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Nested Bounds for the Spectral Radius

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Nested bounds for the spectral radius of a matrix are of great importance in many problems of approximately solving linear systems. Specifically, for the class of nonnegative matrices, these bounds are used to obtain acceleration parameters for iterative methods, as shown in [23, Ch. 9]. The importance of the class of nonnegative matrices has been recently again emphasized by Kulisch [9] in his theory of nonnegative majorants for the approximate solution of linear systems with complex matrices.

A well known principle for obtaining nested bounds for the spectral radius of a nonnegative irreducible matrix is the Collatz "Quotientensatz" [2, 3] (see also [23, p. 32]). This principle has been generalized by many authors in various directions [1, 5, 14].

Another principle for obtaining nested bounds for the spectral radius is Yamamoto's principle [24]. Some generalizations of Yamamoto's principle are contained in [12].

Both these above mentioned principles have been used for obtaining nested bounds for the spectral radius of the polynomial eigenvalue problem

$$\sum_{k=0}^{m-1} \lambda^k A_{m-k}\, x = \lambda^m A_0\, x,$$

where m is a positive integer and $A_0, \ldots, A_m$ are linear transformations on a given Banach space [4, 13].

The methods of Collatz and Yamamoto have been combined recently by Hall and Spanier [6], and a new hybrid method has been derived for obtaining nested bounds for the spectral radius of a certain class of matrices which contains as a subclass the class of nonnegative matrices. Hall and Spanier have shown connections between all the mentioned methods, and they also have discussed their advantages and disadvantages.

The aim of this paper is on one hand to show that the methods of Collatz, Yamamoto, and Hall-Spanier are applicable also in infinite dimensional Banach spaces, and on the other hand to unify the methods of proof. In addition, new results in finite-dimensional cases are also obtained.

§ 1. Definitions and Notations

Let Y be a real Banach space, and let X be the complex extension of the space Y, i.e., $z \in X$ iff $z = x + i y$ where $x, y \in Y$ and $i^2 = -1$. Denoting the norm in the space Y by $\|\cdot\|_Y$, then X can be normed by defining

$$\|z\|_X = \sup_{0 \leq \theta \leq 2\pi} \|\cos\theta \cdot x + \sin\theta \cdot y\|_Y .$$

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Further, let Y' denote the dual space of continu
and let $[Y]$ denote the space of bounded linear op
The norms in Y' and $[Y]$ are defined as usual by

$$\|y'\|_{Y'} = \sup\{|y'(y)|: \ \|y\|_Y = 1\}, \quad \text{where } y$$

$$\|T\|_{[Y]} = \sup\{\|Ty\|_Y: \ \|y\|_Y = 1\}, \quad \text{where } y$$

With these definitions, X, Y', and $[Y]$ are also Ban

We assume that there exists a (closed) cone[1] K in
i.e., for every $x \in Y$, there exist $u, v \in K$ such that
there exists a $\delta > 0$ such that $\|u+v\|_Y \geqq \delta \|u\|_Y$ for all
(cf. [7]). We then write that $x \geqq y$ or equivalently
we denote the *dual cone* by K', i.e.,

$$K' = \{x' \in Y': \ \langle x, x' \rangle \geqq 0 \text{ for all}$$

where $\langle x, x' \rangle$ denotes the number $x'(x)$.

A subset $H' \subset K'$ is called *K-total* [14] iff $\langle x, x'$
that $x \in K$. The fact that K-total sets exist follows f
the extension of a positive linear functional from a s
Thus, K' is itself a K-total set.

An operator $T \in [Y]$ can be extended to an ope
by the formula $\widetilde{T}z = Tx + iTy$, where $z = x + iy \in X$
the space of all bounded linear operators mapping X in

If $T \in [X]$, then $\sigma(T)$ denotes its *spectrum*, i.e.,
the set of all complex numbers λ for which the re
$(\lambda I - T)^{-1}$ is an element of $[X]$, and $r(T)$ further de

$$r(T) = \sup\{|\lambda|: \ \lambda \in \sigma(T)\}.$$

By definition, we put $\sigma(T) = \sigma(\widetilde{T})$ and $r(T) = r(\widetilde{T})$ i

An operator $T \in [X]$ or $T \in [Y]$ is said to have *prop*
$\lambda \in \sigma(T)$ and $|\lambda| = r(T)$, it follows that λ is an iso
operator $R(\lambda, T) = (\lambda I - T)^{-1}$ or $R(\lambda, \widetilde{T}) = (\lambda I - \widetilde{T})$
notes the identity operator. This implies that there
λ in $\sigma(T)$ with $|\lambda| = r(T)$. The restriction to oper
motivated by the fact that such operators occur freq

An operator $T \in [Y]$ is called *positive* (or more
$x \in K$ implies $Tx \in K$, i.e., $T: K \to K$. A positive ope
nonsupporting [17] iff, for every pair $x \in K$, $x' \in K'$ w
0 denotes the zero vectors in both Y and Y'), the
$p = p(x, x')$ such that $\langle T^p x, x' \rangle \neq 0$. A positive o
called *nonsupporting* [17] iff, for every pair $x \in K$, x'
there exists a positive integer $p = p(x, x')$ such that

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1 A nonempty subset $K \subset Y$ is a *cone* iff (i) for an
scalars $\alpha \geqq 0$ and $\beta \geqq 0$, (ii) K is closed, and (iii) if $x \in K$
0 is the zero element of Y.

We remark that in real m-dimensional Euclidean space E^m with the cone K consisting of all vectors with nonnegative components, the class of semi-nonsupporting operators is identical with the class of nonnegative irreducible $m\times m$ matrices, and the class of nonsupporting operators is similarly identical with the set of all primitive nonnegative irreducible $m\times m$ matrices.

An element $x\in K$ is called *quasi-interior* iff $\langle x, x'\rangle \neq 0$ for all $x'\in K'$ with $x'\neq 0$.

For x a fixed vector in Y with $x\neq 0$, let $T\in[Y]$, and let H' be any K-total set. If R denotes the real numbers, we then define

$$(1.1)\qquad \begin{aligned} r_x(T) &= r_x(T, H') = \sup\{\varrho\in R:\ \langle Tx, x'\rangle \geqq \varrho\langle x, x'\rangle \text{ for all } x'\in H'\},\\ r^x(T) &= r^x(T, H') = \inf\{\varrho\in R:\ \langle Tx, x'\rangle \leqq \varrho\langle x, x'\rangle \text{ for all } x'\in H'\},\end{aligned}$$

where $r_x(T)\equiv-\infty$ if $\langle x, x'\rangle = 0$ and $\langle Tx, x'\rangle < 0$, and $r^x(T)\equiv+\infty$ if $\langle x, x'\rangle = 0$ and $\langle Tx, x'\rangle > 0$.

With these functionals, we can further define the following functionals if $T^p x\neq 0$ for all $p=0, 1, 2, \ldots$:

$$(1.2)\qquad \gamma(p)=\gamma(p, x, T, H')=r_{T^p x}(T);\quad \Gamma(p)=\Gamma(p, x, T, H')=r^{T^p x}(T),\quad p=0, 1, 2, \ldots.$$

$$(1.3)\qquad \begin{aligned} \delta(p) &= \delta(p, x, T, H') = [r_x(T^{2^{p-1}})]^{2^{-p+1}}\\ \Delta(p) &= \Delta(p, x, T, H') = [r^x(T^{2^{p-1}})]^{2^{-p+1}}, \qquad p=1, 2, \ldots,\end{aligned}$$

$$(1.4)\qquad \begin{aligned} \eta(p) &= \eta(p, x, T, H') = [r_{T^{2^{p-2}}x}(T^{2^{p-2}})]^{2^{-p+2}}\\ H(p) &= H(p, x, T, H') = [r^{T^{2^{p-2}}x}(T^{2^{p-2}})]^{2^{-p+2}}, \qquad p=2, 3, \ldots,\end{aligned}$$

where $\eta(1)\equiv r_x(T)$, and $H(1)\equiv r^x(T)$.

