JOURNAL OF APPROXIMATION THEORY 2, 50–65 (1969)

Chebyshev Rational Approximations to e^{-x} in $[0, +\infty)$ and Applications to Heat-Conduction Problems

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1. INTRODUCTION

For any nonnegative integer m, let π_m denote the collection of all real polynomials of degree at most m, and for any nonnegative integers m and n, let $\pi_{m,n}$ denote the collection of all real rational functions $r_{m,n}(x)$ of the form

$$r_{m,n}(x) \equiv \frac{p_m(x)}{q_n(x)}, \quad \text{where } p_m \in \pi_m \quad \text{and} \quad q_n \in \pi_n. \tag{1.1}$$

With this notation, let

$$\lambda_{m,n} \equiv \inf_{r_{m,n} \in \pi_{m,n}} \{ \sup_{0 \leqslant x < \infty} |r_{m,n}(x) - e^{-x}| \}, \qquad m \leqslant n, \tag{1.2}$$

be the error associated with the best *Chebyshev rational approximation* in $\pi_{m,n}$ to e^{-x} in $[0, +\infty)$. It is known ([*1*], p. 55) that there exists a unique $\hat{r}_{m,n}(x) \in \pi_{m,n}$ such that

$$\lambda_{m,n} = \sup_{0 \leqslant x < \infty} |\hat{r}_{m,n}(x) - e^{-x}|. \tag{1.3}$$

In this paper, we specifically give (in §4) the value of $\lambda_{0,n}$ and $\lambda_{n,n}$ for $0 \leqslant n \leqslant 9$ and $0 \leqslant n \leqslant 14$, respectively, along with the associated minimizing Chebyshev rational approximations $\hat{r}_{n,n}(x)$. These $\lambda_{m,n}$ and $\hat{r}_{n,n}(x)$ were determined because they can be used in the numerical solution of certain heat-conduction problems, and this is illustrated in §3. In a sense, the results of §§3 and 4 continue the original investigation of [7], where only $\lambda_{1,1}$ and $\lambda_{2,2}$ were given.

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[1] Work performed under the auspices of the U.S. Atomic Energy Commission.
[2] This research was supported in part by AEC Grant AT(11-1)-1702.

where $g_n(x)$ is defined by

$$g_n(x) \equiv \frac{1}{S_n(x)} - e^{-x}. \tag{2.3}$$

We establish

LEMMA 1. *For any integer $n \geqq 0$, we have*

$$0 \leqq g_n(x) \leqq \frac{1}{2^n} \qquad \textit{for } x \geqq 0. \tag{2.4}$$

Proof. Obviously,

$$g_n(0) = 0.$$

Now, let $n \geqq 1$. Since

$$e^x > S_n(x) \qquad \text{for } x > 0,$$

it follows that

$$g_n(x) > 0 \qquad \text{for } x > 0.$$

Let ξ be a positive number at which $g_n(x)$ possesses its maximum in $[0, \infty)$. Then by differentiating (2.3), we have

$$\frac{S_n'(\xi)}{S_n^2(\xi)} = e^{-\xi}.$$

Because of

$$S_n'(x) = S_{n-1}(x),$$

this implies that

$$(g_n(\xi) + e^{-\xi})^2 = e^{-\xi}(g_{n-1}(\xi) + e^{-\xi}).$$

Taking square roots, we derive that

$$0 \leqq g_n(\xi) = e^{-\xi}\{[1 + g_{n-1}(\xi)\, e^{\xi}]^{1/2} - 1\}$$
$$< e^{-\xi} \cdot \tfrac{1}{2} g_{n-1}(\xi)\, e^{\xi} = \tfrac{1}{2} g_{n-1}(\xi).$$

Therefore,

$$0 \leqq g_n(x) < \tfrac{1}{2} \max_{0 \leqq t < \infty} g_{n-1}(x) \qquad \text{for all } x \leqq 0. \tag{2.5}$$

Thus, as

$$\max_{0 \leqq x < \infty} |g_0(x)| = 1,$$

then (2.4) follows by induction. Q.E.D.

LEMMA 2. *For any integer $n \geqq 0$, we have*

$$\max_{0 \leqq x < \infty} \left| \frac{e^{-\alpha n}}{S_n(x - \alpha n)} - e^{-x} \right| \leqq \frac{1}{(2e^{\alpha})^n}, \tag{2.6}$$

α being the real solution of (1.7).

Proof. We shall prove that

$$|g_n(x)| \leqq \frac{1}{2^n}$$

holds not only for $x \geqq 0$, but even for $x \geqq -\alpha n$. Then, putting

$$x = t - \alpha n,$$

it follows that

$$\left|\frac{1}{S_n(t-\alpha n)} - e^{-(t-\alpha n)}\right| \leqslant \frac{1}{2^n} \qquad \text{for all } t \geqq 0,$$

which gives (2.6).

