalently, we now can view $R[w]$ as a Rayleigh quotient of a symmetric matrix defined over M-dimensional Euclidean space. More precisely, let

$$\mathfrak{R}[u] = \mathfrak{R}[u_1, u_2, \ldots, u_M] = R[w = \sum_{i=1}^{M} u_i w_i]$$

$$= \frac{\| \sum_{i=1}^{M} u_i w_i \|_N^2}{\| \sum_{i=1}^{M} u_i w_i \|_D^2} = \frac{\mathfrak{a}[u]}{\mathfrak{B}[u]} .$$

To find the stationary values of $\mathfrak{R}[u]$, we write

(5.12) $$\frac{\partial \mathfrak{a}[u]}{\partial u_i} = \lambda \frac{\partial \mathfrak{B}[u]}{\partial u_i} ,$$

which yields the matrix eigenvalue problem,

$$(5.13) \qquad A_M u = \lambda B_M u ,$$

where the $M \times M$ matrices $A_M = (\alpha_{i,j}^{(M)})$ and $B_M = (\beta_{i,j}^{(M)})$ have their entries given by

$$(5.14) \qquad \alpha_{i,j}^{(M)} = (w_i, w_j)_N , \quad \beta_{i,j}^{(M)} = (w_i, w_j)_D, \quad 1 \le i, j \le M .$$

It is clear, from the assumptions made in (5.4)-(5.5), that the matrices A_M and B_M are real, symmetric, and positive definite. Thus, the matrix eigenvalue problem (5.13) has M positive eigenvalues $0 < \hat{\lambda}_1 \le \hat{\lambda}_2 \le \cdots \le \hat{\lambda}_M$ and M corresponding linearly independent eigenvectors $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_M$. To each eigenvector $\hat{u}_k$, $1 \le k \le M$, we associate the function

$$(5.15) \qquad \hat{\varphi}_k(x) = \sum_{i=1}^{M} \hat{u}_{k,i} w_i(x) ,$$

where $\hat{u}_{k,i}$ is the i-th component of the vector $\hat{u}_k$, and henceforth we will call $\hat{\lambda}_k$ an <u>approximate eigenvalue</u> and $\hat{\varphi}_k(x)$ an <u>approximate eigenfunction</u> for (5.1)-(5.2).

To obtain error bounds for the approximate eigenvalues $\hat{\lambda}_k$, we make use of the following known result of Ciarlet, Schultz, and Varga [17] which extends results of Birkhoff, de Boor, Swartz, and Wendroff [7].

<u>Theorem 5.1.</u> With the assumptions of (5.4)-(5.5), let $\{\varphi_i(x)\}_{i=1}^k$ be the first k eigenfunctions of (5.1)-(5.2), orthonormalized in the sense that

$$(5.16) \qquad (\varphi_i, \varphi_j)_D = \delta_{i,j} , \quad 1 \le i, j \le k .$$

Let $\{\tilde{\varphi}_i(x)\}_{i=1}^k$ be any "globally approximating set of functions" to $\{\varphi_i(x)\}_{i=1}^k$ in H_N, in the sense that

$$(5.17) \qquad \sum_{i=1}^{k} \| \tilde{\varphi}_i - \varphi_i \|_D^2 < 1 .$$

Then, the functions $\{\tilde{\varphi}_i(x)\}_{i=1}^k$ are linearly independent, and if we define

$$(5.18) \qquad \epsilon_i(x) \equiv \tilde{\varphi}_i(x) - \varphi_i(x), \quad 1 \le i \le k,$$

then

$$(5.19) \qquad \lambda_j \le \tilde{\lambda}_j \le \lambda_j + \frac{\sum_{i=1}^{j} \|\epsilon_i\|_N^2}{\left(1 - \sqrt{\sum_{i=1}^{j} \|\epsilon_i\|_D^2}\right)^2} \quad \text{for all } 1 \le j \le k,$$

where

$$(5.20) \qquad \tilde{\lambda}_j \equiv \max_{c_1, c_2, \ldots, c_j} \frac{\|\sum_{i=1}^{j} c_i \tilde{\varphi}_i\|_N^2}{\|\sum_{i=1}^{j} c_i \tilde{\varphi}_i\|_D^2}, \qquad 1 \le j \le k,$$

is the j-th approximate eigenvalue for the finite-dimensional subspace of H_N spanned by $\{\tilde{\varphi}_i(x)\}_{i=1}^{j}$.

