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On Higher-Order Numerical Methods for Nonlinear Two-Point Boundary Value Problems*

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§ 1. Introduction

As a special case of the nonlinear two-point boundary value problems considered in [5], consider

$$D^2 u(x) = f(x, u), \quad 0 < x < 1, \quad D \equiv \frac{d}{dx}, \tag{1.1}$$

subject to the boundary conditions of

$$u(0) = u(1) = 0, \tag{1.2}$$

where it is assumed that $f(x, u) \in C^0([0, 1] \times R)$, that there exists a constant γ such that

$$\begin{gathered} \frac{f(x, u_1) - f(x, u_2)}{u_1 - u_2} \geqq \gamma > -\pi^2 \quad \text{for all} \quad x \in [0, 1] \\ \text{and all} \quad -\infty < u_1, u_2 < +\infty \quad \text{with} \quad u_1 \neq u_2, \end{gathered} \tag{1.3}$$

and that for every positive constant c, there exists a finite constant $M(c)$ such that

$$\begin{gathered} \frac{f(x, u_1) - f(x, u_2)}{u_1 - u_2} \leqq M(c) \quad \text{for all} \quad x \in [0, 1], \\ |u_1| \leqq c, \quad |u_2| \leqq c, \quad u_1 \neq u_2. \end{gathered} \tag{1.3'}$$

By applying the Rayleigh-Ritz method for the variational form of (1.1)—(1.2) to the particular Hermite space $H_0^{(1)}(\Delta(h))$, i.e., the set of all continuous functions $w(x)$ defined on $[0, 1]$ satisfying (1.2) such that $w(x)$ is linear on each subinterval $[ih, (i+1)h]$ of $[0, 1]$, $0 \leqq i \leqq N$, where $h = 1/(N+1)$, one obtains a unique element $\hat{w}_h(x)$ in $H_0^{(1)}(\Delta(h))$ (cf. Theorem 1 of § 2 and [5, Theorem 2]). Assuming that the unique solution $\varphi(x)$ of (1.1)—(1.2) is of class $C^2[0, 1]$, it is known, as a special case [5, Theorem 10], that

$$\|\hat{w}_h - \varphi\|_{L^\infty} \equiv \max_{0 \leqq x \leqq 1} |\hat{w}_h(x) - \varphi(x)| \leqq C_1 h, \tag{1.4}$$

where C_1 is independent of h. Recently, CIARLET [4] has improved this specific result by showing that

$$\|w_h - \varphi\|_{L^\infty} \leqq C_2 h^2, \tag{1.5}$$

provided that γ of (1.3) satisfies $\gamma > -8$, and that

$$\|\hat{w}_h - \varphi\|_{L^\infty} \leqq C_3 h^{\frac{3}{2}} \tag{1.6}$$

when $-8 \geqq \gamma > -\pi^2$, where the constants C_2 and C_3 are independent of h.

In this paper, we show that the improvement of (1.4) in (1.5) and (1.6), given by CIARLET in [4], can be *widely* generalized to higher-order nonlinear two-point boundary value problems. Our approach, following that of [4], depends upon carefully estimating the difference between the interpolation, $\tilde{w}(x)$, and the approximation, $\hat{w}(x)$, of the solution function $\varphi(x)$ in some finite-dimensional space of L-spline functions (cf. §2), but, unlike the method of [4], our approach does not depend upon the Maximum Principle. As a special case of our Theorem 3, we show that the inequality of (1.5) is valid for *all* γ with $\gamma > -\pi^2$, and that the restriction to Hermite subspaces $H_0^{(1)}(\Delta(h))$ with uniform partitions of $[0, 1]$ in (1.5) can be dropped.

In §4, we show (Theorems 4 and 5) that these results can be further generalized when we assume that the solution $\varphi(x)$ possesses additional smoothness properties.

Finally, in §5 we illustrate these theoretical results with actual numerical computations for several two-point boundary value problems.

§ 2. Preliminaries

Consider the numerical approximation of the solution of the two-point boundary value problem

$$-L^* L[u(x)] = f(x, u), \quad a < x < b, \tag{2.1}$$

with homogeneous boundary conditions

$$D^j u(a) = D^j u(b) = 0, \quad 0 \leqq j \leqq n-1, \quad D \equiv \frac{d}{dx}, \tag{2.2}$$

where

$$L[v(x)] \equiv \sum_{j=0}^{n} a_j(x)\, D^j v(x), \quad a \leqq x \leqq b, \quad n \geqq 1, \tag{2.3}$$

for any $v(x) \in C^n[a, b]$, and L^* denotes its formal adjoint, i.e.,

$$L^*[v(x)] \equiv \sum_{j=0}^{n} (-1)^j D^j \big(a_j(x)\, v(x)\big), \tag{2.3'}$$

where we assume that the coefficient function $a_j(x)$ is in $C^j[a, b]$ for each $0 \leqq j \leqq n$. We also assume that there exists a positive real number ω such that

$$a_n(x) \geqq \omega > 0 \quad \text{for all} \quad x \in [a, b]. \tag{2.4}$$

The differential operator $L^* L$ is surely self-adjoint, and upon writing $L^* L$ as

$$L^* L[u(x)] = \sum_{j=0}^{n} (-1)^j D^j \big(p_j(x)\, D^j u(x)\big),$$

it follows from (2.4) that $p_n(x) = a_n^2(x) \geqq \omega^2 > 0$ for all $x \in [a, b]$. Hence, the operator $L^* L$ is by definition *strongly elliptic* (cf. [16, p. 176]).

We now introduce some standard notation. For m a positive integer, the Sobolev space $W^{m,2}[a, b]$ consists of all real-valued functions $f(x)$ defined on

$[a, b]$ such that f and its distributional derivatives $D^j f$ with $0 \leqq j \leqq m$ all belong to $L^2[a, b]$. By virtue of Sobolev's lemma [16, p. 174], $W^{m,2}[a, b]$ can also be described as the collection of all real-valued functions $w(x)$ defined on $[a, b]$ such that $w(x) \in C^{m-1}[a, b]$ and $D^{m-1}w(x)$ is absolutely continuous on $[a, b]$ with $D^m w(x) \in L^2[a, b]$, where we consider equivalence classes of functions which are equal almost everywhere on $[a, b]$. The Sobolev norm associated with any element of $W^{m,2}[a, b]$ is then defined by

$$\|w\|_m \equiv \left\{ \int_a^b \left[\sum_{j=0}^m \left((D^j w(x))^2 \right] dx \right\}^{\frac{1}{2}}, \quad w \in W^{m,2}[a, b], \tag{2.5}$$

and, for the special case $m = n$, $W_0^{n,2}[a, b]$ denotes the subspace of functions of $W^{n,2}[a, b]$ which satisfy the boundary conditions of (2.2).

Because the operator $L^* L$ is strongly elliptic, Gårding's inequality [16, p. 175] gives us that constants $K_1 > 0$ and $\beta \geqq 0$ necessarily exist such that

$$\|w\|_n^2 \leqq K_1 \int_a^b \{(L[w(t)])^2 + \beta [w(t)]^2\} dt \quad \text{for all} \quad w \in W_0^{n,2}[a, b]. \tag{2.6}$$

Next, with the boundary conditions of (2.2), the nonnegative quantity Λ, defined by

$$\Lambda \equiv \inf_{w \in W_0^{n,2}[a,b]} \frac{\int_a^b (L[w(t)])^2 dt}{\int_a^b (w(t))^2 dt}, \tag{2.7}$$

can be shown in fact to be *positive*. For the specific case that $L = D$ and $a = 0$ and $b = 1$, the Rayleigh-Ritz inequality [7, p. 184] directly gives us that $\Lambda = \pi^2$.

For the function $f(x, u)$ of (2.1), we assume that $f(x, u)$ is a real-valued function defined on $[a, b] \times R$, such that $f(x, u_0(x)) \in L^2[a, b]$ for all $u_0(x) \in W_0^{n,1}[a, b]$ and such that f satisfies

$$\left(\frac{f(x, u_1) - f(x, u_2)}{u_1 - u_2} \right) \geqq \gamma > -\Lambda \quad \text{for almost all} \quad x \in [a, b] \quad \text{and all} \quad u_1, u_2 \in R \quad \text{with} \quad u_1 \neq u_2, \tag{2.8}$$

where Λ is defined by (2.7). Finally, we assume that for each constant $c > 0$, there exists a positive constant $M(c)$ such that

$$\frac{f(x, u_1) - f(x, u_2)}{u_1 - u_2} \leqq M(c) \quad \text{for almost all} \quad x \in [a, b] \quad \text{and all } u_1, u_2 \in R \quad \text{with} \quad |u_1| \leqq c, \quad |u_2| \leqq c \quad \text{with} \quad u_1 \neq u_2. \tag{2.8'}$$

The preceding hypotheses are more than sufficient (cf. [6, Theorem 6.1]) to prove

Theorem 1. With the preceding assumptions, there exists a unique (generalized) solution $\varphi(x)$ over $W_0^{n,2}[a, b]$ to (2.1)—(2.2). Moreover, if S_M is any finite-dimensional subspace of $W_0^{n,2}[a, b]$, then there exists a unique $\hat{w}(x) \in S_M$ which minimizes the functional

$$F[w] = \int_a^b \left\{ \tfrac{1}{2} (L[w(t)])^2 + \int_0^{w(t)} f(t, \eta)\, d\eta \right\} dt, \quad w \in W_0^{n,2}[a, b], \tag{2.9}$$

over S_M, i.e.,

$$F[w] \geqq F[\hat{w}] \quad \text{for all} \quad w \in S_M \quad \text{with equality only for} \quad w = \hat{w} \in S_M.$$

If $\{w_i(x)\}_{i=1}^M$ is any basis for S_M, then $\hat{w}(x)$ is also characterized as the solution of

$$(2.10) \qquad \int_a^b \{L[\hat{w}(t)] \cdot L[w_i(t)] + f(t, \hat{w}(t))\, w_i(t)\}\, dt = 0, \qquad 1 \leqq i \leqq M.$$

Finally, there exists a constant K_2, independent of S_M, such that

$$(2.11) \qquad \|\hat{w} - \varphi\|_n \leqq K_2 \inf_{v \in S_M} \|v - \varphi\|_n.$$

Because of their use in later sections, we now give a brief description of the notations and basic results of [14] for "L-splines". Let $\Delta: a = x_0 < x_1 < \cdots < x_{N+1} = b$ denote a partition of $[a, b]$ with knots x_i, and let $\underline{z} = (z_1, z_2, \ldots, z_N)^T$ be an *incidence vector* with integer components satisfying $1 \leqq z_i \leqq n$.