To prove the above reduced proposition, let $n \geqq 1$. Writing

$$S_n(-y) = e^{-y} - \sum_{j=n+1}^{\infty} (-1)^j \frac{y^j}{j!},$$

we see that for $0 < y \leqslant \alpha n < n+1$, the above series is an alternating series whose terms decrease in modulus monotonically to zero. This gives us that

$$e^{-y} < S_n(-y) < e^{-y} + \frac{y^{n+1}}{(n+1)!}, \qquad n \text{ even}, \tag{2.7}$$

and

$$e^{-y} - \frac{y^{n+1}}{(n+1)!} < S_n(-y) < e^{-y}, \qquad n \text{ odd}, \tag{2.8}$$

for $0 < y \leqslant \alpha n$. To obtain a (positive) lower bound for the left side of (2.8) for $0 < y \leqslant \alpha n$, we observe that

$$e^{-y} - \frac{y^{n+1}}{(n+1)!} \geqslant e^{-\alpha n} - \frac{(\alpha n)^{n+1}}{(n+1)!} = \frac{n^{n+1}}{(n+1)!}\left\{\frac{(n+1)!\, e^{-\alpha n}}{n^{n+1}} - \alpha^{n+1}\right\}$$

$$> \frac{n^{n+1}\, e^{-(1+\alpha)n}}{(n+1)!}\{\sqrt{2\pi n} - \alpha(\alpha e^{1+\alpha})^n\},$$

the last inequality following from Stirling's inequality. But since

$$\alpha e^{\alpha+1} = \frac{1}{2e^{\alpha}}$$

from (1.7), and

$$\left\{\sqrt{2\pi n} - \alpha\left(\frac{1}{2e^{\alpha}}\right)^n\right\} > 1$$

for all $n \geqslant 1$, then

$$e^{-y} - \frac{y^{n+1}}{(n+1)!} > \frac{n^{n+1}\, e^{-(1+\alpha)n}}{(n+1)!} \qquad \text{for all } n \geqslant 1, \qquad 0 < y \leqslant \alpha n. \tag{2.9}$$

The inequalities (2.8), (2.9) imply for odd n that

$$0 < \frac{1}{S_n(-y)} - e^y < \frac{1}{e^{-y} - \frac{y^{n+1}}{(n+1)!}} - e^y$$

$$= \frac{y^{n+1}\, e^y}{(n+1)!\left(e^{-y} - \frac{y^{n+1}}{(n+1)!}\right)}$$

$$\leqq \frac{\alpha^{n+1}\, n^{n+1}\, e^{\alpha n}}{n^{n+1}\, e^{-(1+\alpha)n}} = \alpha(\alpha\, e^{(2\alpha+1)})^n = \frac{\alpha}{2^n}.$$

Consequently,

$$0 < \frac{1}{S_n(-y)} - e^y < \frac{1}{2^n} \qquad \text{for } n \text{ odd}, \qquad 0 < y \leqq \alpha n. \tag{2.10}$$

For n even, one similarly arrives at

$$0 > \frac{1}{S_n(-y)} - e^y > -\frac{y^{n+1}}{(n+1)!}\, e^{2y}$$

$$\geqq -\frac{n^{n+1}\, e^{-n}}{(n+1)!}\, \alpha(\alpha\, e^{(2\alpha+1)})^n$$

$$> -\frac{\alpha}{2^n}.$$

Consequently

$$0 > \frac{1}{S_n(-y)} - e^y > -\frac{1}{2^n} \qquad \text{for } n \text{ even}, \qquad 0 < y \leqslant \alpha n, \tag{2.11}$$

and (2.10) and (2.11) imply the desired inequality (2.6) Q.E.D.

Lemma 2 directly gives us

THEOREM 1. *For any integer $n \geqq 0$, we have*

$$0 < \lambda_{n,n} \leqq \lambda_{n-1,n} \leqq \cdots \leqq \lambda_{0,n} \leqq \frac{1}{(2\, e^{\alpha})^n}, \tag{2.12}$$

where α is the solution of (1.7).