We remark that the quantities $\gamma(p)$, $\Gamma(p)$ reduce to the familiar Collatz bounds [2], $\delta(p)$, $\Delta(p)$ to the Yamamoto bounds [24], and $\eta(p)$, $H(p)$ to the Hall and Spanier hybrid bounds [6] for the spectral radius $r(T)$ of a nonnegative irreducible $n\times n$ matrix T, when the cone K is the set of all nonnegative vectors, the K-total set H' is chosen to be $\{e'_j\}_{j=1}^n$ where if $y=(y_1, y_2, \ldots, y_n)$ is any vector in E^n, then $\langle y, e'\rangle \equiv y_j$, $1\leqq j\leqq n$, and x is any vector with positive components. As indicated in (1.2)—(1.4), these bounds in general depend upon the particular choice of x. However, the dependence on H' is only formal, because

$$\begin{aligned} r_x(T) &= \sup\{\varrho\in R:\ (Tx-\varrho x)\in K\},\\ r^x(T) &= \inf\{\tau\in R:\ (\tau x - Tx)\in K\}\end{aligned}$$

and $r_x(T)=-\infty$, $r^x(T)=+\infty$, if the corresponding sets over which the sup and inf are taken are empty respectively.

We further remark that the initial restriction to real Banach spaces is not essential, and can be removed by using results of Schaefer [19].

§ 2. Preliminary Results

In §§ 3—4, some relations between the functionals $\gamma(p)$, $\Gamma(p)$, $\delta(p)$, $\Delta(p)$, $\eta(p)$, and $H(p)$ will be developed for certain classes of linear operators. The purpose of this section is to formulate some preliminary results which will be useful in establishing these relations.

Lemma 1. Let $T \in [Y]$, let $x \in K$ with $x \neq 0$, and a
are both finite. Then

(2.1) $$T x \geqq r_x(T) x \quad \text{and} \quad T x \leqq r^x($$

Hence, if $r_x(T) \geqq 0$, then $T x \in K$, and if $r^x(T) \leqq 0$, th

Proof. Since $r_x(T)$ is finite, we have from (1.1) f

$$\langle T x - \varrho x, x' \rangle \geqq 0 \quad \text{for all} \quad x$$

But since H' is a K-total subset of K', then $(T x -$
the same is evidently true for $r_x(T)$, i.e., $(T x - r_x(T) x)$
which proves the first inequality of (2.1). If $r_x(T) \geqq$
remainder of the lemma follows similarly. Q.E.D.

Lemma 2. Let B and T, both in $[Y]$, be positive
Then, for any $x \in K$ with $B x \neq 0$ such that $r^x(T)$ is f

(2.2) $$r_{Bx}(T) \geqq r_x(T) \quad \text{and} \quad r^{Bx}(T) \leqq$$

Proof. From Lemma 1, $T x \geqq r_x(T) x$ for any $x \in$
positive operator B and using the commutativity of

$$T B x = B T x \geqq r_x(T) B x$$

Consequently, $\langle T B x, x' \rangle \geqq r_x(T) \langle B x, x' \rangle$ for all x
inequality of (2.2) then follows. The second inequalit
lished. Q.E.D.

The following theorem contains the basic prope
$r(T)$ of the operator T for the class of operators chosen
we need some facts concerning spectral properties of
operator (cf. [22, p. 305]).

Let λ_j be any isolated singularity of the resolvent op
where $T \in [X]$ or $T \in [Y]$. Then,

(2.3) $$R(\lambda, T) = \sum_{k=0}^{\infty} A_{j,k} (\lambda - \lambda_j)^k + \sum_{k=1}^{\infty} B_{j,}$$

is a Laurent expansion of $R(\lambda, T)$ in a neighborho
$A_{j,k}$, $B_{j,k+1} \in [X]$ for all $k \geqq 0$, and that

(2.4) $$B_{j,1} = \frac{1}{2\pi i} \int_{C_j} R(\lambda, T)\, d\lambda, \qquad B_{j,k+1} = (T - \lambda_j$$

where $C_j \equiv \{\lambda : |\lambda - \lambda_j| = \varrho_j,\ \varrho_j > 0\}$ is such that if K_j
then $K_j \cap \sigma(T) = \{\lambda_j\}$.

If f is any polynomial, then

(2.5) $$\begin{aligned} f(T) &= \frac{1}{2\pi i} \int_{C} f(\lambda)\, R(\lambda, T)\, d\lambda \\ &= \sum_{k=1}^{\infty} \frac{f^{(k-1)}(\lambda_j)}{(k-1)!} B_{j,k} + \frac{1}{2\pi i} \end{aligned}$$

where $C=\{\lambda: |\lambda|=r(T)+\varepsilon\}$, and C' is any closed Jordan curve which contains in its interior the remaining parts of the spectrum $\sigma(T)-\{\lambda_j\}$. In particular, since T has property S, let $\lambda_1, \ldots, \lambda_s \in \sigma(T)$ denote the points of $|z|=r(T)$ which are poles of $R(\lambda, T)$. Then,

$$(2.5') \qquad f(T)=\sum_{j=1}^{s}\sum_{k=1}^{\infty}\frac{f^{(k-1)}(\lambda_j)}{(k-1)!}B_{j,k}+\frac{1}{2\pi i}\int_{C''} f(\lambda)\,R(\lambda, T)\,d\lambda,$$

where $C''=\{\lambda: |\lambda|=\varrho''\}$ is such that if $L''=\{\lambda: |\lambda|\leqq\varrho''\}$, then $L''\cap\sigma(T)=\sigma(T)-\{\lambda_1, \ldots, \lambda_s\}$. The representation of (2.5) will be used later in §§ 3 and 5.

If $T\in[X]$ has property S, let $\lambda_j\in\sigma(T)$ denote the pole of order g_j of $R(\lambda, T)$ lying on the circumference $|z|=r(T)>0$ for $j=1, 2, \ldots, s$. Then we have [11] that

$$(2.6) \qquad \lim_{n\to\infty}\left\|\frac{\lambda_j^{-(g_j-1)}}{(g_j-1)!}B_{j,g_j}-\frac{1}{n}\sum_{k=1}^{n}k^{-g_j+1}\lambda_j^{-k}T^k\right\|_{[X]}=0,$$

which is written as

$$(2.6') \qquad \frac{\lambda_j^{-(g_j-1)}}{(g_j-1)!}B_{j,g_j}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}k^{-g_j+1}\lambda^{-k}T^k$$

for all λ_j for which the corresponding multiplicity g_j satisfies $g_j\geqq g_\ell$ for $\ell=1, 2, \ldots, s$.