Definition 1. The L-spline space $Sp(L, \Delta, \underline{z})$ is the collection of all real-valued functions $w(x)$ defined on $[a, b]$ such that

$$(2.12) \qquad \begin{array}{ll} \text{i)} & L^* L[w(x)] = 0 \quad \text{on } (x_i, x_{i+1}) \text{ for } \textit{each} \quad 0 \leqq i \leqq N, \\ & \text{where } L \text{ and } L^* \text{ satisfy } (2.3)-(2.4). \\ \text{ii)} & D^k w(x_i -) = D^k w(x_i +) \quad \text{for all} \quad 0 \leqq k \leqq 2n - 1 - z_i, \quad 1 \leqq i \leqq N. \end{array}$$

We remark that since $1 \leqq z_i \leqq n$, each element $w(x)$ of $Sp(L, \Delta, \underline{z})$ is necessarily of class $C^{n-1}[a, b]$. From this, it is readily verified that $Sp(L, \Delta, \underline{z})$ is a *finite-dimensional* subspace of $W_0^{n,2}[a, b]$. We denote by $Sp_0(L, \Delta, \underline{z})$ the subspace of $Sp(L, \Delta, \underline{z})$ whose elements satisfy (2.2).

If we define $z_0 = z_{N+1} = n$, then any element $g(x) \in W_0^{n,2}[a, b]$ possesses a *unique* interpolation (of Type I) in $Sp_0(L, \Delta, \underline{z})$ (cf. [14, Theorem 3]), i.e., there exists a unique $\tilde{w}(x) \in Sp_0(L, \Delta, \underline{z})$ such that

$$(2.13) \qquad D^j \tilde{w}(x_i) = D^j g(x_i) \quad \text{for all} \quad 0 \leqq j \leqq z_i - 1 \quad \text{and all} \quad 0 \leqq i \leqq N + 1.$$

If $\Delta: a = x_0 < x_1 < \cdots < x_{N+1} = b$ is a partition of $[a, b]$, then we define $\bar{\Delta} \equiv \max_{0 \leqq i \leqq N} (x_{i+1} - x_i)$, and $\underline{\Delta} \equiv \min_{0 \leqq i \leqq N} (x_{i+1} - x_i)$. The following error bounds for $g(x) - \tilde{w}(x)$ constitute a slight extension of the results of [14, Theorems 8 and 9].

Theorem 2. Given any $g(x) \in W^{2n,2}[a, b]$ which satisfies the boundary conditions of (2.2), choose any partition $\Delta: a = x_0 < x_1 < \cdots < x_{N+1} = b$ of $[a, b]$ and any incidence vector $\underline{z} = (z_0, z_1, \ldots, z_{N+1})^T$ where $z_0 = z_{N+1} = n$, and let $\tilde{w}(x)$ be the unique interpolation of $g(x)$ in $Sp_0(L, \Delta, \underline{z})$ in the sense of (2.13). Then, with the constants K_1, β, and Λ of (2.6) and (2.7),

$$(2.14) \qquad \|D^j(g - \tilde{w})\|_{L^2} \leqq K_1 \left(1 + \frac{\beta}{\Lambda}\right) \frac{(n!)^2}{j!} \left(\frac{\bar{\Delta}}{\pi}\right)^{n-j} \|L^* L[g]\|_{L^2}, \qquad 0 \leqq j \leqq n,$$

and

$$(2.15) \qquad \|D^j(g - \tilde{w})\|_{L^\infty} \leqq K_1 \left(1 + \frac{\beta}{\Lambda}\right) \frac{(n!)^2 (\bar{\Delta})^{2n - j - (\frac{1}{2})}}{\sqrt{n}\, (\pi)^n} \|L^* L[g]\|_{L^2}, \qquad 0 \leqq j \leqq n - 1.$$

Proof. From the definitions of (2.5) and (2.7), we have that $\|D^n w\|_{L^2}^2 \leqq \|w\|_n^2$ and $\|w\|_{L^2}^2 \leqq \frac{1}{\Delta} \|L[w]\|_{L^2}^2$. Hence, from the inequality of (2.6), we have

$$(2.16)\qquad \|D^n w\|_{L^2}^2 \leqq \|w\|_n^2 \leqq K_1\left(1+\frac{\beta}{\Delta}\right)\|L[w]\|_{L^2}^2 \quad \text{for } all \quad w \in W_0^{n,2}[a, b].$$

Thus, with $H^2 \equiv K_1\left(1+\frac{\beta}{\Delta}\right)$ and the fact that we have $z_0 = z_{N+1} = n$, we can directly apply the bounds derived in Theorems 8 and 9 of [14] to deduce the results of (2.14) and (2.15). Q.E.D.

Since the inequalities of (2.16) were obtained in [14] by means of a mesh restriction, i.e., $\bar{\Delta}$ was forced to be sufficiently small, then the results of (2.14) and (2.15) constitute an improvement of some results of [14]. We also remark that the exponents of $\bar{\Delta}$ in (2.14) and (2.15) are *best possible* in the sense that they cannot in general be increased for the class $W^{2n,2}[a, b]$ (cf. [14, Theorems 11 and 12]).

We conclude this section with a statement of Green's formula for the differential operator L of (2.3):

$$(2.17)\qquad \int_\alpha^\beta \{w(t)\, L[y(t)] - y(t)\, L^*[w(t)]\}\, dt = P\big(y(x), w(x)\big)\Big|_{x=\alpha}^{x=\beta}$$

for any $w(t), y(t) \in W^{n,2}[a, b]$ and any interval $[\alpha, \beta] \subset [a, b]$, and where

$$(2.18)\qquad P\big(y(x), w(x)\big) \equiv \sum_{j=0}^{n-1} D^{n-j-1} y(x) \sum_{k=0}^{j} (-1)^k D^k\{a_{n-j+k}(x)\, w(x)\}.$$

In §4, we shall make use of the analogue of the relation (2.17) for the differential operator $\mathscr{L} = L \cdot D^{2q}$, $q \geqq 0$, where L is defined in (2.3).

§ 3. High-Order Accuracy

We now focus our attention on the particular finite-dimensional subspace $Sp_0(L, \Delta, \underline{z})$ of $W_0^{n,2}[a, b]$. Choosing $S_M = Sp_0(L, \Delta, \underline{z})$ in Theorem 1, there exists a unique element, $\hat{w}(x)$, in $Sp_0(L, \Delta, \underline{z})$ which satisfies the equations of (2.10). The solution $\varphi(x)$ of (2.1)—(2.2) is, by Theorem 1, in $W_0^{n,2}[a, b]$ and satisfies (2.2). Hence, φ possesses a unique interpolation (in the sense of (2.13)), $\tilde{w}(x)$, in $Sp_0(L, \Delta, \underline{z})$, where $z_0 = z_{N+1} = n$. The main idea now is to compare the two functions $\tilde{w}(x)$ and $\hat{w}(x)$ in $Sp_0(L, \Delta, \underline{z})$. Following CIARLET [4], we define

$$(3.1)\qquad k_j \equiv \int_a^b \{L[\tilde{w}(t)] \cdot L[w_j(t)] + f\big(t, \tilde{w}(t)\big)\, w_j(t)\}\, dt, \qquad 1 \leqq j \leqq M,$$

where $\{w_j(t)\}_{j=1}^M$ is any basis for $Sp_0(L, \Delta, \underline{z})$. In analogy with (2.10), we deduce from Green's formula (2.17) that

$$(3.2)\qquad 0 = \int_a^b \{L[\varphi(t)] \cdot L[w_j(t)] + f\big(t, \varphi(t)\big)\, w_j(t)\}\, dt, \qquad 1 \leqq j \leqq M.$$

Hence, subtracting (3.2) from (3.1) gives

$$(3.3)\qquad k_j = \int_a^b \big\{L[\tilde{w}(t) - \varphi(t)] \cdot L[w_j(t)] + \big(f(t, \tilde{w}(t)) - f(t, \varphi(t))\big)\, w_j(t)\big\}\, dt, \qquad 1 \leqq j \leqq M.$$

We now show that the first integral of (3.3) is zero for all j. For any $v(t) \in Sp_0(L, \Delta, \underline{z})$, we can write, using Green's formula (2.17), that

$$\int_a^b L[\tilde{w}(t) - \varphi(t)] \cdot L[v(t)]\,dt = \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} L[\tilde{w}(t) - \varphi(t)] \cdot L[v(t)]\,dt$$
$$= \sum_{i=0}^{N} \left\{ \int_{x_i}^{x_{i+1}} (\tilde{w}(t) - \varphi(t)) \cdot L^* L[v(t)]\,dt + P\left(\tilde{w}(x) - \varphi(x), L[v(x)]\right)\Big|_{x=x_i}^{x=x_{i+1}} \right\}.$$