COROLLARY. *Let $\{m(n)\}_{n=0}^{\infty}$ be any sequence of nonnegative integers such that $0 \leqslant m(n) \leqslant n$ for each $n \geqslant 0$. Then,*

$$\overline{\lim_{n\to\infty}}\, (\lambda_{m(n),n})^{1/n} \leqslant \frac{e^{-\alpha}}{2} = 0.43501\ \ldots. \tag{2.13}$$

2. Now, we want to show that at least in the case $m = 0$, the speed of convergence of the sequence $\lambda_{m,n}$ is not greater than geometric. Again, we need two lemmas. First, we introduce the quantity

$$K_n = \min_{P_n \in \pi_n} \left\{ \max_{0 \leqslant x \leqslant 2n/3} |P_n(x) - e^x| \right\}. \tag{2.14}$$

LEMMA 3. *Suppose that there exists a sequence of polynomials* $\{Q_n(x)\}_{n=0}^{\infty}$ *with* $Q_n(x) \in \pi_n$ *for all* $n \geqslant 0$, *a real number* $q \geqslant 2$, *and an integer* n_0 *such that*

$$\left| \frac{1}{Q_n(x)} - e^{-x} \right| \leqq \frac{1}{q^n} \qquad \text{for all } x \geqq 0 \text{ and for all } n \geqq n_0. \tag{2.15}$$

Then,

$$K_n \leqq \frac{(e^{2/3})^n}{(q e^{-2/3})^n - 1} \qquad \text{for } n \geqq n_0. \tag{2.16}$$

Proof. First, observe that $e^{2/3} < 2$. Then, from (2.15), it follows for $0 \leqq x \leqq \frac{2}{3}n$, $n > n_0$, that

$$0 < e^{-x} - q^{-n} \leqq \frac{1}{Q_n(x)} \leqq e^{-x} + q^{-n},$$

and therefore

$$-\frac{e^x}{e^{-x}q^n + 1} \leqq Q_n(x) - e^x \leqq \frac{e^x}{e^{-x}q^n - 1}.$$

Thus,

$$|Q_n(x) - e^x| \leqq \frac{(e^{2/3})^n}{(q e^{-2/3})^n - 1} \qquad \text{for } 0 \leqq x \leqq \tfrac{2}{3}n, \qquad n \leqq n_0,$$

from which (2.16) is evident. Q.E.D.

LEMMA 4. *For any integer* $n \geqq 0$,

$$K_n > \frac{e^{n/3}\, n^{n+1}}{3 \cdot 6^n (n+1)}. \tag{2.17}$$

Proof. Writing

$$x = \frac{n}{3}(t+1),$$

we see that

$$K_n = \inf_{\tilde{P}_n \in \pi_n} \left\{ \sup_{-1 \leqq t \leqq +1} |\tilde{P}_n(t) - e^{n(t+1)/3}| \right\}.$$

For $t \in [-1, +1]$, we have the representation

$$e^{n(t+1)/3} = e^{n/3}\left(I_0\left(\frac{n}{3}\right) + 2\sum_{\nu=1}^{\infty} I_\nu\left(\frac{n}{3}\right) T_\nu(t)\right).$$

Here, $T_\nu(t)$ denotes the ν-th-Chebyshev polynomial of the first kind and

$$I_\nu(z) \equiv \sum_{\mu=0}^{\infty} \frac{(z/2)^{2\mu+\nu}}{\mu!(\nu+\mu)!}$$

is the Bessel function of order ν with so-called purely imaginary argument. Obviously,

$$I_\nu(x) > 0 \qquad \text{for } x > 0.$$

By a theorem of Hornecker (cf. [*4*], Theorem 66), then

$$K_n \geqq 2\,e^{n/3} \sum_{\mu=0}^{\infty} I_{(2\mu+1)(n+1)}\left(\frac{n}{3}\right).$$

Since

$$I_{n+1}\left(\frac{n}{3}\right) > \frac{n^{n+1}}{6^{n+1}(n+1)!},$$

it follows that

$$K_n > 2\,e^{n/3}\, I_{n+1}\left(\frac{n}{3}\right) > \frac{e^{n/3}\, n^{n+1}}{3\cdot 6^n(n+1)!},$$

which establishes (2.17). Q.E.D.

Now, we are able to prove

THEOREM 2. *For the quantity* σ_1 *defined in* (1.4), *we have*

$$\sigma_1 \geqq \tfrac{1}{6}. \tag{2.18}$$

Proof. By Theorem 1, we know already

$$\sigma_1 < \tfrac{1}{2}.$$

For every number q with

$$\sigma_1 < \frac{1}{q} < \tfrac{1}{2}, \tag{2.19}$$

there exists, by the definition of σ_1, a sequence of polynomials $Q_n(x)$ and an integer n_0 such that the assumptions of Lemma 3 are satisfied. Combining (2.16) and (2.17) we see that for all $n \geqq n_0$, the inequality

$$\frac{e^{n/3}\, n^{n+1}}{3\cdot 6^n(n+1)!} < \frac{e^{2/3n}}{(q\,e^{-2/3})^n - 1}$$

must hold. Using Stirling's formula, i.e.,

$$n! < \sqrt{2\pi n}\, n^n e^{-n}\left(1+\frac{1}{4n}\right),$$

leads to

$$g^n < (e^{2/3})^n + 3\left[\left(\frac{n+1}{n}\right)6^n\sqrt{2\pi n}\right]\left(1+\frac{1}{4n}\right), \qquad n \geqq n_0.$$