In particular, if $T\in[Y]$ is a positive operator with property S and $r(T)>0$, then it is known from Schaefer's extension [20] of the Pringsheim Theorem that the eigenvalue $\lambda=r(T)$ has maximal multiplicity, say g, with respect to all singularities of $R(\lambda, T)$ on $|\lambda|=r(T)$. Thus, all the operators

$$S_n^{(g)}=\frac{1}{n}\sum_{k=1}^{n}k^{-g+1}[r(T)]^{-k}T^k,$$

are *positive*, and hence so is the limit as n tends to infinity. Evidently, from (2.6),

$$(2.7) \qquad \frac{(r(T))^{-(g-1)}}{(g-1)!}B_{1,g}=\lim_{n\to\infty}S_n^{(g)} \quad (r(T)>0),$$

where $B_{1,g}$ will always denote the member of the spectral decomposition corresponding to the eigenvalue $r(T)$ having maximum multiplicity g. Since the operator $S_n^{(g)}$ commutes with T for every n, we remark that the same is also true for $B_{1,g}$ from (2.6).

In the special case that $T\in[Y]$ is a positive operator with property S but with $r(T)=0$, then since T has property S, $R(\lambda, T)$ has a finite Neumann series development:

$$R(\lambda, T)=\frac{I}{\lambda}+\frac{T}{\lambda^2}+\cdots+\frac{T^{g-1}}{\lambda^g} \text{ for any } \lambda\neq 0.$$

Thus, from the expansions of (2.3), it follows that $B_{1,1}=I$, and

$$(2.7') \qquad B_{1,g}=T^{g-1} \quad (r(T)=0),$$

which is also a positive operator which commutes with T.

If g_j is the multiplicity of λ_j as a pole of $R(\lambda, T$
$B_{j,k+1}=0$, where 0 denotes the zero operator, for
$B_{j,g_j}x$ for any $x\in K$ is either the zero vector, or an
to the eigenvalue λ_j. In particular, $B_{1,g}x$ for $x\in K$,
an eigenvector of T corresponding to the spectral ra

Theorem 2.1. Let $T\in[Y]$ be a positive operator h

(2.8) $\max\{r_x(T):\ x\in K \text{ and } B_{1,g}x\neq 0\}=\min\{r^x(T)$
$=r(T).$

Proof. We consider the case where $r(T)>0$, the
being similar. Since T is a positive operator with pr
in (2.7), is a positive operator which commutes wi
Lemma 2 for any $x\in K$ with $B_{1,g}x\neq 0$, such that $r^x($

(2.9) $r^x(T)\geqq r^{B_{1,g}x}(T)=r(T)=r_{B_{1,g}x}($

Consequently,

(2.9') $\inf\{r^x(T): x\in K \text{ and } B_{1,g}x\neq 0\}\geqq r(T)\geqq \sup\{$

However, for the particular choice $y\equiv B_{1,g}x\neq 0$ in K
$r(T)=r_y(T)$, which, together with (2.9'), establishes

For $T\in[Y]$ a positive operator with property S
$x\in K$ with $B_{1,g}\neq 0$ such that $r^x(T)$ is finite, we have

(2.10) $0\leqq r_x(T)\leqq r(T)\leqq r^x(T)$

If $r(T)>0$ and $B_{1,g}x\neq 0$, then since T^n and $B_{1,g}$ co

$$B_{1,g}(T^n x)=T^n(B_{1,g}x)=(r(T))^n B_{1,g}x\neq($$

Thus, $T^n x\neq 0$ for all $n\geqq 1$, and applying Lemma 2

(2.11) $0\leqq r_x(T)\leqq\cdots\leqq r_{T^n x}(T)\leqq\cdots\leqq r(T)\leqq\cdots$

for all $n\geqq 1$. If $r(T)=0$, then since $B_{1,g}=T^{g-1}$ from (
for any $x\in K$ and any $k\geqq g$. It is then convenient to
for all $k\geqq g-1$. With this definition, the inequalitie
this case $r(T)=0$ as well.

We conclude this section with a sequence of lem
the following sections.

Lemma 3. Let $T\in[Y]$ be a positive operator. Th

(2.12) $r_x(T)=\inf\left\{\frac{\langle Tx, x'\rangle}{\langle x, x'\rangle}:\ x'\in H' \text{ with}\right.$

and

(2.13) $r^x(T)=\sup\left\{\frac{\langle Tx, x'\rangle}{\langle x, x'\rangle}:\ x'\in\right.$

where $r^x(T)=+\infty$ if $\langle x, x'\rangle=0$ and $\langle Tx, x'\rangle>0$ f

Proof. Define

$$\varrho = \inf\left\{\frac{\langle T x, x'\rangle}{\langle x, x'\rangle}:\ x'\in H' \text{ with } \langle x, x'\rangle \neq 0\right\}.$$

Then, since T is a positive operator and $x\in K$, we evidently have

$$\langle T x, x'\rangle \geqq \varrho \langle x, x'\rangle$$

for *all* $x'\in H'$. Hence, from (1.1), $\varrho \leqq r_x(T)$. If $\varrho < r_x(T)$, then, from the definition of ϱ, there would exist at least one $\hat{x}'\in H'$ with $\langle x, \hat{x}'\rangle \neq 0$ such that

$$\langle T x, \hat{x}'\rangle < r_x(T)\,\langle x, \hat{x}'\rangle.$$

But this contradicts the fact (cf. Lemma 1) that $r_x(T)\,\langle x, x'\rangle \leqq \langle T x, x'\rangle$ for all $x'\in H'$. The remainder of this lemma follows similarly. Q.E.D.

Lemma 4. Let $T\in[Y]$ be a positive operator. Let $x\in K$, $x\neq 0$. Then,

$$r_x(T) \leqq r(T). \tag{2.14}$$

If $r(T)$ is an isolated singularity of the resolvent operator $R(\lambda, T)$, $x\in K$ is such that

$$B_{1,g}\,x \neq 0, \qquad B_{1,g+1}\,x = 0 \tag{2.15}$$

for some positive integer g, where $B_{1,1}, B_{1,2}, \ldots$ are defined in (2.4) (see also (2.7)), and if

$$\lim_{n\to\infty} \|\gamma_n T^n x - B_{1,g}\,x\|_X = 0 \tag{2.16}$$

for some sequence $\{\gamma_n\}_{n=1}^{\infty}$ for which

$$\lim_{n\to\infty} \frac{\gamma_n}{\gamma_{n+1}} = r(T), \tag{2.17}$$

then

$$r^{T^n x}(T) \geqq r(T) \tag{2.18}$$

for $n = 0, 1, \ldots$.

Proof. If $r_x(T) > r(T)$, then

$$\left(I - \frac{1}{r_x(T)}\,T\right)^{-1} = \sum_{k=0}^{\infty} (r_x(T))^{-k}\,T^k$$

is a positive operator, and hence, according to Lemma 1,

$$-x = \frac{1}{(r_x(T))}\left(I - \frac{1}{r_x(T)}\,T\right)^{-1}(T x - r_x(T)\,x) \geqq 0,$$

from which it follows that $x = 0$, a contradiction proving (2.14).