Because $v(t) \in Sp_0(L, \Delta, \underline{z})$, then $L^* L[v(t)] = 0$ on each subinterval (x_i, x_{i+1}) defined by Δ (cf. (2.12i)), and hence the integrals of the last sum vanish. The last sum, involving $P\left(\tilde{w}(x) - \varphi(x), L[v(x)]\right)\Big|_{x=x_i}^{x=x_{i+1}}$, also vanishes when one takes into account the continuity properties of $\tilde{w}(x)$ in $Sp_0(L, \Delta, \underline{z})$ (cf. (2.12ii)), and the interpolation properties of $\tilde{w}(x)$ with respect to $\varphi(x)$ (cf. (2.13)). Put differently, $\tilde{w}(t)$ is the *orthogonal projection* (cf. [3]) of $\varphi(t)$ onto $Sp_0(L, \Delta, \underline{z})$ with respect to the inner product on $W_0^{n,2}[a, b]$ defined by

$$\langle w, v \rangle \equiv \int_a^b L[w(t)] \cdot L[v(t)]\,dt, \qquad w, v \in W_0^{n,2}[a, b].$$

If we define the function $p(t)$ in $[a, b]$ through

$$(3.4) \qquad \begin{cases} p(t)\{\tilde{w}(t) - \varphi(t)\} = f(t, \tilde{w}(t)) - f(t, \varphi(t)), & \tilde{w}(t) \neq \varphi(t), \\ p(t) = \gamma, & \tilde{w}(t) = \varphi(t), \end{cases}$$

then we have shown that the quantity k_j of (3.3) reduces to

$$(3.5) \qquad k_j = \int_a^b p(t)\left(\tilde{w}(t) - \varphi(t)\right) w_j(t)\,dt, \qquad 1 \leqq j \leqq M.$$

Similarly, if we subtract (2.10) from (3.1), we obtain

$$(3.6) \qquad k_j = \int_a^b \{L[\tilde{w}(t) - \hat{w}(t)] \cdot L[w_j(t)] + q(t)\left(\tilde{w}(t) - \hat{w}(t)\right) \cdot w_j(t)\}\,dt, \quad 1 \leqq j \leqq M,$$

where the function $q(t)$ is defined in $[a, b]$ through

$$(3.7) \qquad \begin{cases} q(t)\left(\tilde{w}(t) - \hat{w}(t)\right) = f(t, \tilde{w}(t)) - f(t, \hat{w}(t)), & \tilde{w}(t) \neq \hat{w}(t), \\ q(t) = \gamma, & \tilde{w}(t) = \hat{w}(t), \end{cases}$$

We now obtain bounds for the functions $p(t)$ of (3.4) and $q(t)$ of (3.7). From [5, Lemma 4], an a priori bound for $\|\varphi\|_{L^\infty}$ (valid also for $\|\hat{w}\|_{L^\infty}$) can be determined, which is valid for any choice of subspace S_M of $W_0^{n,2}[a, b]$. Since $\bar{\Delta} \leqq b - a$, it follows that the case $j = 0$ of (2.15) gives an a priori bound for $\|\varphi - \tilde{w}\|_{L^\infty}$ which is independent of the partitioning Δ and the incidence vector $\underline{z}$. Thus, as $\|\tilde{w}\|_{L^\infty} \leqq \|\tilde{w} - \varphi\|_{L^\infty} + \|\varphi\|_{L^\infty}$, an a priori bound for $\|\tilde{w}\|_{L^\infty}$ can be determined which is valid for all subspaces $Sp_0(L, \Delta, \underline{z})$. Now, using the hypotheses of (2.8) and (2.8′), it follows that a constant Γ exists such that

$$(3.8) \qquad -\Lambda < \gamma \leqq p(t),\ q(t) \leqq \Gamma \quad \text{for all} \quad a \leqq t \leqq b,$$

and all choices of subspaces $Sp_0(L, \Delta, \underline{z})$ of $W_0^{n,2}[a, b]$.

We now make use of the fact that $\tilde{w}(t)$ and $\hat{w}(t)$ are both elements of $Sp_0(L, \Delta, \underline{z})$. Writing

$$\tilde{w}(t) = \sum_{j=1}^{M} \tilde{\alpha}_j w_j(t) \quad \text{and} \quad \hat{w}(t) = \sum_{j=1}^{M} \hat{\alpha}_j w_j(t),$$

then multiplying by $\tilde{\alpha}_j - \hat{\alpha}_j$ in (3.5) and summing on j gives

$$\sum_{j=1}^{M} (\tilde{\alpha}_j - \hat{\alpha}_j) k_j = \int_a^b p(t) \left(\tilde{w}(t) - \varphi(t)\right) \left(\tilde{w}(t) - \hat{w}(t)\right) dt. \tag{3.9}$$

The same method applied to (3.6) similarly gives

$$\sum_{j=1}^{M} (\tilde{\alpha}_j - \hat{\alpha}_j) k_j = \int_a^b \left\{\left(L[\tilde{w}(t) - \hat{w}(t)]\right)^2 + q(t) \left(\tilde{w}(t) - \hat{w}(t)\right)^2\right\} dt. \tag{3.10}$$

We now remark, as in [5, Corollary 2], that the inequality of (2.6) is valid also for any $\beta' > -\Lambda$. In particular, with the constant γ of (2.8), this means that there is a positive constant K_3 such that

$$\|w\|_n^2 \leqq K_3 \int_a^b \{(L[w(t)])^2 + \gamma (w(t))^2\} dt \quad \text{for all} \quad w \in W_0^{n,2}[a, b]. \tag{3.11}$$

Hence, using the lower bound for $q(t)$ from (3.8), then (3.10) and (3.11) combine to give

$$\sum_{j=1}^{M} (\tilde{\alpha}_j - \hat{\alpha}_j) k_j \geqq \frac{\|\tilde{w} - \hat{w}\|_n^2}{K_3}. \tag{3.12}$$

On the other hand, the integral of (3.9) can be bounded above from (3.8) and Schwarz's inequality by

$$\sum_{j=1}^{M} (\tilde{\alpha}_j - \hat{\alpha}_j) k_j \leqq \Gamma' \|\tilde{w} - \varphi\|_{L^2} \cdot \|\tilde{w} - \hat{w}\|_{L^2}, \quad \text{where} \quad \Gamma' \equiv \max(|\Gamma|, |\gamma|), \tag{3.13}$$

and as $\|\tilde{w} - \hat{w}\|_{L^2} \leqq \|\tilde{w} - \hat{w}\|_n$ by definition, the inequalities of (3.12) and (3.13) combine to give

$$\|\tilde{w} - \hat{w}\|_n \leqq \Gamma' K_3 \|\tilde{w} - \varphi\|_{L^2}. \tag{3.14}$$

If $\varphi \in W^{2n,2}[a, b]$ and satisfies the boundary conditions of (2.2), we can then find an upper bound for $\|\tilde{w} - \varphi\|_{L^2}$ from the special case $j = 0$ of (2.14) of Theorem 2. Thus, there exists a constant K_4, independent of the choice of Δ and $\underline{z}$ in $Sp_0(L, \Delta, \underline{z})$, such that

$$\|\tilde{w} - \hat{w}\|_n \leqq K_4 (\bar{\Delta})^{2n} \|L^* L[\varphi]\|_{L^2}. \tag{3.15}$$

Then, using the triangle inequality and the inequalities of (3.15) and (2.14), we evidently have

$$\begin{aligned}
\|D^j(\hat{w} - \varphi)\|_{L^2} &\leqq \|D^j(\hat{w} - \tilde{w})\|_{L^2} + \|D^j(\tilde{w} - \varphi)\|_{L^2} \leqq \|\tilde{w} - \hat{w}\|_n + \|D^j(\tilde{w} - \varphi)\|_{L^2} \\
&\leqq \left\{K_4 (\bar{\Delta})^{2n} + K_1 \left(1 + \frac{\beta}{\Lambda}\right) \frac{(n!)^2}{j!} \left(\frac{\bar{\Delta}}{\pi}\right)^{2n-j}\right\} \|L^* L[\varphi]\|_{L^2} \\
&\leqq K_5 (\bar{\Delta})^{2n-j} \|L^* L[\varphi]\|_{L^2}, \qquad 0 \leqq j \leqq n.
\end{aligned}$$

For bounds in the uniform norm, we use the fact that, from Sobolev's lemma in one dimension [16, p. 174], $\|D^j(\tilde{w} - \hat{w})\|_{L^\infty} \leqq K_6 \|\tilde{w} - \hat{w}\|_n$ for all $0 \leqq j \leqq n-1$. Thus, using the triangle inequality and the inequalities of (3.15) and (2.15), we similarly obtain

$$\begin{aligned}
\|D^j(\hat{w} - \varphi)\|_{L^\infty} &\leqq \|D^j(\hat{w} - \tilde{w})\|_{L^\infty} + \|D^j(\tilde{w} - \varphi)\|_{L^\infty} \\
&\leqq \left\{K_4 K_6 (\bar{\Delta})^{2n} + K_1 \left(1 + \frac{\beta}{\Lambda}\right) \frac{(n!)^2}{\sqrt{n}\, \pi^n} (\bar{\Delta})^{2n-j-(\frac{1}{2})}\right\} \|L^* L[\varphi]\|_{L^2} \\
&\leqq K_7 (\bar{\Delta})^{2n-j-(\frac{1}{2})} \|L^* L[\varphi]\|_{L^2}, \qquad 0 \leqq j \leqq n-1.
\end{aligned}$$

We state these results as

Theorem 3. Given any partition $\Delta: a=x_0<x_1<\cdots<x_{N+1}=b$ of $[a,b]$ and any incidence vector $\underline{z}=(z_0, z_1, \ldots, z_{N+1})^T$ where $z_0=z_{N+1}=n$, let $\hat{w}(x)$ be the unique function in the L-spline space $Sp_0(L, \Delta, \underline{z})$ which minimizes the functional $F[w]$ of (2.9) over $Sp_0(L, \Delta, \underline{z})$. If $\varphi(x)$, the generalized solution of (2.1)—(2.2) is of class $W^{2n,2}[a,b]$, then there exist constants K_5 and K_7, independent of Δ and $\underline{z}$, such that

$$\|D^j(\hat{w}-\varphi)\|_{L^2} \leqq K_5(\bar{\Delta})^{2n-j}\|L^*L[\varphi]\|_{L^2} \qquad \text{for all} \quad 0\leqq j\leqq n, \tag{3.16}$$

and

$$\|D^j(\hat{w}-\varphi)\|_{L^\infty} \leqq K_7(\bar{\Delta})^{2n-j-(\frac{1}{2})}\|L^*L[\varphi]\|_{L^2} \qquad \text{for all} \quad 0\leqq j\leqq n-1. \tag{3.17}$$