Thus, as $e^{2/3} < 2$, it is clear that the above inequality is valid for all $n \geqq n_0$ only if

$$q \leqq 6.$$

Since q is an arbitrary number which has only to satisfy the inequalities (2.19), it is obvious that

$$\sigma_1 \geqq \tfrac{1}{6}. \qquad \text{Q.E.D.}$$

3. Applications to Heat-Conduction Problems

We begin with the matrix differential equation

$$B\frac{d\mathbf{c}(t)}{dt} = -A\mathbf{c}(t) + \mathbf{g}, \qquad t > 0, \tag{3.1}$$

subject to the initial condition

$$\mathbf{c}(0) = \tilde{\mathbf{c}}. \tag{3.2}$$

Here, A and B are assumed to be *commuting Hermitian and positive definite* $N \times N$ matrices, and $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_N)^T$. The solution $\mathbf{c}(t)$ of (3.1)–(3.2) can be verified to be

$$\mathbf{c}(t) = A^{-1}\mathbf{g} + \exp(-tB^{-1}A)\{\tilde{\mathbf{c}} - A^{-1}\mathbf{g}\}, \qquad \text{for all } t \geqslant 0. \tag{3.3}$$

For any fixed nonnegative integers m and n with $0 \leqslant m \leqslant n$, let $\hat{r}_{m,n}(x) \equiv \hat{p}_{m,n}(x)/\hat{q}_{m,n}(x)$ denote the (m,n)-th Chebyshev rational approximation of e^{-x} in $[0, +\infty)$, i.e.,

$$\sup_{0 \leqslant x < \infty} |\hat{r}_{m,n}(x) - e^{-x}| = \lambda_{m,n}, \tag{3.4}$$

where $\lambda_{m,n}$ is defined in (1.2). Then, we define the (m,n)-th *Chebyshev approximation* $\mathbf{c}_{m,n}(t)$ of $\mathbf{c}(t)$ in (3.3), by

$$\mathbf{c}_{m,n}(t) = A^{-1}\mathbf{g} + \hat{r}_{m,n}(tB^{-1}A)\{\tilde{\mathbf{c}} - A^{-1}\mathbf{g}\}, \quad \text{for all } t \geqslant 0, \tag{3.5}$$

where $\hat{r}_{m,n}(tB^{-1}A)$ is the matrix formally given by

$$(\hat{q}_{m,n}(tB^{-1}A))^{-1} \cdot (\hat{p}_{m,n}(tB^{-1}A)).$$

From (3.3) and (3.5), we have

$$\mathbf{c}_{m,n}(t) - \mathbf{c}(t) = (\hat{r}_{m,n}(tB^{-1}A) - \exp(-tB^{-1}A))\{\tilde{\mathbf{c}} - A^{-1}\mathbf{g}\}, \quad t \geqslant 0. \quad (3.6)$$

We now associate with the positive definite Hermitian matrix B of (3.1), the particular vector norm

$$\|\mathbf{c}\|_B{}^2 \equiv \mathbf{c}^* B\mathbf{c} = \|B^{1/2}\mathbf{c}\|_2{}^2, \quad \text{where } \|\mathbf{v}\|_2{}^2 \equiv \mathbf{v}^* \cdot \mathbf{v}. \quad (3.7)$$

For any $N \times N$ matrix D, the induced operator norm of D is then

$$\|D\|_B \equiv \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|D\mathbf{x}\|_B}{\|\mathbf{x}\|_B} = \|B^{1/2} D B^{-1/2}\|_2 \equiv \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|B^{1/2} D B^{-1/2}\mathbf{x}\|_2}{\|\mathbf{x}\|_2}.$$

Using the facts that A and B are commuting Hermitian matrices, and the polynomials $\hat{p}_{m,n}(x)$ and $\hat{q}_{m,n}(x)$ are both real, we can write

$$\|\hat{r}_{m,n}(tB^{-1}A) - \exp(-tB^{-1}A)\|_B = \|\hat{r}_{m,n}(tB^{-1}A) - \exp(-tB^{-1}A)\|_2$$
$$= \max_{1 \leqslant i \leqslant N} |\hat{r}_{m,n}(t\lambda_i) - e^{-t\lambda_i}|, \quad \text{for all } t \geqslant 0,$$

where $\{\lambda_i\}_{i=1}^N$ denote the positive eigenvalues of $B^{-1}A$. But as $t\lambda_i \in [0, +\infty)$ for any nonnegative t and any eigenvalue λ_i, we evidently have from (3.4) that

$$\|\hat{r}_{m,n}(tB^{-1}A) - \exp(-tB^{-1}A)\|_B \leqslant \lambda_{m,n}, \quad \text{for all } t \geqslant 0. \quad (3.8)$$