Similarly, to obtain error bounds for the approximate eigenfunctions $\hat{\varphi}_k(x)$, we again make use of a result of Ciarlet, Schultz, and Varga [17], which extends results of Birkhoff, de Boor, Swartz, and Wendroff [7].

<u>Theorem 5.2</u>. With the assumptions of (5.4)-(5.5), let $\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x)$ be the first k eigenfunctions of (5.1)-(5.2), where it is assumed that the corresponding eigenvalues λ_j satisfy $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$. If S_M is any finite-dimensional subspace of H_N with dim $S_M = M \ge k$, let $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_k$ and $\hat{\varphi}_1(x), \hat{\varphi}_2(x), \ldots, \hat{\varphi}_k(x)$ be the first k approximate eigenvalues and approximate eigenfunctions obtained by applying the Rayleigh-Ritz method for (5.1)-(5.2) to S_M. Then, there exists a constant C, dependent on k but independent of S_M, such that

$$(5.21) \qquad \|\hat{\varphi}_k - \varphi_k\|_N \le C \left\{ \sum_{j=1}^{k} (\hat{\lambda}_j - \lambda_j) \right\}^{1/2}.$$

If, in addition, there exists a positive constant K such that

$$(5.22) \qquad \|w\|_{L^\infty[0,1]} \le K \|w\|_N \quad \text{for all } w(x) \in H_N,$$

then

$$(5.23)\quad \|\hat{\varphi}_k - \varphi_k\|_{L^\infty[0,1]} \le K\|\hat{\varphi}_k - \varphi_k\|_N \le KC\{\sum_{j=1}^{k}(\hat{\lambda}_j - \lambda_j)\}^{1/2}.$$

To give a concrete illustration of the foregoing material in this section, suppose we consider the numerical approximation of the eigenvalues and eigenfunctions of the following second-order eigenvalue problem

$$(5.24)\qquad -D^2u(x) = \lambda u(x), \qquad 0 < x < 1,$$

with boundary conditions

$$(5.25)\qquad u(0) = u(1) = 0,$$

which is a special case of second-order eigenvalue problems considered by Birkhoff, de Boor, Swartz, and Wendroff [7], and and Wendroff [51]. For this particular problem, $\mathfrak{L} = -D^2$ and $\mathfrak{M} = I$, and consequently,

$$(\mathfrak{L}u, v)_{L^2[0,1]} = (u, \mathfrak{L}v)_{L^2[0,1]} = \int_0^1 Du(x)\cdot Dv(x)\,dx \quad \text{for all } u, v \in \mathfrak{D},$$

and

$$(\mathfrak{M}u, v)_{L^2[0,1]} = (u, \mathfrak{M}v)_{L^2[0,1]} = \int_0^1 u(x)\cdot v(x)\,dx \quad \text{for all } u, v \in \mathfrak{D}.$$

With the boundary conditions of (5.25), we have from the Rayleigh-Ritz inequality (cf. [27, p. 184]) that

$$(\mathfrak{L}u, u)_{L^2[0,1]} = \int_0^1 (Du(x))^2 dx \ge \pi^2 \int_0^1 (u(x))^2 dx = \pi^2(\mathfrak{M}u, u)_{L^2[0,1]}.$$

Thus, the inequalities of (5.5) are valid for $K = \pi^2$ and $d = 1$. For this example, the space H_N can be shown to be topologically equivalent to the Sobolev space $W_0^{1,2}[0,1]$.