If, moreover, $\varphi(x)$ is in $C^{2n}[a,b]$, and $\mathscr{F}$ is a collection of L-spline spaces $Sp_0(L, \Delta, \underline{z})$ such that $\tilde{w}(x)$, the unique $Sp_0(L, \Delta, \underline{z})$-interpolate of $\varphi(x)$ (in the sense of (2.13)), satisfies

$$\|D^j(\tilde{w}-\varphi)\|_{L^\infty} \leqq K'(\bar{\Delta})^{2n-j}\|L^*L[\varphi]\|_{L^\infty} \qquad \text{for all} \quad 0\leqq j\leqq n-1 \quad \text{and all} \quad Sp_0(L,\Delta,\underline{z})\in\mathscr{F} \tag{3.18}$$

for some constant K', then

$$\|D^j(\hat{w}-\varphi)\|_{L^\infty} \leqq K_8(\bar{\Delta})^{2n-j}\|L^*L[\varphi]\|_{L^\infty} \qquad \text{for all} \quad 0\leqq j\leqq n-1 \quad \text{and all} \quad Sp_0(L,\Delta,\underline{z})\in\mathscr{F}, \tag{3.19}$$

where K_8 is a positive constant.

We remark that the inequality of (3.19) is established in the manner of that for (3.17). For the Hermite spaces[1] $H_0^{(n)}(\Delta)$, which correspond to the special case $L=D^n$ and $\underline{z}=(n, n, \ldots, n)^T$ in $Sp_0(L, \Delta, \underline{z})$, the inequality of (3.18) is known [2, Theorem 2] to be valid for all $n\geqq 1$ and any partition Δ of $[a,b]$. In this case, the collection $\mathscr{F}$ of Theorem 3 can be chosen to be all Hermite spaces $H^{(n)}(\Delta)$, where Δ is an *arbitrary* partitioning of $[a,b]$. Other applications of the second part of Theorem 3 are also possible. For example, if we choose $\mathscr{F}$ to be the collection of all *cubic* spline subspaces $Sp_0(L, \Delta, \underline{z})$, i.e., where $L=D^2$ and the components of $\underline{z}$ satisfy $z_i=1$, $1\leqq i\leqq N$, subject to the restriction that there is a positive number σ such that $\bar{\Delta}\leqq\sigma\underline{\Delta}$ for *all* partitions Δ in $\mathscr{F}$, then the inequality of (3.18) is known to be valid (cf. [1] and [13]).

From the results of Theorem 2, we know that the results of (3.16) and (3.17) are *best possible* in the sense that the exponents of $\bar{\Delta}$ in these inequalities cannot in general be improved for the class $W^{2n,2}[a,b]$. Similarly, if we consider the collection $\mathscr{F}$ of all Hermite spaces $H^{(n)}(\Delta)$, where Δ is an arbitrary partition of $[a,b]$, then it is known [2, Theorem 3] that the exponent of $\bar{\Delta}$ in (3.18) cannot in general be improved for the class $C^{2n}[a,b]$, and thus the exponent of $\bar{\Delta}$ in (3.19) is also best possible.

The result of Theorem 3, which will be generalized in the next section, itself constitutes an improvement over analogous published results for the numerical approximation of the solution of the two-point boundary value problem of (2.1) to (2.2). In [5, Theorem 10], it was shown that $\hat{w}_\Delta(x)$, the unique function in the Hermite space $H_0^{(n)}(\Delta)$ which minimizes the functional of (2.9) over $H_0^{(n)}(\Delta)$, satisfies

$$\|\hat{w}_\Delta-\varphi\|_{L^\infty}\leqq K(\bar{\Delta})^n,$$

1. The case of high order accuracy in the uniform norm for the specific Hermite space $H_0^{(n)}(\Delta)$ is treated in detail in [11]. Computational aspects are also included.

assuming that $\varphi(x)$, the solution of (2.1)—(2.2), is of class $C^{2n}[a, b]$. But from (3.19) for the case $j=0$, we see that the exponent of $\bar{\Delta}$ in this inequality can be *increased* to $2n$. Next, for the special case $n=1$, $a=0$, and $b=1$ of (2.1)—(2.2), we have that the quantity Λ defined in (2.7) is π^2. In this case, (3.19) of Theorem 3 gives us that

$$(3.20) \qquad \|\hat{w}_\Delta - \varphi\|_{L^\infty} \leqq K_8(\bar{\Delta})^2, \quad \hat{w}_\Delta \in H_0^{(1)}(\Delta),$$

under the assumption (2.8) that $\gamma > -\pi^2$ and that $\varphi(x) \in C^2[a, b]$, which improves the recent results of CIARLET [4]. Note also that the inequality of (3.20) is valid for *all* partitions of $[a, b]$, which further improves the results of [4].

In the special case that f of (2.1) is *independent* of u, the result of Theorem 3 can be strengthened. Since f is independent of u, it follows from the definition in (3.4) that $p(t) \equiv 0$, in $[a, b]$. Thus, from (3.9) and (3.12), we deduce that $\|\tilde{w} - \hat{w}\|_n = 0$, i.e., $\tilde{w}(x) \equiv \hat{w}(x)$ in $[a, b]$, and consequently, $\tilde{w}(x)$ is the *interpolation* of $\varphi(x)$ in $Sp_0(L, \Delta, \underline{z})$ (cf. (2.13)). This gives us the

Corollary. Given any partition $\Delta: a = x_0 < x_1 < \cdots < x_{N+1} = b$ of $[a, b]$ and any incidence vector $\underline{z} = (z_0, z_1, \ldots, z_{N+1})^T$ where $z_0 = z_{N+1} = n$, let f of (2.1) be independent of u, and let $\hat{w}(x)$ be the unique function in $Sp_0(L, \Delta, \underline{z})$ which minimizes the functional of (2.9) over $Sp_0(L, \Delta, \underline{z})$. Then, in addition to the inequalities of (3.16), (3.17), and (3.19), we have that

$$(3.21) \qquad D^j \hat{w}(x_i) = D^j \varphi(x_i) \quad \text{for all} \quad 0 \leqq j \leqq z_i - 1 \quad \text{and all} \quad 0 \leqq i \leqq N+1.$$

We remark that the interpolation result of (3.21) of the Corollary of Theorem 1, which was proved independently by HULME [9], is a generalization of a result of ROSE [12] for the case $n=1$.

If we consider in particular partitions Δ of $[a, b]$ such that $\bar{\Delta} \leqq \sigma \underline{\Delta}$ for a fixed $\sigma > 0$, then we know [14, Theorems 10 and 13] that the interpolation $\tilde{w}(x)$ in $Sp_0(L, \Delta, \underline{z})$ of $\varphi(x)$ in $W^{2n,2}[a, b]$ satisfies the following strengthened forms of (2.14) and (2.15):

$$(3.22) \qquad \|D^j(\tilde{w} - \varphi)\|_{L^2} \leqq K_5'(\bar{\Delta})^{2n-j} \qquad \text{for all} \quad 0 \leqq j \leqq 2n-1,$$

and

$$(3.23) \qquad \|D^j(\tilde{w} - \varphi)\|_{L^\infty} \leqq K_7'(\bar{\Delta})^{2n-j-(\frac{1}{2})} \quad \text{for all} \quad 0 \leqq j \leqq 2n-1,$$

i.e., error bounds for higher derivatives of $\tilde{w}(x) - \varphi(x)$ are available. This means that since $\hat{w}(x) = \tilde{w}(x)$ in the linear case, the above inequalities of (3.22) and (3.23) are then valid for $\hat{w}(x) - \varphi(x)$ for such partitions, which strengthens the result of the previous Corollary.

We finally remark that there are known *regularity theorems* (cf. [10, Chapter 4]), giving sufficient conditions on the smoothness of the coefficient functions in (2.1), which guarantee that the unique generalized solution $\varphi(x)$ of (2.1)—(2.2) is of class $W^{2n,2}[a, b]$. In particular, $a_i(x) \in C^n(a, b) \cap W^{n,2}[a, b]$ for all $0 \leqq i \leqq n$ is such a sufficient condition.

§ 4. Higher-Order Accuracies

The results of the previous section assumed that $\varphi(x)$, the solution of (2.1) to (2.2), was either of class $W^{2n,2}[a, b]$ or $C^{2n}[a, b]$. As in [5], one expects that more accurate Rayleigh-Ritz approximations of $\varphi(x)$ are possible if $\varphi(x)$ is smoother, i.e., if $\varphi(x) \in W^{2m,2}[a, b]$ or $\varphi(x) \in C^{2m}[a, b]$, where $m \geqq n$. We shall prove in this section a result, Theorem 4, which generalizes the results of Theorem 3, under the assumption that $\varphi(x)$, the unique solution of (2.1)—(2.2), is in $W^{2m,2}[a, b]$ or in $C^{2m}[a, b]$, where $m = n + q$, $q \geqq 0$. The construction for this result is similar to the recent work of HULME [9].