Thus, taking norms in (3.6), gives us the *global* error bound

$$\|\mathbf{c}_{m,n}(t) - \mathbf{c}(t)\|_B \leqslant \lambda_{m,n}\|\tilde{\mathbf{c}} - A^{-1}\mathbf{g}\|_B, \quad \text{for all } t \geqslant 0. \quad (3.9)$$

To indicate how the inequality (3.9) can be used in the numerical solution of parabolic partial differential equations, we consider here the solution $u(x,t)$ of the simple one-dimensional heat-conduction problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + r(x), \quad 0 < x < 1, \quad t > 0, \quad (3.10)$$

subject to the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad \text{for all } t > 0, \quad (3.11)$$

and the initial condition

$$u(x,0) = \tilde{u}(x), \quad 0 \leqslant x \leqslant 1, \quad (3.12)$$

where $r(x)$ and $\tilde{u}(x)$ are given real functions on $[0,1]$. We remark that similar applications are valid in higher dimensions.

For any fixed positive integer N, let $h = 1/(N+1)$, and let $\{w_i(x)\}_{i=1}^N$ be the piecewise-linear functions defined by

$$w_i(x) = \left\{ \begin{array}{ll} 1 - \left(\frac{x - ih}{h}\right), & ih \leqslant x \leqslant (i+1)h, \\ 1 + \left(\frac{x - ih}{h}\right), & (i-1)h \leqslant x \leqslant ih, \\ 0, & x \notin [(i-1)h, (i+1)h] \end{array} \right\}, \qquad 1 \leqslant i \leqslant N. \tag{3.13}$$

The set S of all real linear combinations of the $w_i(x)$'s is known in the literature as an *Hermite space* (cf. [2], §6). All functions of S vanish at the endpoints of $[0,1]$.

The *semi-discrete Galerkin approximation* (cf. [6])

$$\hat{w}(x,t) \equiv \sum_{i=1}^{N} c_i(t)\, w_i(x), \qquad 0 \leqslant x \leqslant 1, \qquad t \geqslant 0, \tag{3.14}$$

of the solution $u(x,t)$ of (3.10)–(3.12), is determined by solving the matrix differential equation (3.1)–(3.2) for the functions $c_i(t)$, $1 \leqslant i \leqslant N$, where the matrices $B = (b_{i,j})$ and $A = (a_{i,j})$ have their entries explicitly defined by

$$b_{i,j} = \int_0^1 w_i(x)\, w_j(x)\, dx; \qquad a_{i,j} = \int_0^1 w_i'(x)\, w_j'(x)\, dx, \qquad 1 \leqslant i,j \leqslant N, \tag{3.15}$$

and where the vector $\mathbf{g}$ of (3.1) has components g_i defined by

$$g_i = \int_0^1 r(x)\, w_i(x)\, dx, \qquad 1 \leqslant i \leqslant N. \tag{3.16}$$

The vector $\tilde{\mathbf{c}}$ of (3.2) is determined from the coefficients of the best L^2-approximation in S of $\tilde{u}(x)$ of (3.12), i.e.,

$$\inf_{s \in S} \|\tilde{u} - s\|_{L^2[0,1]} = \left\| \tilde{u}(x) - \sum_{i=1}^{N} \tilde{c}_i\, w_i(x) \right\|_{L^2[0,1]}. \tag{3.17}$$

From (3.15), it can be verified that A and B are commuting real tridiagonal symmetric positive definite matrices, so that the inequality of (3.9) is applicable.

Based on energy-type inequalities, it can be deduced from [6], Theorem 1, that for $r(x)$ of (3.10) and $\tilde{u}(x)$ of (3.12) sufficiently smooth, there exists a constant K, independent of h and t, such that

$$\|\hat{w}(\cdot,t) - u(\cdot,t)\|_{L^2[0,1]} \leqslant Kh^2, \qquad \text{for all } t \geqslant 0. \tag{3.18}$$