For any $x'\in H'$ for which $\langle T^n x, x'\rangle > 0$, we have from Lemma 3

$$r^{T^n x}(T) \geqq \frac{\langle T^{n+1} x, x'\rangle}{\langle T^n x, x'\rangle}.$$

Choose $\varepsilon > 0$ arbitrarily. The first assumption in (2.15) guarantees the existence of an element $\hat{x}'\in H'$ for which $\langle B_{1,g}\,x, \hat{x}'\rangle \neq 0$. According to the assumptions of (2.16) and (2.17),

$$\frac{\langle T^{n+1}x, \hat{x}'\rangle}{\langle T^n x, \hat{x}'\rangle} > r(T) - \varepsilon$$

because $\langle B_{1,g}x, \hat{x}'\rangle \neq 0$ implies that $\langle T^n x, \hat{x}'\rangle > 0$
Summarizing, we obtain

$$r^{T^n x}(T) \geqq \frac{\langle T^{n+1}x, \hat{x}'\rangle}{\langle T^n x, \hat{x}'\rangle} > r(T$$

Since $\varepsilon > 0$ was arbitrary, (2.18) then follows from

$$r^{T^{n-1}x}(T) \geqq r^{T^n x}(T),$$

which is a consequence of the relation

$$r^x(T)\, x \geqq T x$$

which follows from Lemma 1 if $r^x(T) < \infty$. For $r^{T^n x}$
there is nothing to prove in (2.18). Q.E.D.

Remark. Condition (2.17) is fulfilled with

$$\gamma_n = \frac{(g-1)!}{n^{g-1}} \left(r(T)\right)^{-n+g-}$$

if $r(T)$ is an isolated dominant eigenvalue of T. (Se

Lemma 5. Let $T \in [Y]$ be a positive operator wi
isolated singularity of $R(\lambda, T)$, and let $\{x_n\}_{n=1}^{\infty} \in K$ a

(2.19) $$\lim_{n\to\infty} \|x_n - B_{1,g}\, x\|_X = 0,$$

and the relations

(2.20) $$x_n \leqq \omega\, B_{1,g}\, x \neq 0$$

hold for n sufficiently large with $0 < \omega < +\infty$, ω in

(2.21) $$r_{x_n}(T) \leqq r_{x_{n+1}}(T),$$

and

(2.21′) $$r^{x_{n+1}}(T) \leqq r^{x_n}(T)$$

for all $n \geqq 1$.

Then,

(2.22) $$\lim_{n\to\infty} r_{x_n}(T) = \lim_{n\to\infty} r^{x_n}(T) =$$

Proof. From the assumption (2.21) of monotoni

$$\varrho = \lim_{n\to\infty} r_{x_n}(T)$$

exists, and, from (2.14) of Lemma 4,

(2.23) $$\varrho \leqq r(T).$$

We shall prove that $\varrho \geqq r(T)$. For every $\varepsilon > 0$, there exists an element $x'_\varepsilon \in H'$ such that

$$\langle T x_n, x'_\varepsilon \rangle < (\varrho + \varepsilon) \langle x_n, x'_\varepsilon \rangle \tag{2.24}$$

for all n sufficiently large; otherwise, $\langle T x_n, x' \rangle \geqq (\varrho + \varepsilon) \langle x_n, x' \rangle$ would hold for *all* $x' \in H'$. But then, it would follow that $r_{x_n}(T) \geqq \varrho + \varepsilon$, a contradiction. It follows from (2.24) that $\langle x_n, x'_\varepsilon \rangle > 0$, and consequently from (2.20), $\langle B_{1,g} x, x'_\varepsilon \rangle \neq 0$. But then, using (2.19),

$$\lim_{n \to \infty} \frac{\langle T x_n, x'_\varepsilon \rangle}{\langle x_n, x'_\varepsilon \rangle} = r_{B_{1,g} x}(T) = r(T),$$

and thus, to every $\varepsilon_1 > 0$ there exists a positive integer N such that

$$\frac{\langle T x_n, x'_\varepsilon \rangle}{\langle x_n, x'_\varepsilon \rangle} > r(T) - \varepsilon_1$$

for $n > N$. According to (2.24), we obtain

$$\varrho + \varepsilon > r(T) - \varepsilon_1$$

and, since $\varepsilon > 0$ and $\varepsilon_1 > 0$ were arbitrary, $r(T) \leqq \varrho$. We have thus proved the first part of (2.22). The remaining part can be proved similarly using the fact that (2.19) and (2.21′) guarantee, as in the proof of Lemma 4 the validity of the relation

$$r^{x_n}(T) \geqq r(T)$$

for all $n \geqq 1$. Q.E.D.

Remark. It is easy to see that Lemma 5 fails in general if (2.20) does not hold, as the following example shows. Let Y be two-dimensional Euclidean space E^2, let K be the cone of vectors with nonnegative components, let $H' = \{x'_1 = (1, 0),\ x'_2 = (0, 1)\}$, and let

$$T = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

with $0 < \alpha < \beta$. Defining

$$x_n = \begin{bmatrix} 1/n \\ 1 \end{bmatrix}, \quad n \geqq 1, \quad x = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we find that $g = 1$ and $B_{1,1} x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lim_{n \to \infty} x_n$. Moreover, $r_{x_n}(T) = \alpha$ and $r^{x_n}(T) = \beta = r(T)$ for all $n \geqq 1$. Thus, (2.19), (2.21), and (2.21′) are all satisfied, while (2.20) is not, and it is clear that the conclusion (2.22) of Lemma 5 fails in this case.

§ 3. Convergence Theorems

In this section, we shall give some conditions which guarantee the convergence of the sequences defined in (1.2)—(1.4) to the spectral radius $r(T)$ of a positive operator $T \in [Y]$.

Theorem 3.1. Let $T \in [Y]$ be a positive operator with property S. Let $x \in K$ be such that $B_{1,g} x \neq 0$, where $B_{1,g}$ is defined in (2.7) or (2.7′), and such that

$r^x(T)$ is finite. Then,

(3.1) $\gamma(0) \leqq \cdots \leqq \gamma(p) \leqq \cdots \leqq r(T) \leqq \cdots \leqq I$

(3.2) $\delta(1) \leqq \cdots \leqq \delta(p) \leqq \cdots \leqq r(T) \leqq \cdots \leqq \Delta$

and

(3.3) $\eta(1) \leqq \cdots \leqq \eta(p) \leqq \cdots \leqq r(T) \leqq \cdots \leqq H$

Proof. According to our definitions, the inequaliti in the trivial case when $r(T)=0$. To prove any c (3.1)—(3.3) in the nontrivial case, it suffices to pr left, i.e., for example $\eta(p-1) \leqq \eta(p)$, since the p equalities are similar.

That $\gamma(p-1) \leqq \gamma(p)$, follows directly from (2.11). consider the positive operator $T^{2^{p-2}}$. Then, from L $B_{1,g}\, x \neq 0$,

$$r_x(T^{2^{p-2}})\, x \leqq T^{2^{p-2}} x.$$

Applying the operator $T^{2^{p-2}}$ results in

$$r_x(T^{2^{p-2}})\, T^{2^{p-2}} x \leqq T^{2^{p-1}} x$$

from which it follows that

$$[r_x(T^{2^{p-2}})]^2 x \leqq T^{2^{p-1}} x.$$

Consequently, $\langle T^{2^{p-1}} x, x' \rangle \geqq [r_x(T^{2^{p-2}})]^2 \langle x, x' \rangle$ for a

$$r_x(T^{2^{p-1}}) \geqq [r_x(T^{2^{p-2}})]^2.$$

Thus, from the definition of (1.3), $[\delta(p)]^{2^{p-1}} \geqq [\delta(p$ $\delta(p-1)$.