Given positive integers m and n with $m = n + q$, $q \geqq 0$, let $\Delta: a = x_0 < x_1 < \cdots < x_{N+1} = b$ be any partition of $[a, b]$, and let $\underline{z} = (z_0, z_1, \ldots, z_{N+1})^T$ be any associated incidence vector with integer components satisfying $1 \leqq z_i \leqq m + q$ for all $0 \leqq i \leqq N + 1$. From the n-th order differential operator L of (2.3), we now consider the $\mathscr{L}$-spline space $Sp(\mathscr{L}, \Delta, \underline{z})$ where $\mathscr{L} \equiv L D^{2q}$, whose elements are defined through (2.12) of Definition 1 of §2. As in §2, we fix the components $z_0 = z_{N+1} = m + q$ (corresponding to Type I interpolation in [14]). It is easy to see that there are z_i basis functions for the $\mathscr{L}$-spline space $Sp(\mathscr{L}, \Delta, \underline{z})$ associated with each knot x_i of Δ. Thus, it follows that

$$\dim Sp(\mathscr{L}, \Delta, \underline{z}) = \tau \equiv 2(m + q) + \sum_{i=1}^{N} z_i .$$

Now, let $\hat{S}p(\mathscr{L}, \Delta, \underline{z})$ be defined as the subspace of $Sp(\mathscr{L}, \Delta, \underline{z})$ of elements with the particular boundary behavior

$$(4.1) \qquad \begin{aligned} D^\ell s(a) = D^\ell s(b) = 0 \quad &\text{for all} \quad 0 \leqq \ell \leqq q - 1 \quad \text{if} \quad q \geqq 1 \\ &\text{and for all} \quad 2q \leqq \ell \leqq n - 1 + 2q . \end{aligned}$$

Evidently,

$$\dim \hat{S}p(\mathscr{L}, \Delta, \underline{z}) = \tau - 2m .$$

Next, let $H(L, \Delta, \underline{z})$ be the set of all real-valued functions $w(x)$ defined on $[a, b]$ such that

$$(4.2) \qquad w(x) = D^{2q} s(x), \quad a \leqq x \leqq b, \quad \text{where} \quad s(x) \in \hat{S}p(\mathscr{L}, \Delta, \underline{z}) .$$

Because of the boundary conditions of (4.1) for $0 \leqq \ell \leqq q - 1$ if $q \geqq 1$, there is a 1 to 1 correspondence between elements of $H(L, \Delta, \underline{z})$ and elements of $\hat{S}p(\mathscr{L}, \Delta, \underline{z})$, and hence

$$(4.3) \qquad \dim H(L, \Delta, \underline{z}) = \tau - 2m .$$

Next, because of the boundary conditions of (4.1) for $2q \leqq \ell \leqq n - 1 + 2q$, it follows from (4.2) that each element $w(x)$ of $H(L, \Delta, \underline{z})$ satisfies $D^\ell w(a) = D^\ell w(b) = 0$ for $0 \leqq \ell \leqq n - 1$. Because each element $s(x)$ of $\hat{S}p(\mathscr{L}, \Delta, \underline{z})$ is necessarily of class $W^{m+q,2}[a, b]$, it follows that each $w(x)$ of $H(L, \Delta, \underline{z})$ is of class $W^{n,2}[a, b]$, and from this, we conclude that $H(L, \Delta, \underline{z})$ is evidently *a finite-dimensional subspace* of $W_0^{n,2}[a, b]$. To give examples of this, let $L = D^n$, so that $\mathscr{L} = D^{m+q}$. If we choose the components of the incidence vector $\underline{z}$ to satisfy $z_i = m + q$ for all $0 \leqq i \leqq N + 1$, then $H(L, \Delta, \underline{z})$ in this case can be verified to be the *nonsmooth* Hermite space $H(\Delta; n; 2m)$, considered in [5, p. 413]. Similarly, if we choose $z_i = 1$, $1 \leqq i \leqq N$, then $H(L, \Delta, \underline{z})$ in this case can be verified to be the natural spline subspace $Sp^{(m)}(\Delta)$, considered in [5].

Assume now that the solution $\varphi(x)$ of (2.1)—(2.2) is of class $W^{2m,2}[a, b]$, where $m = n + q$, $q \geqq 0$. Then, let $\psi(x)$ be the unique real-valued function defined on $[a, b]$ such that

$$D^{2q}\psi(x) = \varphi(x), \quad a \leqq x \leqq b, \tag{4.4}$$

and

$$D^{\ell}\psi(a) = D^{\ell}\psi(b) = 0 \quad \text{for all} \quad 0 \leqq \ell \leqq q - 1 \quad \text{if} \quad q \geqq 1. \tag{4.5}$$

From (4.4), it follows that $\psi(x) \in W^{2m+2q,2}[a, b]$. Next, let $\tilde{s}(x)$ be the unique interpolation of $\psi(x)$ in $Sp(\mathscr{L}, \Delta, \underline{z})$, i.e., (cf. (2.13))

$$D^j\tilde{s}(x_i) = D^j\psi(x_i) \quad \text{for all} \quad 0 \leqq j \leqq z_i - 1 \quad \text{and all} \quad 0 \leqq i \leqq N + 1. \tag{4.6}$$

Because $\varphi(x)$ satisfies the boundary conditions of (2.2) by Theorem 2, it follows from (4.4) and (4.5) that $\tilde{s}(x)$ satisfies the boundary conditions of (4.1), i.e., $\tilde{s}(x) \in \hat{S}p(\mathscr{L}, \Delta, \underline{z})$. Thus, if we define the function $\tilde{w}(x)$ by

$$\tilde{w}(x) = D^{2q}\tilde{s}(x), \quad a \leqq x \leqq b, \tag{4.7}$$

then $\tilde{w}(x) \in H(L, \Delta, \underline{z})$. Moreover, we have that $\tilde{w}(x)$ interpolates $\varphi(x)$ in the sense that

$$D^j\tilde{w}(x_i) = D^{j+2q}\tilde{s}(x_i) = D^{j+2q}\psi(x_i) = D^j\varphi(x_i), \quad 0 \leqq j \leqq z_i - 1 - 2q, \tag{4.8}$$

provided that $z_i \geqq 1 + 2q$. Note that if $1 + 2q > z_i$, $\tilde{w}(x)$ need *not* interpolate $\varphi(x)$ at the knot x_i.

We now use the finite-dimensional subspace $H(L, \Delta, \underline{z})$ of $W_0^{n,2}[a, b]$ in the Rayleigh-Ritz procedure of §2. From Theorem 1, there exists a unique element $\hat{w}(x)$ in $H(L, \Delta, \underline{z})$ which minimizes the functional $F[w]$ of (2.9) over $H(L, \Delta, \underline{z})$, and we can similarly define the quantities k_j of (3.3), namely

$$k_j = \int_a^b \{L[\tilde{w}(t) - \varphi(t)] \cdot L[w_j(t)] + p(t)\left(\tilde{w}(t) - \varphi(t)\right) w_j(t)\}\, dt, \quad 1 \leqq j \leqq M, \tag{4.9}$$

where $\tilde{w}(x)$ is defined in (4.7), $p(t)$ is defined in (3.4), and $\{w_j(t)\}_{j=1}^{M}$ is any basis for $H(L, \Delta, \underline{z})$. Because of the identification (4.2) between elements of $H(L, \Delta, \underline{z})$ and elements of $\hat{S}p(\mathscr{L}, \Delta, \underline{z})$, the first integral of (4.9) can be written as

$$\sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} LD^{2q}[\tilde{s}(t) - \psi(t)] \cdot LD^{2q}[s_j(t)]\, dt = \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} \mathscr{L}[\tilde{s}(t) - \psi(t)] \cdot \mathscr{L}[s_j(t)]\, dt,$$

where $s_j(t)$ is the element in $\hat{S}p(\mathscr{L}, \Delta, \underline{z})$ uniquely associated with $w_j(t)$ in $H(L, \Delta, \underline{z})$ through (4.2). Exactly as in §3, we use Green's formula (cf. (2.17)), the interpolation properties of $\tilde{s}(t)$ of (4.6), the continuity properties of $\tilde{s}(t)$ (cf. (2.12ii)), and the fact that $\mathscr{L}^*\mathscr{L}[s_j(t)] = 0$ on each subinterval (x_i, x_{i+1}) of $[a, b]$ (cf. (2.12i)) to show that the above sum is necessarily *zero* for any $1 \leqq j \leqq M$. Thus, the expression for k_j in (4.9) reduces to

$$k_j = \int_a^b p(t)\left(\tilde{w}(t) - \varphi(t)\right) w_j(t)\, dt, \quad 1 \leqq j \leqq M, \tag{4.10}$$

where $p(t)$ can be seen to satisfy the bounds of (3.8). The argument now closely follows that of §3. Writing

$$\tilde{w}(t) = \sum_{j=1}^{M} \tilde{\alpha}_j w_j(t) \quad \text{and} \quad \hat{w}(t) = \sum_{j=1}^{M} \hat{\alpha}_j w_j(t),$$

then multiplying by $(\tilde{\alpha}_j - \hat{\alpha}_j)$ in (4.10) and summing on j gives

$$\sum_{j=1}^{M} (\tilde{\alpha}_j - \hat{\alpha}_j)\, k_j = \int_a^b p(t)\,(\tilde{w}(t) - \varphi(t))\,(\tilde{w}(t) - \hat{w}(t))\,dt. \tag{4.11}$$

Now, because $\tilde{s}(x)$ is the $Sp(\mathscr{L}, \Delta, \underline{z})$-interpolate of Type I of $\psi(x)$, where $\psi(x) \in W^{2m+2q,2}[a, b]$, then it follows from (2.14) of Theorem 2 that

$$\|D^{\ell}(\tilde{s} - \psi)\|_{L^2} \leqq K_9 (\bar{\Delta})^{2m+2q-\ell} \|\mathscr{L}^* \mathscr{L}[\psi]\|_{L^2}, \quad 0 \leqq \ell \leqq m+q, \tag{4.12}$$

where K_9 is independent of Δ and $\underline{z}$. Choosing $\ell = 2q + j$, and using the definitions of (4.4) and (4.7) gives us that

$$\|D^j(\tilde{w} - \varphi)\|_{L^2} \leqq K_9 (\bar{\Delta})^{2m-j} \|\mathscr{L}^* L[\varphi]\|_{L^2}, \quad 0 \leqq j \leqq n. \tag{4.13}$$