Thus, by virtue of the Sobolev imbedding theorem in one dimension (cf. [53, p. 174]), we have that

$$\|u\|_{L^\infty[0,1]} \leq \frac{1}{2}\|u\|_N \quad \text{for all } u(x) \in H_N ,$$

so that the inequality of (5.22) is valid with $K = 1/2$. Since the eigenvalues of (5.24)-(5.25) are $\lambda_k = (k\pi)^2$, $k \geq 1$, and are thus distinct, all the assumptions of Theorems 5.1 and 5.2 are fulfilled.

Continuing with the particular example of (5.24)-(5.25), consider a uniform mesh Δ: $0 = x_0 < x_1 < x_2 < \cdots < x_{N+1} = 1$ on $[0,1]$ where $x_i = ih$, $0 \leq i \leq N+1$, and where $h = 1/(N+1)$. Choosing the particular Lg-spline space $Sp(D, \Delta, z)$ of $W_0^{1,2}[0,1]$ of continuous piecewise linear functions with knots at the x_i which vanish at $x = 0$ and $x = 1$, we use the basis $\{t_i(x)\}_{i=1}^N$ for $Sp(D, \Delta, z)$ defined in (4.14). For this choice of basis, the matrix eigenvalue problem of (5.13) is then $Ax = \lambda Bx$, where the $N \times N$ matrix A is explicitly given by (4.15), and the $N \times N$ matrix B is given by

$$(5.26) \qquad B = \frac{h}{6}\begin{bmatrix} 4 & 1 & & 0 \\ 1 & 4 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 4 \end{bmatrix}.$$

Since the matrices A and B in this case are both $N \times N$ real symmetric and positive definite tridiagonal matrices, the eigenvalues of $Ax = \lambda Bx$ can be accurately computed by Givens' method (cf. [52, p. 340]).

To show how the error bounds of Theorems 5.1 and 5.2 can be applied in this case, we know that the eigenfunctions $\varphi_j(x)$ are explicitly given by $\sin(j\pi x)$, $j \geq 1$, and are hence of of class $C^\infty[0,1]$. Consequently, fixing $k \geq 1$, we know that there exist positive constants K_1 and K_2 dependent on k, but independent of h, such that if $\tilde{\varphi}_i(x)$ is the $Sp(D,\Delta,z)$-interpolation of $\varphi_i(x)$ for $1 \leq i \leq k$, then

$$\|\tilde{\varphi}_i - \varphi_i\|_{L^2[0,1]} \le K_1 h^2, \qquad 1 \le i \le k,$$

and

$$\|D(\tilde{\varphi}_i - \varphi_i)\|_{L^2[0,1]} \le K_2 h, \qquad 1 \le i \le k.$$

Thus, as $\|\tilde{\varphi}_i - \varphi_i\|_D = \|\tilde{\varphi}_i - \varphi_i\|_{L^2[0,1]}$ and $\|\tilde{\varphi}_i - \varphi_i\|_N = \|D(\tilde{\varphi}_i - \varphi_i)\|_{L^2[0,1]}$ in this case, we have that

$$(5.27) \quad \left\|\sum_{i=1}^{k} (\tilde{\varphi}_i - \varphi_i)\right\|_D \le kK_1 h^2 < 1 \text{ for } h^2 < \frac{1}{kK_1}.$$

In other words, for h sufficiently small, the $Sp(D,\Delta,z)$-interpolates $\tilde{\varphi}_i(x)$ of $\varphi_i(x)$, $1 \le i \le k$, are globally approximating, and from (5.19) of Theorem 5.1, we obtain the result of Wendroff [51]:

$$(5.28) \quad \lambda_j \le \hat{\lambda}_j \le \lambda_j + \frac{jK_2^2 h^2}{\left(1 - \sqrt{j}\, K_1 h^2\right)^2} \le \lambda_j + M_1 h^2, \qquad 1 \le j \le k,$$

for all h sufficiently small, where M_1 is independent of h. With this inequality, it also follows from (5.23) of Theorem 5.2 that

$$(5.29) \quad \|\hat{\varphi}_j - \varphi_j\|_{L^\infty[0,1]} \le \frac{1}{2} \|\hat{\varphi}_j - \varphi_j\|_N \le M_2 h, \qquad 1 \le j \le k,$$

for all h sufficiently small, where M_2 is independent of h.