Next, for the positive operator $T^{2^{p-2}}$, Lemma 1 ap gives

$$T^{2^{p-1}} x \geqq r_{T^{2^{p-2}}x}(T^{2^{p-2}})\, T^{2^{p-}}$$

Applying the operator $T^{2^{p-1}}$ further results in

$$T^{2^p} x \geqq r_{T^{2^{p-2}}x}(T^{2^{p-2}})\, T^{2^{p-1}}\, T^{2^{p-2}} x \geqq [r_{T^{2^{p-2}}}$$

from which it follows that

$$[r_{T^{2^{p-2}}x}(T^{2^{p-2}})]^2 \leqq r_{T^{2^{p-1}}x}(T^{2^p}$$

Consequently, $\eta(p) \leqq \eta(p+1)$.

It remains to prove that $\gamma(p)$, $\delta(p)$, and $\eta(p)$ From (2.11), we know this is true for $\gamma(p)$. For $\delta($ (2.10) that

$$\delta(p) = [r_x(T^{2^{p-1}})]^{2^{-(p-1)}} \leqq [r(T^{2^{p-1}})]^{2^{-(}}$$

and

$$\eta(p) = [r_{T^{2^{p-2}}x}(T^{2^{p-2}})]^{2^{-(p-2)}} \leqq [r(T^{2^{p-}}$$

which completes the proof. Q.E.D.

The next result gives the convergence of the sequences $\{\gamma(p)\}_{p=0}^{\infty}$, $\{\delta(p)\}_{p=1}^{\infty}$, etc. to $r(T)$ in the dominant case for a class of positive operators which includes operators whose spectral radius is a pole of $R(\lambda, T)$, i.e., operators with property S.

Theorem 3.2. Let T be a positive operator with $r(T)>0$ such that $r(T)$ is a dominant isolated eigenvalue. Suppose that $x \in K$ is such that (2.15) holds. Then we have

$$\lim_{p\to\infty} \gamma(p) = \lim_{p\to\infty} \Gamma(p) = r(T), \tag{3.4}$$

$$\lim_{p\to\infty} \delta(p) = \lim_{p\to\infty} \Delta(p) = r(T), \tag{3.5}$$

and

$$\lim_{p\to\infty} \eta(p) = \lim_{p\to\infty} H(p) = r(T). \tag{3.6}$$

Proof. With $f_n(\lambda) \equiv n^{-g+1}\left(\frac{\lambda}{r(T)}\right)^n$ and the hypothesis that $B_{1,g+1}x=0$, it follows from (2.5) that

$$f_n(T)\,x = \sum_{k=1}^{g} \frac{f_n^{(k-1)}(r(T))}{(k-1)!} B_{1,k}\,x + \frac{1}{2\pi i}\left(\int_{C'} f_n(\lambda)\,R(\lambda, T)\,d\lambda\right)x,$$

where $C' = \{\lambda: |\lambda| = \varrho'' < r(T)\}$ is such that it contains in its interior $\sigma(T) - \{r(T)\}$. Regrouping the terms of $f_n(T)$, we write $f_n(T)\,x$ as

$$f_n(T)\,x = \frac{f_n^{(g-1)}(r(T))}{(g-1)!} B_{1,g}\,x + U_n\,x$$

where U_n, an element of $[X]$, is given by

$$U_n \equiv \sum_{k=1}^{g-1} \frac{f_n^{(k-1)}(r(T))}{(k-1)!} B_{1,k} + \frac{1}{2\pi i}\int_{C'} f_n(\lambda)\,R(\lambda, T)\,d\lambda$$

for $g>1$. Since $\|R(\lambda, T)\|_{[X]} \leqq K$ on C', it follows that the integral term of U_n is bounded in the $[X]$-norm by $\frac{(\varrho'')^{n+1}K}{(r(T))^n n^{g-1}}$, which tends to zero as $n\to\infty$. From the definition of $f_n(\lambda)$, the same is true for the remaining terms of U_n, and thus, $\lim_{n\to\infty} \|U_n\|_{[X]} = 0$. Hence,

$$\lim_{n\to\infty} n^{-g+1}[r(T)]^{-n}\,T^n\,x = \frac{[r(T)]^{-g+1}}{(g-1)!} B_{1,g}\,x.$$

Using the fact that $r_{T^n x}(T) \leqq r_{T^{n+1}x}(T) \leqq r(T)$ for all $n \geqq 1$ (Lemmas 2 and 4), then an application of Lemma 5 gives

$$\lim_{n\to\infty} r_{T^n x}(T) = r(T).$$

From this, (3.4) follows. To prove (3.5) and (3.6), we know that

$$r(T^\ell) = [r(T)]^\ell, \qquad \ell = 1, 2, \ldots,$$

and using the established result of (3.4), then (3.5) and (3.6) follow. Q.E.D.

Remark. It is obvious that the assumptions on $x \in K$ such that $B_{1,g}x \neq 0$ while $B_{1,g+1}x=0$, are fulfilled if $r(T)$ is a pole of the resolvent operator $R(\lambda, T)$ of order g, since then $B_{1,k}=0$ for all $k \geqq g+1$.

Theorem 3.3. Let $T \in [Y]$ be a positive operator h
$x \in K$ be a vector for which $r^x(T) < +\infty$ and for whic

(3.7) $\quad \nu' \equiv r_x(B_{1,g}) > 0 \quad$ and $\quad \tau' \equiv r^x(B_{1,g}$

Then, (3.1)—(3.3) and (3.5)—(3.6) are valid.

Proof. Since $B_{1,g}x \neq 0$ from (3.7) and $r^x(T) < +\infty$
to (3.3) are valid from Theorem 3.1. Next, by virtue of
of (3.7) are equivalent to the existence of positive nu
for which

(3.7′) $\quad \nu x \leqq B_{1,g}x \quad$ and $\quad B_{1,g}x \leqq \tau x$

Hence, as T is a positive operator which commutes
(3.7′) that

(3.8) $\quad T^\ell x \leqq \frac{(r(T))^\ell}{\nu} B_{1,g}x \quad$ and $\quad \frac{(r(T))^\ell B_{1,g}x}{\tau} \leqq T^\ell x$

From the second inequality of (3.8) with $\ell = 2^{p-1}$, a
(3.7′), we have

$$\langle T^{2^{p-1}}x, x'\rangle \geqq \frac{(r(T))^{2^{p-1}}}{\tau} \langle B_{1,g}x, x'\rangle \geqq \left(\frac{\nu}{\tau}\right)($$

for all $x' \in H'$. Thus,

$$r_x(T^{2^{p-1}}) \geqq \left(\frac{\nu}{\tau}\right)(r(T))^{2^{p-1}}.$$

Similarly,

$$r^x(T^{2^{p-1}}) \leqq \frac{(r(T))^{2^{p-1}}\tau}{\nu}.$$

From these inequalities, (3.5) follows. Further, the sa

$$r_{T^{2^{p-2}}x}(T^{2^{p-2}}) \geqq \left(\frac{\nu}{\tau}\right)(r(T))^{2^{p-2}},$$

and

$$r_{T^{2^{p-2}}x}(T^{2^{p-2}}) \leqq \left(\frac{\tau}{\nu}\right)(r(T))^{2^{p-2}},$$

from which (3.6) follows. Q.E.D.

We now give some examples which illustrate the r
first remark that if $T \in [Y]$ is a positive semi-nonsu
property S, then it is known [17] that $g = 1$, i.e., $B_{1,1}$ is
of the spectral decomposition corresponding to the ei
maximum multiplicity unity. Moreover, if $x \in K$ is non
interior [14].