With this inequality for the case $j = 0$, and the upper bound for $p(t)$ from (3.8), the application of Schwarz's inequality to the integral of (4.11) gives

$$\sum_{j=1}^{M} (\tilde{\alpha}_j - \hat{\alpha}_j)\, k_j \leqq \Gamma' K_9 (\bar{\Delta})^{2m} \|\mathscr{L}^* L[\varphi]\|_{L^2} \cdot \|\tilde{w} - \hat{w}\|_{L^2}. \tag{4.14}$$

As in (3.12), the sum $\sum_{j=1}^{M} (\tilde{\alpha}_j - \hat{\alpha}_j)\, k_j$ can be bounded below by $\|\tilde{w} - \hat{w}\|_n^2 / K_3$. Thus, we deduce that

$$\|\tilde{w} - \hat{w}\|_n \leqq K_{10} (\bar{\Delta})^{2m} \|\mathscr{L}^* L[\varphi]\|_{L^2}, \tag{4.15}$$

where K_{10} is independent of Δ and $\underline{z}$. By means of the triangle inequality and the inequalities of (4.13) and (4.15), we have in analogy with Theorem 3 the result of

Theorem 4. Assume that the solution $\varphi(x)$ of (2.1)—(2.2) is of class $W^{2m,2}[a, b]$, where $m = n + q$, $q \geqq 0$, and let $H(L, \Delta, \underline{z})$ be the finite-dimensional subspace of $W_0^{n,2}[a, b]$ defined through (4.1) and (4.2). Then, given any partition $\Delta: a = x_0 < x_1 < \cdots < x_{N+1} = b$ of $[a, b]$ and any incidence vector $\underline{z} = (z_0, z_1, \ldots, z_{N+1})^T$ where $z_0 = z_{N+1} = m + q$ and $1 \leqq z_i \leqq m + q$ for $1 \leqq i \leqq N$, let $\hat{w}(x)$ be the unique function in $H(L, \Delta, \underline{z})$ which minimizes the functional $F[w]$ of (2.9) over $H(L, \Delta, \underline{z})$. Then, there exist constants K_{11} and K_{12}, independent of Δ and $\underline{z}$, such that

$$\|D^j(\hat{w} - \varphi)\|_{L^2} \leqq K_{11} (\bar{\Delta})^{2m-j} \|\mathscr{L}^* L[\varphi]\|_{L^2} \quad \text{for all} \quad 0 \leqq j \leqq n, \tag{4.16}$$

and

$$\|D^j(\hat{w} - \varphi)\|_{L^\infty} \leqq K_{12}) \bar{\Delta})^{2m-j-(\frac{1}{2})} \|\mathscr{L}^* L[\varphi]\|_{L^2} \quad \text{for all} \quad 0 \leqq j \leqq n-1. \tag{4.17}$$

If, moreover, $\varphi(x)$ is in $C^{2m}[a, b]$ and $\mathscr{F}$ is a collection of subspaces $H(L, \Delta, \underline{z})$ of $W_0^{n,2}[a, b]$ such that $\tilde{w}(x)$, the unique $H(L, \Delta, \underline{z})$-interpolation of $\varphi(x)$ (in the sense of (4.8)), satisfies

$$\|D^j(\tilde{w} - \varphi)\|_{L^\infty} \leqq K' (\bar{\Delta})^{2m-j} \|\mathscr{L}^* L[\varphi]\|_{L^\infty} \quad \text{for all} \quad 0 \leqq j \leqq n-1 \quad \text{and all} \quad H(L, \Delta, \underline{z}) \in \mathscr{F} \tag{4.18}$$

for some constant K', then

$$\|D^j(\hat{w} - \varphi)\|_{L^\infty} \leqq K_{13} (\bar{\Delta})^{2m-j} \|\mathscr{L}^* L[\varphi]\|_{L^\infty} \quad \text{for all} \quad 0 \leqq j \leqq n-1 \quad \text{and all} \quad H(L, \Delta, \underline{z}) \in \mathscr{F}, \tag{4.19}$$

where K_{13} is a positive constant.

Corollary. Given any partition $\Delta: a=x_0<x_1<\cdots<x_{N+1}=b$ of $[a, b]$ and any incidence vector $\underline{z}=(z_0, z_1, \ldots, z_{N+1})^T$ with $z_0=z_{N+1}=m+q$ and $1\leqq z_i\leqq m+q$ for $1\leqq i\leqq N$, let f of (2.1) be independent of u, and let $\hat{w}(x)$ be the unique function in $H(L, \Delta, \underline{z})$ which minimizes the functional of (2.9) over $H(L, \Delta, \underline{z})$. Then, in addition to the inequalities of (4.16), (4.17), and (4.19), we have that

$$D^j \hat{w}(x_i) = D^j \varphi(x_i) \quad \text{for all} \quad 0\leqq j\leqq z_j-1-2q, \quad 0\leqq i\leqq N+1, \tag{4.20}$$

provided that $z_i\geqq 1+2q$.

As our final result, let us now consider any $2n$-th order differential operator M of the form

$$M[v(x)] = \sum_{i=0}^{n} (-1)^i D^i(p_i(x)\, D^i v(x)), \qquad v\in C^{2n}[a, b]. \tag{4.21}$$

We seek now to solve, as in [5], the nonlinear two-point boundary value problem

$$-M[u(x)] = f(x, u(x)), \qquad a<x<b, \tag{4.22}$$

subject to the boundary conditions of

$$D^j u(a) = D^j u(b) = 0, \qquad 0\leqq j\leqq n-1. \tag{4.22'}$$

We assume, in analogy with (2.6), that constants $K_1>0$ and $\beta\geqq 0$ exist such that

$$\|w\|_n^2 \leqq K_1 \int_a^b \left\{\sum_{i=0}^{n} p_i(t)\, (D^i w(t))^2 + \beta\,(w(t))^2\right\} dt \quad \text{for all} \quad w\in W_0^{n,2}[a, b]. \tag{4.23}$$

As we know from Gårding's inequality [16, p. 175], $p_n(x)>0$ in $[a, b]$ is sufficient for (4.23). We remark that the inequality of (4.23) implies that

$$\Lambda \equiv \inf_{w\in W_0^{n,2}[a,b]} \frac{\int_a^b \left\{\sum_{i=0}^{n} p_i(t)\,(D^i w(t))^2\right\} dt}{\int_a^b (w(t))^2\, dt} \tag{4.24}$$

is *finite* (cf. [5, Lemma 1]). We then assume as in §2 that $f(x, u)$ of (4.22) satisfies the hypotheses of (2.8) and (2.8').

Suppose now that it is possible to find an n-th order differential operator

$$\ell[u(x)] \equiv \sum_{j=0}^{n} \beta_j(x)\, D^j u(x), \tag{4.25}$$

such that

$$M[v(x)] \equiv \ell^* \ell[v(x)] + \sum_{i=0}^{k} (-1)^i D^i(\sigma_i(x)\, D^i v), \qquad v\in C^{2n}[a, b], \tag{4.26}$$

where $\sigma_i(x)\in C^i[a, b]$, $0\leqq i\leqq k$, and $0\leqq k\leqq n$. As a special case of interest, suppose that the operator M of (4.21) is such that $p_n(x)\equiv 1$. Then, it is clear that the choice $\ell[u(x)]\equiv D^n u(x)$ surely satisfies (4.26) with $0\leqq k<n$. This choice of the operator ℓ is computationally attractive since the elements of the subspace $Sp_0(\ell, \Delta, \underline{z})$ of $W_0^{n,2}[a, b]$ are piecewise-*polynomial* functions of local degree $2n-1$.

Given any finite-dimensional subspace S_M of $W_0^{n,2}[a, b]$, then integration by parts, coupled with the expression of (4.26), gives us that the unique element

$\hat{w}(x) \in S_M$ which minimizes the functional of

$$F[w] = \int_a^b \left\{ \frac{1}{2} \sum_{i=0}^{n} p_i(t) \left(D^i w(t)\right)^2 + \int_0^{w(t)} f(t, \eta)\, d\eta \right\} dt \tag{4.27}$$

over S_M satisfies (cf. (2.10))

$$\int_a^b \left\{ \ell[\hat{w}(t)] \cdot \ell[w_j(t)] + \sum_{i=0}^{k} \sigma_i(t) D^i \hat{w}(t) \cdot D^i w_j(t) + f\left(t, \hat{w}(t)\right) w_j(t) \right\} dt = 0, \quad 1 \leqq j \leqq M, \tag{4.28}$$

where $\{w_j(x)\}_{j=1}^{M}$ is any basis for S_M. Similarly, the solution $\varphi(t)$ of (4.22)—(4.22′) satisfies

$$\int_a^b \left\{ \ell[\varphi(t)] \cdot \ell[w_j(t)] + \sum_{i=0}^{k} \sigma(t) D^i \varphi(t) \cdot D^i w_j(t) + f\left((t, \varphi(t)\right) w_j(t) \right\} dt = 0, \quad 1 \leqq j \leqq M. \tag{4.29}$$