The argument above connecting the error bounds of (5.19) and (5.23) with interpolation errors for Lg-spline interpolation can be easily extended to the general case. For simplicity, we restrict our attention to the particular boundary conditions of (5.2'). If L is a differential operator (cf. (2.1)) of order $m \ge n$, and $\{\Delta_j\}_{j=1}^{\infty}$ is any sequence of partitions of $[0,1]$ where $\Delta_j: 0 = x_0^{(j)} < x_1^{(j)} < \cdots < x_{N_j+1}^{(j)} = 1$ such that $\lim_{j \to \infty} \bar{\Delta}_j = 0$, let $\{z_j\}_{j=1}^{\infty}$ be any sequence of associated incidence vectors, where $z_j = (z_1^{(j)}, z_2^{(j)}, \ldots, z_{N_j}^{(j)})$ and $1 \leq z_\ell^{(j)} \leq r$ for all $1 \leq \ell \leq N_j$ and all $j \geq 1$. Then, the Lg-spline space $Sp(L, \Delta_j, z_j)$ of functions which satisfy the boundary conditions of (5.2'), is a finite-dimensional subspace of the space H_N for all $j \geq 1$. Applying the Rayleigh-Ritz method to the subspace $Sp(L, \Delta_j, z_j)$ gives $\hat{\lambda}_{k,j}$ and $\hat{\varphi}_{k,j}(x)$ as approximations to λ_k and $\varphi_k(x)$, respectively. We now apply the L^2-interpolation results of Theorems 3.2 and 3.4 of §3, just as in the example above. Combining this with the error bounds of (5.19) of Theorem 5.1 and (5.23) of Theorem 5.2 results in

Theorem 5.3. With the assumptions of (5.4)-(5.5), let $\{\Delta_j\}_{j=1}^{\infty}$ be a sequence of partitions of $[0,1]$, with $\lim_{j \to \infty} \bar{\Delta}_j = 0$, let $\{z_j\}_{j=1}^{\infty}$ be a corresponding sequence of incidence vectors associated with $\{\Delta_j\}_{j=1}^{\infty}$, and let $\hat{\lambda}_{k,j}$ and $\hat{\varphi}_{k,j}(x)$ be the k-th approximate eigenvalue and the k-th approximate eigenfunction of (5.1)-(5.2'), obtained by applying the Rayleigh-Ritz method to the subspace $Sp(L, \Delta_j, z_j)$ of H_N. If the eigenfunctions $\{\varphi_i(x)\}_{i=1}^{k}$ of (5.1)-(5.2') are of class $W^{t,2}[0,1]$, with $t \geq 2m \geq 2n$, there exists a positive constant K_1, independent of j, and a positive integer j_0 such that

$$(5.30) \quad \lambda_k \leq \hat{\lambda}_{k,j} \leq \lambda_k + K_1(\bar{\Delta}_j)^{2(2m-n)} \quad \text{for all } j \geq j_0 .$$

Moreover, if the first k eigenvalues are simple, i.e., $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$, and the inequality of (5.22) is valid, there exists a positive constant K_2, independent of j, and a positive integer j_0 such that

$$(5.31) \quad \|\hat{\varphi}_{k,j} - \varphi_k\|_{L^\infty[a,b]} \leq K\|\hat{\varphi}_{k,j} - \varphi_k\|_N \leq K_2(\bar{\Delta}_j)^{2m-n} \quad \text{for all } j \geq j_0 .$$

We remark that since subspaces of cubic splines and cubic Hermite piecewise polynomial functions correspond to particular special choices of $Sp(D^2, \Delta, z)$, the result of Theorem 5.3 generalizes the results of Birkhoff, de Boor, Swartz and Wendroff [7], which correspond to the case $m = 2$ and $n = 1$ of Theorem 5.3, as well as the results of Wendroff [51], which correspond to the case $m = n = 1$ of Theorem 5.3.