Example 1. Let $Y = R^m$ be real Euclidean space, a
all elements in R^m with nonnegative components. T
interior elements of K is simply the set of vectors in R^m
it follows that if x and y are quasi-interior, there exist co
such that (cf. (3.7′))

$$\nu x \leqq y \leqq \tau x.$$

Consequently, if $T \in [Y]$ is any positive semi-nonsupporting operator having property S, i.e., T is a nonnegative irreducible $m \times m$ matrix, and if x is quasi-interior, then the inequalities of (3.7′) hold. Thus, (3.1)—(3.3) and (3.5)—(3.6) are valid. In particular, if T is a *cyclic* irreducible nonnegative matrix, we deduce as in [6] and [24] that the methods of Yamamoto and Hall and Spanier are necessarily convergent for any initial vector x with positive components. Furthermore, Theorem 3.3 gives conditions ((3.7) or (3.7′)) on the initial vector x which ensure the convergence of the indicated methods when T is a nonnegative *reducible* matrix. In this sense, Theorem 3.3 gives new information in finite-dimensional cases.

Example 2. Let $Y = C^0[0, 1]$ be the Banach space of all real-valued continuous functions on $[0, 1]$, with the uniform norm, and let K be the cone of all continuous nonnegative functions on $[0, 1]$. Clearly, any positive function in K is quasi-interior. Then, for each pair of positive functions x and y in K, there exist constant $\alpha > 0$ and $\beta < +\infty$ such that

$$\alpha x \leqq y \leqq \beta x .$$

Consequently, if $T \in [Y]$ is any positive semi-nonsupporting operator having property S, and if x is any positive function in K, the inequalities of (3.7′) hold, and thus the results of (3.1)—(3.3) and (3.5)—(3.6) are again valid.

Example 3. Let $Y = L_2[0, 1]$ be the Hilbert space of equivalence classes of all Lebesgue square integrable functions on $[0, 1]$, and let K be the cone of equivalence classes of functions of Y which are nonnegative almost everywhere on $[0, 1]$. Suppose that $T \in [Y]$ is a positive semi-nonsupporting operator having property S such that for any quasi-interior element y, there exists a positive integer $\ell(y) = \ell$ and a positive real number $\alpha(y) = \alpha$ such that

$$\alpha e \leqq T^{\ell} y, \quad \text{where} \quad e(s) \equiv 1 \quad \text{for all} \quad s \in [0, 1]. \tag{3.9}$$

Furthermore, for every $y \in Y$, assume that there is a positive integer m such that $T^m y$ is a bounded function almost everywhere on $[0, 1]$. Then, we assert that constants $\nu > 0$ and $\tau < +\infty$ exist such that (cf. (3.7′))

$$\nu e \leqq B_{1,1} e \leqq \tau e . \tag{3.10}$$

To show this, the hypothesis that T is a positive semi-nonsupporting operator coupled with the remark following the proof of Theorem 3.3 gives us that $B_{1,1} x$ is quasi-interior if $x \in K$ is nonzero. Thus, it follows from (3.9) with y set equal to $B_{1,1} e$ that

$$\alpha e \leqq T^{\ell}(B_{1,1} e) = (r(T))^{\ell} B_{1,1} e ,$$

and hence

$$\frac{\alpha}{[r(T)]^{\ell}} \leqq B_{1,1} e ,$$

which gives the first inequality of (3.10).

On the other hand, by hypothesis, there is a positive integer m for which $T^m B_{1,1} e$ is a bounded function almost everywhere on $[0, 1]$, and hence

$$T^m B_{1,1} e \leqq \sigma e$$

with $\sigma < +\infty$. But, $T^m B_{1,1} = [r(T)]^m B_{1,1}$, and hence th

$$B_{1,1} e \leqq \frac{\sigma}{[r(T)]^m} e,$$

which gives the second inequality of (3.10). Thus, fro
of (3.1)—(3.3) and (3.5)—(3.6) are again valid.

Corollary 3.1. Let Y be a Banach lattice, and let
operator whose spectral radius $r(T)$ is a pole of the r
Let x be any element of the cone K such that $r_x($
Then, (3.1)—(3.4) and (3.5)—(3.6) are valid.

Proof. A well known conjecture of Schaefer [21],
Niiro and Sawashima [16], says that if $r(T)$ is a po
semi-nonsupporting, then all singularities of $R(\lambda, T)$
$|\lambda| = r(T)$ are simple poles. Thus, T has property S a
3.1 follows from Theorem 3.3. Q.E.D.

Corollary 3.2. If T is a nonsupporting operator ha
is an arbitrary element of K, then (3.1)—(3.6) hold.

Proof. The relations (3.1)—(3.6) are direct consequ
3.2, the assumptions of which are fulfilled according t

Remark. It is easy to see that the strongly K-posi
absolutely K-positive operators [15] are nonsupportin
interior operators [21] are semi-nonsupporting. It can
operators [7, p. 60] can be treated as nonsupporting
cone TK in the space $Y_1 = TK - TK$ generated b
Also, the strongly K-positive, absolutely K-positive a
in finite-dimensional spaces, with K being the set o
components, essentially coincide with primitive irredu
and quasi-interior or semi-nonsupporting operators co
irreducible nonnegative matrices, the u_0-positive ma
ducible.

§ 4. Comparison Theorems

In the case that two of the methods discussed i
computing upper and lower bounds for $r(T)$ are conve
is which converges more rapidly. A partial answer to
in the following theorems.

Theorem 4.1. Let $T \in [Y]$, not necessarily positive
that $r^x(T) < +\infty$, and assume that

(4.1) $\qquad 0 \leqq \gamma(p) \leqq \gamma(p+1) \quad$ and $\quad \Gamma(p+1) \leqq \Gamma(p)$

Then,

(4.2) $\qquad \gamma(2^{p-2}) \leqq \eta(p) \qquad$ and $\qquad H(p) \leqq \Gamma(2^{p-}$

Thus, if $\lim_{p\to\infty} \gamma(p) = \lim_{p\to\infty} \Gamma(p) = r(T)$, then

(4.3) $$\lim_{p\to\infty} \eta(p) = \lim_{p\to\infty} H(p) = r(T)$$

Proof. It again suffices to consider only the first inequality of (4.2). From the definitions of (1.1) and (1.2), it follows that $\langle T^{\ell+1}x, x'\rangle \geqq \gamma(\ell)\langle T^{\ell}x, x'\rangle$ for all $x'\in H'$, for any nonnegative integer ℓ. Since the $\gamma(p)$'s are all nonnegative from (4.1), we can take products over ℓ, giving

$$\langle T^{2^{p-1}}x, x'\rangle \geqq \left(\prod_{k=0}^{\nu}\gamma(2^{p-2}+k)\right)\langle T^{2^{p-2}}x, x'\rangle \quad \text{for all} \quad x'\in H',$$

where $\nu = 2^{p-2}-1$. Thus, from the definitions of (1.1) and (1.4), we have

$$(\eta(p))^{2^{p-2}} \geqq \prod_{k=0}^{\nu}\gamma(2^{p-2}+k) \geqq (\gamma(2^{p-2}))^{2^{p-2}},$$

the last inequality following from the monotonicity assumption of (4.1). Thus, we have

$$\eta(p) \geqq \gamma(2^{p-2}),$$

the desired inequality of (4.2). The result of (4.3) is then an obvious consequence of (4.2). Q.E.D.