On the other hand, for any $0 \leqslant m \leqslant n$, define the (m,n)th *Chebyshev-Galerkin approximation* of the solution of (3.10)–(3.12), as

$$\hat{w}_{m,n}(x,t) \equiv \sum_{i=1}^{N} c_{m,n,i}(t)\, w_i(x), \tag{3.19}$$

where the functions $c_{m,n,i}(t)$ are the components of the vector $\mathbf{c}_{m,n}(t)$ of (3.5). Now, using the definitions of (3.7) and (3.15), we verify that

$$\|\hat{w}_{m,n}(\cdot,t) - \hat{w}(\cdot,t)\|^2_{L^2[0,1]} = \int_0^1 \left\{ \sum_{i=1}^{N} (c_{m,n,i}(t) - c_i(t))\, w_i(x) \right\}^2 dx$$
$$= \|\mathbf{c}_{m,n}(t) - \mathbf{c}(t)\|_B^2. \qquad (3.20)$$

Hence, from (3.9), we have

$$\|\hat{w}_{m,n}(\cdot,t) - \hat{w}(\cdot,t)\|_{L^2[0,1]} \leqslant \lambda_{m,n} \|\tilde{\mathbf{c}} - A^{-1}\mathbf{g}\|_B, \qquad \text{for all } t \geqslant 0. \quad (3.21)$$

Thus, combining (3.18) and (3.21) gives

$$\|\hat{w}_{m,n}(\cdot,t) - u(\cdot,t)\|_{L^2[0,1]} \leqslant Kh^2 + \lambda_{m,n} \|\tilde{\mathbf{c}} - A^{-1}\mathbf{g}\|_B, \qquad \text{for all } t \geqslant 0. \quad (3.22)$$

The point of this global error analysis is that $\hat{w}_{m,n}(x,t)$ can be calculated for *any* $t \geqslant 0$ in just one step, in contrast with standard difference methods which arrive at an approximation for $u(x, m\Delta t)$ only after all intermediate approximations of $u(x, j\Delta t)$, $1 \leqslant j \leqslant m$, are computed.

We also remark that the difficult part in determining $\mathbf{c}_{m,n}(t)$ of (3.5) consists of solving the linear system of equations:

$$\hat{q}_{m,n}(tB^{-1}A)(\mathbf{c}_{m,n}(t) - A^{-1}\mathbf{g}) = \hat{p}_{m,n}(tB^{-1}A)(\tilde{\mathbf{c}} - A^{-1}\mathbf{q}). \qquad (3.23)$$

Since $\hat{p}_{m,n}(tB^{-1}A)$ enters into the computation of $\mathbf{c}_{m,n}(t)$ only as a matrix factor, there is little to be gained computationally by choosing $m < n$ in (3.5). For this basic reason, we were initially interested in the values of $\lambda_{n,n}$, as in [7].

4. The Constants $\lambda_{n,n}$ and $\lambda_{0,n}$

In this section, we give the explicit values of $\lambda_{0,n}$, $0 \leqslant n \leqslant 9$, in Table I, and of $\lambda_{n,n}$, $0 \leqslant n \leqslant 14$, in Table II. These numbers (and the associated rational functions $\hat{r}_{n,n}(x)$) were determined by using a Remez-type algorithm ([9], p. 173). The actual algorithm used is fully described in Cody, Fraser, and Hart [3].

Table I

n	$\lambda_{0,n}$
0	5.000 (−01)
1	9.357 (−02)
2	2.307 (−02)
3	6.353 (−03)
4	1.848 (−03)
5	5.553 (−04)
6	1.703 (−04)
7	5.294 (−05)
8	1.663 (−05)
9	5.264 (−06)

TABLE II

n	$\lambda_{n,n}$
0	5.000 (−01)
1	6.685 (−02)
2	7.359 (−03)
3	7.994 (−04)
4	8.653 (−05)
5	9.346 (−06)
6	1.008 (−06)
7	1.087 (−07)
8	1.172 (−08)
9	1.263 (−09)
10	1.361 (−10)
11	1.466 (−11)
12	1.579 (−12)
13	1.701 (−13)
14	1.832 (−14)

The following functions $r_{n,n}(x)$, $0 \leqslant n \leqslant 14$, constitute a partial *Walsh Table* (cf. [*4*], p. 162) for Chebyshev rational approximations of e^{-x} in $[0, +\infty)$.

TABLE III

$$e^{-x} \simeq \sum_{i=0}^{n} p_i x^i \Big/ \sum_{i=0}^{n} q_i x^i, \qquad 0 \leqslant x < \infty$$

i	p_i		q_i	
		$n = 1$		
0	1.0669	(00)	1.0000	(00)
1	−1.1535	(−01)	1.7275	(00)
		$n = 2$		
0	9.92641	(−01)	1.00000	(00)
1	−1.88332	(−01)	6.69295	(−01)
2	4.21096	(−03)	5.72258	(−01)
		$n = 3$		
0	1.00079 9	(00)	1.00000 0	(00)
1	−2.23657 8	(−01)	7.98292 3	(−01)
2	1.24996 2	(−02)	2.20411 5	(−01)
3	−9.98100 9	(−05)	1.24858 0	(−01)