As in Theorem 4, we assume that $\varphi(x)$ is of class $W^{2m,2}[a, b]$ and satisfies (4.22′), where $m = n + q$, $q \geqq 0$, but now we minimize the functional of (4.27) over the subspace $H(\ell, \Delta, \underset{\sim}{z})$ of $W_0^{n,2}[a, b]$, whose elements $w(x)$ are determined by means of (4.2) from elements $s(x)$ in the space $\hat{S}p(\ell D^{2q}, \Delta, \underset{\sim}{z})$. Let the interpolate of $\varphi(x)$ in $H(\ell, \Delta, \underset{\sim}{z})$, in the sense of (4.8), be denoted again by $\tilde{w}(x)$. Then, defining the quantities k_j as in (4.9), we can express k_j both as

$$k_j = \int_a^b \left\{ \sum_{i=0}^{n} p_i(t) D^i \left(\tilde{w}(t) - \hat{w}(t)\right) \cdot D^i w_j(t) + q(t) \left(\tilde{w}(t) - \hat{w}(t)\right) \cdot w_j(t) \right\} dt, \quad 1 \leqq j \leqq M, \tag{4.30}$$

as well as

$$\begin{aligned} k_j = \int_a^b \Big\{ & \ell[\tilde{w}(t) - \varphi(t)] \cdot \ell[w_j(t)] + \sum_{i=0}^{k} \sigma_i(t) D^i \left(\tilde{w}(t) - \varphi(t)\right) \cdot D^i w_j(t) \\ & + p(t) \left(\tilde{w}(t) - \varphi(t)\right) w_j(t) \Big\} dt, \quad 1 \leqq j \leqq M. \end{aligned} \tag{4.31}$$

The first integral of (4.31), as before, vanishes for each j, and writing

$$\tilde{w}(t) = \sum_{j=1}^{M} \tilde{\alpha}_j w_j(t) \quad \text{and} \quad \hat{w}(t) = \sum_{j=1}^{M} \hat{\alpha}_j w_j(t),$$

then multiplying by $(\hat{\alpha}_j - \tilde{\alpha}_j)$ in (4.30) and (4.31) and summing on j yields

$$\sum_{j=1}^{M} (\tilde{\alpha}_j - \hat{\alpha}_j) k_j = \int_a^b \left\{ \sum_{i=1}^{n} p_i(t) \left[D^i \left(\tilde{w}(t) - \hat{w}(t)\right)\right]^2 + q(t) \left(\tilde{w}(t) - \hat{w}(t)\right)^2 \right\} dt, \tag{4.32}$$

and

$$\begin{aligned} \sum_{j=1}^{M} (\tilde{\alpha}_j - \hat{\alpha}_j) k_j = \int_a^b \Big\{ & \Big(\sum_{i=0}^{k} \sigma_i(t) D^i \left(\tilde{w}(t) - \varphi(t)\right) \cdot D^i \left(\tilde{w}(t) - \hat{w}(t)\right) \Big) \\ & + p(t) \left(\tilde{w}(t) - \varphi(t)\right) \cdot \left(\tilde{w}(t) - \hat{w}(t)\right) \Big\} dt. \end{aligned} \tag{4.33}$$

As before, $\sum_{j=1}^{M} (\tilde{\alpha}_j - \hat{\alpha}_j)\, k_j \geqq \frac{\|\tilde{w} - \hat{w}\|_n^2}{K_3}$ follows from (4.23) and (4.32). We now obtain an upper bound for the sum of (4.33). Because $\sigma_i(t) \in C^i[a, b]$ by assumption, we can integrate the following term by parts:

$$(4.33')\qquad \begin{aligned}&\int_a^b \sigma_i(t)\, D^i(\widetilde{w}(t)-\varphi(t))\, D^i(\widetilde{w}(t)-(\widehat{w}(t))\,dt\\ &\quad=(-1)^j\int_a^b D^{i-j}(\widetilde{w}(t)-\varphi(t))\cdot D^j\{\sigma_i(t)\,D^i(\widetilde{w}(t)-\widehat{w}(t))\}\,dt\end{aligned}$$

where $j\leqq i$, and $i+j\leqq n$. Since $0\leqq k\leqq n$, it is easy to see from this that the integral in (4.33′) can be expressed as a sum of integrals of the form

$$(4.34)\qquad I\equiv\int_a^b D^{\mu_1}(\widetilde{w}(t)-\varphi(t))\, D^{\mu_2}\sigma_i(t)\, D^{\mu_3}(\widetilde{w}(t)-\widehat{w}(t))\,dt,$$

where $0\leqq\mu_3\leqq n$, $0\leqq\mu_2\leqq i$, and $0\leqq\mu_1\leqq\max\{2k-n,0\}\equiv\delta$. Note that if $0\leqq k\leqq[n/2]$, then $\delta=0$. Because $D^{\mu_2}\sigma_i(t)$ is a bounded factor in (4.34), we can obtain an upper bound for I in (4.34) from Schwarz's inequality, i.e.,

$$(4.35)\qquad |I|\leqq K\|\widetilde{w}-\widehat{w}\|_n\cdot\|D^{\mu_1}(\widetilde{w}-\varphi)\|_{L^2},\qquad 0\leqq\mu_1\leqq\delta=\max(2k-n,0).$$

In particular, using the error bounds of (4.13), we thus find that

$$(4.36)\qquad \sum_{j=1}^{M}(\tilde{\alpha}_j-\hat{\alpha}_j)\,k_j\leqq K_{14}(\bar{\Delta})^{2m-\delta}\|\widetilde{w}-\widehat{w}\|_n,$$

so that

$$(4.37)\qquad \|\widetilde{w}-\widehat{w}\|_n\leqq K'_{14}(\bar{\Delta})^{2m-\delta}.$$

This gives us, in the manner of previous proofs, the result of

Theorem 5. Assume that the unique (generalized) solution $\varphi(x)$ of (4.22) to (4.22′) is of class $W^{2m,2}[a,b]$, where $m=n+q$, $q\geqq 0$, let Δ and $\underline{z}$ satisfy the hypotheses of Theorem 4, and let $H(\ell,\Delta,\underline{z})$ be the finite-dimensional subspace of $W_0^{n,2}[a,b]$ defined by means of (4.2) from elements $s(x)$ in the space $\hat{S}p(\ell D^{2q},\Delta,\underline{z})$ where the differential operator ℓ of (4.25) satisfies (4.26), and the functions $\sigma_i(t)$ of (4.26) are of class $C^i[a,b]$, $0\leqq i\leqq k\leqq n$. Let $\widehat{w}(x)$ be the unique function in $H(\ell,\Delta,\underline{z})$ which minimizes the functional of (4.27) over $H(\ell,\Delta,\underline{z})$. Then, there exist constants K_{15} and K_{16}, independent of Δ and $\underline{z}$, such that

$$(4.38)\qquad \|D^j(\widehat{w}-\varphi)\|_{L^2}\leqq K_{15}(\bar{\Delta})^{2m-\max(\delta,j)},\qquad 0\leqq j\leqq n,$$

where $\delta\equiv\max\{2k-n,0\}$, and

$$(4.39)\qquad \|D^j(\widehat{w}-\varphi)\|_{L^\infty}\leqq K_{16}(\bar{\Delta})^{2m-\max(\delta,j)-(\frac{1}{2})},\qquad 0\leqq j\leqq n-1.$$

If, moreover $\varphi(x)$ is in $C^{2m}[a,b]$ and $\mathscr{F}$ is a collection of subspaces $H(\ell,\Delta,\underline{z})$ of $W_0^{n,2}[a,b]$ such that $\widetilde{w}(x)$, the unique $H(\ell,\Delta,\underline{z})$-interpolate of $\varphi(x)$ (in the sense of (4.8)), satisfies

$$(4.40)\qquad \|D^j(\widetilde{w}-\varphi)\|_{L^\infty}\leqq K'(\bar{\Delta})^{2m-j}\quad\text{for all}\quad 0\leqq j\leqq n-1\quad\text{and all}\quad H(\ell,\Delta,\underline{z})\in\mathscr{F}$$

for some constant K', then

$$(4.41)\qquad \|D^j(\widehat{w}-\varphi)\|_{L^\infty}\leqq K_{17}(\bar{\Delta})^{2m-\max(\delta,j)}\quad\text{for all}\quad 0\leqq j\leqq n-1\quad\text{and all}\quad H(\ell,\Delta,\underline{z})\in\mathscr{F},$$

where K_{17} is a positive constant.

The importance of this last result is that one can have some freedom in the selection of the operator ℓ of (4.25) in order to make numerical computations as simple as possible, with little or no loss of accuracy in the approximation of the solution $\varphi(x)$ of (4.22)—(4.22′).

We remark that most of the results of [5] correspond to the special case $k=n$ of Theorem 5. We also remark that the results of Theorem 5 can be further generalized, but this will be considered in a later paper.

§ 5. Computational Results

In this section, we discuss numerical results obtained from some concrete examples. These numerical results confirm the theoretical accuracies established in previous sections, and also illustrate the computational superiority of variational techniques over standard finite difference techniques for such two-point boundary value problems.

As our first example, consider the numerical approximation of the solution of

$$(5.1)\qquad -D^4 u(x) = f(x, u) = u(x) + g(x), \quad 0 < x < 1,$$

with boundary conditions

$$(5.2)\qquad u(0) = Du(0) = u(1) = Du(1) = 0,$$

where

$$(5.3)\qquad g(x) = \begin{cases} +6x^4 - 5x^3 + 144, & 0 \leqq x \leqq \frac{1}{2}, \\ -945(2x-1)^{\frac{1}{2}} - (2x-1)^{\frac{9}{2}} - 5x^3 + 6x^4 + 144, & \frac{1}{2} \leqq x \leqq 1. \end{cases}$$

For this example, the quantity Λ in (2.7) is bounded below by π^4, and as $f(x, u)$ is linear in u, we see that the inequalities of (2.8) are satisfied for $\gamma = 1$.

As is easily verified, the unique solution, $\varphi(x)$, of (5.1) and (5.2) is explicitly given by

$$(5.4)\qquad \varphi(x) = \begin{cases} -6x^4 + 5x^3, & 0 \leqq x \leqq \frac{1}{2}, \\ -6x^4 + 5x^3 + (2x-1)^{\frac{9}{2}}, & \frac{1}{2} \leqq x \leqq 1. \end{cases}$$

Note that $\varphi(x)$ is in $C^4[0, 1]$, but is *not* in $C^5[0, 1]$. More precisely, the modulus of continuity ω of its fourth derivative satisfies

$$(5.5)\qquad \omega(D^4\varphi; \delta) \equiv \sup_{\substack{x,y\in[0,1] \\ |x-y|\leqq\delta}} |D^4\varphi(x) - D^4\varphi(y)| = 945(2\delta)^{\frac{1}{2}}$$

for all $0 \leqq \delta \leqq \frac{1}{2}$.