Theorem 4.2. Let $T\in[Y]$, not necessarily positive, be such that T^{2^q} is a positive operator for some nonnegative integer q, and let $x\in K$ be such that $T^{2^k}x \neq 0$ for all $k \geqq q$. Then,

$$\delta(p-1) \leqq \eta(p) \quad \text{and} \quad H(p) \leqq \Delta(p-1) \quad \text{for all} \quad p \geqq q+2. \tag{4.4}$$

Proof. To establish the first inequality of (4.4), we have from Lemma 1 that

$$r_x(T^{\ell})\, x \leqq T^{\ell}x$$

for any $\ell \geqq 0$. Since T^{2^q} is a positive operator, so is T^{2^k} for all $k \geqq q$. Applying the positive operator $T^{2^{p-2}}$ where $p-2 \geqq q$ to the above inequality for $\ell = 2^{p-2}$, we have

$$r_x(T^{2^{p-2}})\, T^{2^{p-2}}x \leqq T^{2^{p-1}}x, \quad p \geqq q+2.$$

Thus, by directly appealing to the definition of (1.1), this gives that

$$r_x(T^{2^{p-2}}) \leqq r_{T^{2^{p-2}}x}(T^{2^{p-2}}),$$

which from the definitions of (1.3) and (1.4) gives the first inequality of (4.4) Q.E.D.

As an immediate corollary of Theorems 4.1 and 4.2 and the inequalities of (2.11), we have

Corollary 4.1. Let $T\in[Y]$ be a positive operator. Then, for any $x\in K$ such that $T^p x \neq 0$ for all $p \geqq 0$, the inequalities of (4.2) and (4.4) are valid for all $p \geqq 2$.

Several remarks are now in order. First, Theorems 4.1 and 4.2 compare the various methods of obtaining nested bounds for $r(T)$ *without* the assumption that these methods are convergent. Next, Hall and Spanier [6, Theorems 6 and 7] have proved in the matrix case inequalities like those of (4.2) and (4.4), but for all p sufficiently large. Because of our slightly modified enumeration in Yamamoto's method, the inequalities of (4.4) compare the $\{\delta, \Delta\}$ and $\{\eta, H\}$ methods in the case T is a positive operator for *all* $p \geqq 2$.

From Theorem 3.2, we know that all three methods $\{\gamma, \Gamma\}$, $\{\delta, \Delta\}$, and $\{\eta, H\}$ of computing upper and lower bounds for the spectral radius $r(T)$ of a positive

operator $T \in [Y]$ having property S are convergent if
value of T, i.e., $\lambda \in \sigma(T)$ with $|\lambda| = r(T)$ implies $\lambda =$
of the resolvent operator $R(\lambda, T)$ of multiplicity g. W
the asymptotic convergent rates of these methods in
$r(T) > 0$. We shall also assume that the elements x' of

(4.5) $$\|x'\|_Y = 1,$$

and that $x \in K$ is such that

(4.6) $$0 < \varkappa \leqq \langle B_{1,g} x, x' \rangle$$

for all $x' \in H'$ for which $\langle B_{1,g} x, x' \rangle \neq 0$, where $\varkappa$ is ind
that the normalization in H' of (4.5) and the inequali
restrictions for finite-dimensional cases. In infinite di
these assumptions can exclude some K-total sets.

With $f_n(\lambda) \equiv n^{-g+1} \left(\frac{\lambda}{r(T)}\right)^n$, we proceed as in the
$r(T) > 0$ is a pole of multiplicity g of $R(\lambda, T)$, we ha

(4.7) $$f_n(T) = \sum_{k=1}^{g} \frac{f_n^{(k-1)}(r(T))}{(k-1)!} B_{1,k} + \frac{1}{2\pi i} \int_{C'} f_n($$

This can be written as

(4.7') $$\frac{(g-1)!\, n^{-g+1}}{(r(T))^{n-g+1}} T^n = B_{1,g} + V_n$$

where V_n, an element of $[X]$, satisfies, as in the proof of T
With the expression of (4.7') and the result of Lemm

(4.8) $$r_{T^n x}(T^p) = (r(T))^p \inf_{\substack{x' \in H' \\ \langle T^n x, x' \rangle \neq 0}} \left\{ \frac{\left\langle \left(B_{1,g} + \left(\frac{T}{r(T)}\right)^p V_n \right) x \right.}{\langle (B_{1,g} + V_n) x, x' \rangle} \right.$$

and

(4.9) $$r_x(T^p) = (r(T))^p \inf_{\substack{x' \in H' \\ \langle x, x' \rangle \neq 0}} \left\{ \frac{\langle (B_{1,g} + V_p) x, x}{(g-1)!\, p^{-g+1} (r(T))^{g-}} \right.$$

Hence, by the definition of (1.2)—(1.4),

(4.10) $$\gamma(2^{p-2}) = r(T) \inf_{\substack{x' \in H' \\ \langle T^{2^{p-2}} x, x' \rangle \neq 0}} \left\{ \frac{\left\langle \left(B_{1,g} + \left(\frac{T}{r(T)}\right. \right. \right.}{\langle (B_{1,g} + V_{2^{p-}}} \right.$$

(4.11) $$\delta(p) = r(T) \left[\inf_{\substack{x' \in H' \\ \langle x, x' \rangle \neq 0}} \left\{ \frac{\langle (B_{1,g} + V}{(g-1)!\, 2^{-(g-1)(p-}} \right. \right.$$

and

(4.12) $$\eta(p) = r(T) \left[2^{g-1} \inf_{\substack{x' \in H' \\ \langle T^{2^{p-2}} x, x' \rangle \neq 0}} \left\{ \frac{\langle (B_{1,g} +}{\langle (B_{1,g} +} \right. \right.$$

Now, from the definition of the operator V_n of (4.7'), th
of $\|V_n\|_{[X]}$ can be verified:

(4.13) $$\|V_n\|_{[X]} \leqq \frac{c}{n} \qquad \text{if} \quad g$$

and

$$\|V_n\|_{[X]} \leqq c\left(\frac{\varrho''}{r(T)}\right)^n \quad \text{if} \quad g=1, \tag{4.13'}$$

where c is a constant, and ϱ'' is such that $\lambda \in \sigma(T)$ with $|\lambda| \neq r(T)$ implies that $|\lambda| < \varrho'' < r(T)$. By direct computation with (4.10) and (4.11), we have, using (4.13) and (4.13'), the following general asymptotic convergence rates:

$$\gamma(2^{p-2}) = r(T)\left\{1 + \mathcal{O}\left(\frac{1}{2^{p-2}}\right)\right\} \quad \text{as} \quad p \to \infty \quad \text{for} \quad g > 1, \tag{4.14}$$

$$\gamma(2^{p-2}) = r(T)\left\{1 + \mathcal{O}\left(\left(\frac{\varrho''}{r(T)}\right)^{2^{p-2}}\right)\right\} \quad \text{as} \quad p \to \infty \quad \text{for} \quad g = 1, \tag{4.14'}$$

and

$$\delta(p) = r(T)\left\{1 + \mathcal{O}\left(\frac{p-1}{2^{p-1}}\right)\right\} \quad \text{as} \quad p \to \infty \quad \text{for} \quad g > 1, \tag{4.15}$$

and

$$\delta(p) = r(T)\left\{1 + \mathcal{O}\left(\frac{1}{2^{p-1}}\right)\right\} \quad \text{as} \quad p \to \infty \quad \text{for} \quad g = 1. \tag{4.15'}$$