TABLE III—*continued*

i	p_i		q_i	
		$n = 4$		
0	9.99913 47	(−01)	1.00000 00	(00)
1	−2.40253 73	(−01)	7.56683 22	(−01)
2	1.84005 09	(−02)	2.91754 68	(−01)
3	−4.49812 30	(−04)	4.57502 12	(−02)
4	1.67651 42	(−06)	1.93769 80	(−02)
		$n = 5$		
0	1.00000 935	(00)	1.00000 000	(00)
1	−2.50230 902	(−01)	7.50174 555	(−01)
2	2.24805 919	(−02)	2.69910 157	(−01)
3	−8.33629 264	(−04)	6.76687 392	(−02)
4	1.07797 622	(−05)	6.93457 968	(−03)
5	−2.19125 327	(−08)	2.34468 866	(−03)
		$n = 6$		
0	9.99998 991	(−01)	1.00000 000	(00)
1	−2.56774 988	(−01)	7.43173 208	(−01)
2	2.53896 499	(−02)	2.68982 436	(−01)
3	−1.17690 441	(−03)	6.15930 326	(−02)
4	2.48209 105	(−05)	1.13649 362	(−02)
5	−1.90699 255	(−07)	8.25674 222	(−04)
6	2.34264 503	(−10)	2.32303 566	(−04)
		$n = 7$		
0	1.00000 0109	(00)	1.00000 0000	(00)
1	−2.61399 8104	(−01)	7.38606 6424	(−01)
2	2.75489 3180	(−02)	2.66094 7238	(−01)
3	−1.46758 9943	(−03)	6.22100 6831	(−02)
4	4.06054 4787	(−05)	1.02296 0372	(−02)
5	−5.37067 6308	(−07)	1.48784 8134	(−03)
6	2.65391 0891	(−09)	8.08876 9796	(−05)
7	−2.11893 3743	(−12)	1.94833 4848	(−05)

TABLE III—*continued*

i	p_i		q_i	
		$n = 8$		
0	9.99999 98828	(−01)	1.00000 00000	(00)
1	−2.64834 06521	(−01)	7.35165 14516	(−01)
2	2.92069 90785	(−02)	2.64380 63240	(−01)
3	−1.71076 69530	(−03)	6.17188 04886	(−02)
4	5.63076 21623	(−05)	1.05208 36925	(−02)
5	−1.01477 31374	(−06)	1.32834 62347	(−03)
6	9.00129 46140	(−09)	1.59103 92054	(−04)
7	−3.03122 44065	(−11)	6.72702 00039	(−06)
8	1.66078 92788	(−14)	1.41677 26615	(−06)
		$n = 9$		
0	1.00000 00012 6	(00)	1.00000 00000 0	(00)
1	−2.67485 66991 9	(−01)	7.32514 42527 7	(−01)
2	3.05175 28366 6	(−02)	2.63030 80259 5	(−01)
3	−1.91477 63922 5	(−03)	6.15308 52837 4	(−02)
4	7.11036 34252 9	(−05)	1.04926 24819 4	(−02)
5	−1.56780 17352 5	(−06)	1.39500 66714 0	(−03)
6	1.95356 66646 4	(−08)	1.41160 60689 2	(−04)
7	−1.22095 56914 1	(−10)	1.43514 07391 1	(−05)
8	2.92870 66373 4	(−13)	4.85965 89227 3	(−07)
9	−1.14850 40902 2	(−16)	9.09160 46659 0	(−08)
		$n = 10$		
0	9.99999 99986 39	(−01)	1.00000 00000 00	(00)
1	−2.69593 55382 19	(−01)	7.30406 43483 33	(−01)
2	3.15778 64047 17	(−02)	2.61984 45116 28	(−01)
3	−2.08723 02875 56	(−03)	6.13598 78524 81	(−02)
4	8.46946 26115 79	(−05)	1.05222 25667 19	(−02)
5	−2.15295 78934 24	(−06)	1.40044 18324 30	(−03)
6	3.35954 01052 85	(−08)	1.51629 42863 89	(−04)
7	−3.02437 91657 93	(−10)	1.26706 04218 95	(−05)
8	1.38351 22001 13	(−12)	1.11798 57266 64	(−06)
9	−2.44794 47827 24	(−15)	3.10240 81250 45	(−08)
10	7.10595 74433 07	(−19)	5.22077 71857 74	(−09)