The solution of (5.1) and (5.2) was first approximated by a five-point central difference method on a uniform mesh, i.e., with $h = 1/(N+1)$, the following linear system was solved:

$$(5.6)\qquad \frac{6w_i - 4(w_{i-1} + w_{i+1}) + (w_{i-2} + w_{i+2})}{h^4} = -w_i - g_i, \quad 1 \leqq i \leqq N,$$

where $g_i \equiv g(ih)$, with boundary conditions

$$(5.7)\qquad w_0 = w_{N+1} = 0 \quad \text{and} \quad w_{-1} = w_1,\ w_{N+2} = w_N.$$

The results are given in Table 1, where α, which corresponds to the computed exponent of $\bar{\Delta}=h$, is generated from

$$\ln\left(\frac{\|w_{h_2}-\varphi\|'_{L^\infty}}{\|w_{h_1}-\varphi\|'_{L^\infty}}\right)\Big/\ln\left(\frac{h_2}{h_1}\right), \tag{5.8}$$

where $\|w_h-\varphi\|'_{L^\infty}\equiv\max_i|w_i-\varphi(ih)|$ denotes the *discrete* norm.

Eqs. (5.1) and (5.2) were also solved by applying the variational method of §2 to the cubic Hermite space $H_0^{(2)}(\Delta(h))$, where $\Delta(h)$ denotes a *uniform* partition of $[0, 1]$ with $h=1/(N+1)$. In this case, $\bar{\Delta}(h)=h=1/(N+1)$. Since the solution

Table 1. *Finite differences*

N	$\max_i\left\|w_i-\varphi\left(\frac{i}{N+1}\right)\right\|$	α
9	$3.25\cdot10^{-2}$	—
10	$4.10\cdot10^{-2}$	-2.30
16	$1.81\cdot10^{-2}$	1.88
19	$1.18\cdot10^{-2}$	2.60
30	$5.91\cdot10^{-3}$	1.58
39	$4.53\cdot10^{-3}$	1.06
79	$1.71\cdot10^{-3}$	1.41

Table 2. *Variational method for* $H_0^{(2)}(\Delta(h))$

N	$\dim H_0^{(2)}(\Delta(h))$	$\|\hat{w}_h-\varphi\|_{L^\infty}$	α	$\|D(\hat{w}_h-\varphi)\|_{L^\infty}$	α'
3	6	$6.84\cdot10^{-3}$	—	$8.53\cdot10^{-2}$	—
4	8	$2.92\cdot10^{-3}$	3.81	$4.53\cdot10^{-2}$	2.84
6	12	$7.93\cdot10^{-4}$	3.88	$1.72\cdot10^{-2}$	2.88
8	16	$2.97\cdot10^{-4}$	3.86	$8.26\cdot10^{-3}$	2.91
10	20	$1.35\cdot10^{-4}$	3.95	$4.58\cdot10^{-3}$	2.94
16	32	$2.43\cdot10^{-5}$	3.94	$1.27\cdot10^{-3}$	2.95
19	38	$1.28\cdot10^{-5}$	3.93	$7.88\cdot10^{-4}$	2.95

$\varphi(x)$ of (5.1)—(5.2) is of class $C^4[0, 1]$, and the subspace chosen is the cubic Hermite space $H_0^{(2)}(\Delta(h))$, then the result of (3.19) can be applied with $n=2$, i.e., if $\hat{w}_h(x)$ denotes the approximate solution in $H_0^{(2)}(\Delta(h))$, then

$$\|D^j(\hat{w}_h-\varphi)\|_{L^\infty}\leqq K_8 h^{4-j} \quad \text{for} \quad 0\leqq j\leqq 1. \tag{5.9}$$

The numerical results are given in Table 2, where α is computed from (5.8) in the *continuous* norm, and α' is similarly computed in the continuous norm for the values of $\|D(\hat{w}_h-\varphi)\|_{L^\infty}$. Note that the $\mathcal{O}(h^4)$ convergence for $\|\hat{w}_h-\varphi)\|_{L^\infty}$ and the $\mathcal{O}(h^3)$ convergence for $\|D(\hat{w}_h-\varphi)\|_{L^\infty}$ from (5.9) is well illustrated by the results of Table 2, whereas the results of Table 1 show erratic convergence for the finite difference method.

The next example was chosen to illustrate the results attainable when $f(x, u)$ is not continuous, but only L_2-integrable. Consider the simple linear problem:

$$D^4u=\begin{cases}24.0, & 0\leqq x\leqq\frac{1}{2},\\ 48.0, & \frac{1}{2}< x\leqq 1,\end{cases} \tag{5.10}$$

with the same boundary conditions as given by (5.2). It can be easily verified that $\varphi(x)$, given by

$$(5.11)\qquad \varphi(x)=\begin{cases} x^4-\frac{19}{8}x^3+\frac{21}{16}x^2, & 0\leq x\leq\frac{1}{2}\\ 2(x-1)^4+\frac{29}{8}(x-1)^3+\frac{27}{16}(x-1)^2, & \frac{1}{2}<x\leq 1,\end{cases}$$

is a weak solution of problem (5.10) and (5.2) in $W^{4,2}[0, 1]$.

This problem was approximated by finite differences (cf. (5.7) and (5.8)), and by applying the variational method of § 2 to the cubic Hermite space $H_0^{(2)}(\Delta(h))$. From the result of $j=0$ of (3.17) of Theorem 3, it would appear that

$$\|\hat{w}_h-\varphi\|_{L^\infty}\leq K_7 h^{\frac{7}{2}},$$

but if a mesh point x_i coincides with the discontinuity of $D^4\varphi$ at $x=\frac{1}{2}$, the sharper result

$$\|\hat{w}_h-\varphi\|_{L^\infty}\leq K_7' h^4$$

Table 3. *Finite differences*

N	$\max_i \left\vert w_i-\varphi\left(\frac{i}{N+1}\right)\right\vert$	α
9	$2.97\cdot 10^{-2}$	—
10	$6.16\cdot 10^{-2}$	−0.73
16	$2.49\cdot 10^{-2}$	2.08
19	$1.50\cdot 10^{-2}$	3.09
30	$1.56\cdot 10^{-2}$	−0.23
39	$1.29\cdot 10^{-2}$	0.68
79	$8.53\cdot 10^{-3}$	0.56

Table 4. *Variational method for* $H_0^{(2)}(\Delta(h))$

N	$\dim H_0^{(2)}(\Delta(h))$	$\|\hat{w}_h-\varphi\|_{L^\infty}$	α
3	6	$4.88\cdot 10^{-4}$	—
4	8	$2.00\cdot 10^{-4}$	4.00
5	10	$9.65\cdot 10^{-5}$	4.00
6	12	$5.21\cdot 10^{-5}$	3.99
7	14	$3.05\cdot 10^{-5}$	4.00
9	18	$1.25\cdot 10^{-5}$	4.00

can be readily established. The numerical results for the cubic Hermite space $H_0^{(2)}(\Delta(h))$ are given in Table 4. Note again the erratic convergence of the finite difference method in Table 3.

The final example was chosen to illustrate the results of Theorem 4 of §4. The example selected here was taken from [8]. Consider the second order nonlinear boundary value problem

$$(5.12)\qquad D^2u(x)=e^{u(x)},\quad 0<x<1;\quad u(0)=u(1)=0.$$

The unique solution $\varphi(x)$ of (5.12) is given by

$$(5.13)\qquad \varphi(x)=-\ln 2+2\ln\{c\sec(c(x-\tfrac{1}{2})/2)\},\quad 0\leq x\leq 1,$$

where $c=1.3360557$. Since φ is actually analytic in some neighborhood of the segment $[0, 1]$, then in particular $\varphi\in C^4[0, 1]$. Thus, we apply Theorem 4 for the case $m=2$, $n=q=1$, and $L=D$. In order that $H(D, \Delta, \underline{z})$ of Theorem 4 is the particular Hermite space $H_0^{(2)}(\Delta)$, it is necessary to choose the space $\hat{S}p(D^3, \Delta, \underline{z})$ in (4.2), where the components of the incidence vector $\underline{z}=(z_0, z_1, \ldots, z_N, z_{N+1})^T$ are specifically $z_0=z_{N+1}=3$, $z_1=z_2=\cdots=z_N=2$. For such quintic splines spaces $\hat{S}p(D^3, \Delta, \underline{z})$, the results of (4.16) and (4.17) of Theorem 4 are valid, and we thus have that $\|\hat{w}_h-\varphi\|_{L^2}\leq K_{11}h^4$ and $\|\hat{w}_h-\varphi\|_{L^\infty}\leq K_{12}h^{\frac{7}{2}}$. Although studied in [13], the stronger interpolation results of (4.18) are not yet known to be valid, so that we cannot assert that $\|\hat{w}_h - \varphi\|_{L^\infty} \leqq K_{13} h^4$. Nonetheless, the numerical results of Table 5 show $\mathcal{O}(h^4)$ convergence.

Table 5. *Variational method for* $H_0^{(2)}(\Delta(h))$

N	$\dim H_0^{(2)}(\Delta(h))$	$\|\hat{w}_h - \varphi\|_{L^\infty}$	α
1	4	$5.10 \cdot 10^{-5}$	—
2	6	$1.21 \cdot 10^{-5}$	3.54
3	8	$4.24 \cdot 10^{-6}$	3.65
5	12	$9.58 \cdot 10^{-7}$	3.65
7	16	$3.10 \cdot 10^{-7}$	3.93
9	20	$1.28 \cdot 10^{-7}$	3.96
11	24	$6.28 \cdot 10^{-8}$	3.91

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