Explicit calculations of eigenvalues by Birkhoff and de Boor [6], show the exponent of $\bar{\Delta}$ in (5.10) is best possible. The analogue of this for the inequality of (5.11) is similarly true for the eigenfunction approximation in the norm $\|\cdot\|_N$. However, in the norm $\|\cdot\|_{L^\infty[0,1]}$, the exponent of $\bar{\Delta}$ in (5.11) is not in general best possible, and can in fact be improved using particular Lg-spline techniques. Specifically, it has been shown by Pierce and Varga [39] for particular cases that the exponent of Δ_j in (5.11) can be increased to $2m$.

There are extensive numerical results (cf. [7]) for cubic splines and cubic Hermite subspaces as applied to the Mathieu equation. However, we shall now give complementary numerical results here for a simpler eigenvalue problem of (5.24)-(5.25) (cf. [17, 28]). If the quintic Hermite subspace $H_0^{(3)}(\Delta(h)) = Sp(D^3, \Delta(h), z)$, where $z_i = 3$, $1 \le i \le N$, and where $\Delta(h)$ denotes a uniform partition of $[0,1]$ is applied to the particular eigenvalue problem of (5.24)-(5.25), then the results of Theorem 5.3 are valid with $m = 3$ and $n = 1$, i.e., the exponent of Δ_j in (5.30) is 10. The numerical results are given in Table V.

We finally remark that computational results for second-order eigenvalue problems using Rayleigh-Ritz methods for subspaces other than Lg-splines are also given in Ciarlet, Schultz, and Varga [17] and Farrington, Gregory, and Taub [22]. Theoretical results for approximate eigenvalues by finite-difference methods, which are not in general Rayleigh-Ritz methods can be found in Gary [23], and Weinberger [50].

h	$\dim(H_0^{(3)}(\Delta(h)))$	$\hat{\lambda}_1(h)-\pi^2$	$\hat{\lambda}_2(h)-4\pi^2$	$\hat{\lambda}_3(h)-9\pi^2$	$\hat{\lambda}_4(h)-16\pi^2$
1/2	7	$1.27 \cdot 10^{-7}$	$1.65 \cdot 10^{-3}$	$3.51 \cdot 10^{-2}$	$3.83 \cdot 10^{-1}$
1/3	10	$3.66 \cdot 10^{-9}$	$1.18 \cdot 10^{-5}$	$5.98 \cdot 10^{-3}$	$3.59 \cdot 10^{-2}$
1/4	13	$2.42 \cdot 10^{-10}$	$9.96 \cdot 10^{-7}$	$1.18 \cdot 10^{-4}$	$1.32 \cdot 10^{-2}$
1/5	16	$7.41 \cdot 10^{-11}$	$9.53 \cdot 10^{-8}$	$1.62 \cdot 10^{-5}$	$5.06 \cdot 10^{-4}$

Table V - Quintic Hermite Subspaces $H^{(3)}(\Delta(h))$

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INSERTS

p. 105 Footnote to Title

* This research was supported in part by AEC Grant AT(11-1)-1702.

p. 117 line + 3

p. 205]), which holds, provided the derivatives through order m - 1, evaluated at the points a and b, are functionals in Λ.

p. 122 eq. (3.17)

, with the previous assumptions on Λ. These assumptions on Λ are assumed to hold in each of the next four theorems.

Theorem 3.3. ...