TABLE III—*continued*

i	p_i		q_i	
		$n = 11$		
0	1.00000 00000 147	(00)	1.00000 00000 000	(00)
1	−2.71308 69737 149	(−01)	7.28691 30396 820	(−01)
2	3.24525 83980 923	(−02)	2.61143 86818 246	(−01)
3	−2.23434 38385 867	(−03)	6.12306 52273 991	(−02)
4	9.70327 53192 328	(−05)	1.05372 96909 746	(−02)
5	−2.74176 69166 461	(−06)	1.41501 45103 337	(−03)
6	5.02362 65041 453	(−08)	1.53331 52911 587	(−04)
7	−5.77549 91658 630	(−10)	1.39348 62063 194	(−05)
8	3.88619 42441 125	(−12)	9.83157 36139 162	(−07)
9	−1.34133 12302 919	(−14)	7.65774 19206 722	(−08)
10	1.80098 07948 555	(−17)	1.77400 34787 683	(−09)
11	−3.97762 94455 404	(−21)	2.71280 51110 139	(−10)
		$n = 12$		
0	9.99999 99999 8420	(−01)	1.00000 00000 0000	(00)
1	−2.72732 01038 1007	(−01)	7.27267 98946 2094	(−01)
2	3.31862 74887 8945	(−02)	2.60454 26687 4069	(−01)
3	−2.36102 86093 3434	(−03)	6.11258 94397 4162	(−02)
4	1.08182 04721 4783	(−04)	1.05516 59564 1775	(−02)
5	−3.31706 70455 2847	(−06)	1.42483 77048 4151	(−03)
6	6.85640 06647 2736	(−08)	1.56277 80799 5398	(−04)
7	−9.40255 67465 0549	(−10)	1.41984 90819 4392	(−05)
8	8.21592 17852 2494	(−12)	1.10762 17782 0598	(−06)
9	−4.24605 37294 1828	(−14)	6.71223 49299 4341	(−08)
10	1.13357 45322 5507	(−16)	4.67724 66320 0848	(−09)
11	−1.18241 93272 9819	(−19)	9.18591 91007 0282	(−11)
12	2.03287 74252 3846	(−23)	1.28708 77757 5584	(−11)
		$n = 13$		
0	1.00000 00000 00170	(00)	1.00000 00000 00000	(00)
1	−2.73931 40321 02750	(−01)	7.26068 59680 79845	(−01)
2	3.38101 16410 46875	(−02)	2.59878 71290 03795	(−01)
3	−2.47106 93187 70823	(−03)	6.10400 14004 83738	(−02)
4	1.18246 13397 91637	(−04)	1.05636 62149 46976	(−02)
5	−3.86917 56932 66464	(−06)	1.43339 64981 47585	(−03)
6	8.78467 34854 71303	(−08)	1.58368 07943 13270	(−04)
7	−1.37667 89576 47893	(−09)	1.46094 52103 19625	(−05)
8	1.45460 93630 79049	(−11)	1.13726 98219 71446	(−06)
9	−9.90411 24433 78351	(−14)	7.74828 74238 93217	(−08)
10	4.02014 83515 52472	(−16)	4.08864 75641 62874	(−09)
11	−8.47538 24867 61699	(−19)	2.57666 92606 40894	(−10)
12	7.00505 36680 34527	(−22)	4.34551 90724 16265	(−12)
13	−9.55806 72950 74149	(−26)	5.61852 05844 48164	(−13)

TABLE III—*continued*

i	p_i		q_i	
	$n = 14$			
0	9.99999 99999 99816 8	(−01)	1.00000 00000 00000 0	(00)
1	−2.74956 04296 30004 3	(−01)	7.25043 95703 48866 6	(−01)
2	3.43469 84175 67147 5	(−02)	2.59390 94125 01801 2	(−01)
3	−2.56744 39819 02861 8	(−03)	6.09681 85127 28359 5	(−02)
4	1.27340 70715 23318 1	(−04)	1.05740 49161 69115 6	(−02)
5	−4.39328 08492 51123 6	(−06)	1.44055 27154 58731 6	(−03)
6	1.07532 02054 48522 7	(−07)	1.60192 11440 61219 0	(−04)
7	−1.87102 55961 08945 3	(−09)	1.49082 34724 80024 2	(−05)
8	2.28495 15765 30015 5	(−11)	1.18202 91576 35577 2	(−06)
9	−1.90386 40942 83534 5	(−13)	8.01639 97982 35750 3	(−08)
10	1.03101 51365 35049 5	(−15)	4.83618 00878 64828 1	(−09)
11	−3.34902 80333 66753 3	(−18)	2.24720 30400 42852 9	(−10)
12	5.67438 25539 52350 1	(−21)	1.29228 02705 79277 9	(−11)
13	−3.77945 23874 50329 5	(−24)	1.89218 38854 02244 9	(−13)
14	4.16098 26642 37661 3	(−28)	2.27106 80218 89129 5	(−14)

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