Similarly, we deduce from (4.12) that

$$\eta(p) = r(T)\left\{1 + \mathcal{O}\left(\frac{1}{(2^{p-2})}\right)\right\} \quad \text{as} \quad p \to \infty \quad \text{for} \quad g > 1, \tag{4.16}$$

and

$$\eta(p) = r(T)\left\{1 + \frac{1}{2^{p-2}}\,\mathcal{O}\left(\left(\frac{\varrho''}{r(T)}\right)^{2^{p-2}}\right)\right\} \quad \text{as} \quad p \to \infty \quad \text{for} \quad g = 1. \tag{4.16'}$$

We remark that the asymptotic convergence rates of (4.14') and (4.16') agree with the finite-dimensional results of Hall and Spanier [6] for the special case $g=1$ and operators whose spectra have the following structure: all spectral points lying on the circumference $|\lambda| = \varrho$, where ϱ is such that $\lambda \in \sigma(T)$, $\lambda \neq r(T)$, implies $|\lambda| \leqq \varrho$, are *simple* poles of the resolvent operator $R(\lambda, T)$. But the rate (4.15') improves upon their estimate, and (4.14), (4.15) and (4.16) extend moreover their results.

Thus, in the dominant case under consideration, the results of (4.14)—(4.16') show that Collatz's method and the hybrid method of Hall and Spanier have *essentially* the same asymptotic rates of convergence. Because of this, Collatz's method is probably in general preferable because of its inherent simplicity. If, however, the powers T^{2^p} can be easily determined, then the hybrid method of Hall and Spanier is preferable because of the additional factor of 2^{p-2} in (4.16') for the case $g=1$. For Yamamoto's method, it is clear from (4.14)—(4.14') and (4.15)—(4.15') that its asymptotic rate of convergence is *never better* than that for Collatz's method in the dominant case. Moreover, because the inequalities of (4.6) are valid when T is a positive operator, we know in this case that the hybrid method of Hall and Spanier is always superior to Yamamoto's method. We finally remark that because of the generality of the initial vector $x \in K$, we expect that the estimates of (4.13) and (4.13') produce realistic asymptotic convergence rates in (4.14)—(4.16').

The expressions in (4.14)—(4.16') also reveal the important fact that, in the dominant case, one can expect *worse* asymptotic convergence rates for the various methods of estimating $r(T)$ only if one has the case where $r(T)$ is a pole of $R(\lambda, T)$ of multiplicity $g > 1$.

For the *nondominant* case, i.e., there exist $\lambda \in \sigma(T)$
$\lambda \neq r(T)$, we again assume that $T \in [Y]$ is a positive op
and we assume that $x \in K$ is such that the inequalities
from (3.8), there exist positive numbers $\alpha > 0$ and $\beta <$

$$\alpha B_{1,g} x \leqq \left(\frac{T}{r(T)}\right)^{2p-1} x \leqq \beta B_{1,g} x \quad \text{for all}$$

From this, it follows that

$$\alpha \langle B_{1,g} x, x' \rangle \leqq \left\langle \left(\frac{T}{r(T)}\right)^{2p-1} x, x' \right\rangle \leqq \beta \langle$$

for any $x' \in H'$, and hence

$$\frac{\alpha}{\beta} \leqq \frac{\left\langle \left(\frac{T}{r(T)}\right)^{2p-1} x, x' \right\rangle}{\left\langle \left(\frac{T}{r(T)}\right)^{2p-2} x, x' \right\rangle} \leqq \frac{\beta}{\alpha}$$

for any $x' \in H'$ for which $\langle B_{1,g} x, x' \rangle > 0$. From this e

$$r(T) \left(\frac{\alpha}{\beta}\right)^{2^{-(p-2)}} \leqq \eta(p) \leqq r(T) \left(\frac{\beta}{\alpha}\right)^{2}$$

and consequently,

$$\eta(p) = r(T)\left\{1 + \mathcal{O}\left(\frac{1}{2^{p-2}}\right)\right\} \quad \text{as} \quad p \tag{4.17}$$

In other words, the rate of convergence of the meth
the nondominant case *coincides* with the rate of converg
in the dominant case when $g > 1$ (cf. (4.14)). Thus, if $g =$
of the Collatz method is larger in the dominant cas
always try to be in the dominant case, with $g = 1$ w

If $T \in [X]$ is a positive operator with property
preceding discussion suggests considering the *shifted o*

$$W(\tau) \equiv T + \tau I, \quad \tau > 0$$

with shift τ. For any real $\tau > 0$, it is clear that $W(\tau)$ i
with property S, but now $r(W(\tau)) > 0$ is a *dominant*
of $W(\tau)$. Thus, the three methods of obtaining upper an
are, from Theorem 3.2, convergent for any fixed $\tau >$
to the determination of a best possible shift τ for T.

For $T \in [X]$ a positive operator with property S an
optimal shift τ_0 for T can be defined as the $\tau_0 \geqq 0$ for

$$\inf_{\tau \geqq 0} \left(\sup_{\substack{\lambda \in \sigma(T) \\ \lambda \neq r(T)}} \left| \frac{\lambda + \tau}{r(T) + \tau} \right| \right) = \sup_{\substack{\lambda \in \sigma(T) \\ \lambda \neq r(T)}} \left| \frac{\lambda}{r(T} \right. \tag{4.18}$$

The problem of determining τ_0 seems to be in general
of determining $r(T)$. But, for particular classes of o
τ_0 can be easily determined.

With the above assumptions on T, assume that all
lying on the circumference $|\lambda| = r(T) > 0$ are given p

(4.19) $\lambda_j = r(T) e^{i\alpha_j}, \quad 1 \leqq j \leqq s, \quad s > 1, \quad$ where $0 =$

It is readily verified by direct calculation that

$$\inf_{\tau \geqq 0}\left(\max_{2\leqq j\leqq s}\left|\frac{\lambda_j+\tau}{r(T)+\tau}\right|\right)=\max_{2\leqq j\leqq s}\left|\frac{\lambda_j+r(T)}{2r(T)}\right|=q_s,$$

where

$$q_s=\left[\frac{1+\max\limits_{2\leqq j\leqq s}(\cos\alpha_j)}{2}\right]^{\frac{1}{2}}<1 \tag{4.20}$$

i.e., if the only points of the spectrum $\sigma(T)$ are those of (4.19), then $\tau_0=r(T)$ is the optimal shift for T. Continuing, suppose that all points λ of $\sigma(T)$, except for $\lambda_1=r(T)$, lie in the disk in the complex plane with center $-r(T)$ and radius $2q_s r(T)$, i.e.,

$$|\lambda+r(T)|\leqq 2q_s r(T), \quad \lambda \neq r(T). \tag{4.21}$$

It is clear from the definition of τ_0 in (4.18) that $\tau_0=r(T)$ is the optimal shift for T. In particular, if $2q_s\geqq 1$, then since $|\lambda+r(T)|\leqq|\lambda|+r(T)$, it follows that $\tau_0=r(T)$ also if all the points λ of $\sigma(T)$, other than those of (4.19), lie in the disk

$$|\lambda|\leqq(2q_s-1)\,r(T), \tag{4.22}$$

assuming that those $\lambda\in\sigma(T)$ for which equality holds in (4.22) are *poles* of $R(\lambda, T)$.

We state this as

Theorem 4.3. Let $T\in[Y]$ be a positive operator with property S and with $r(T)>0$, and let all the singularities λ of $R(\lambda, T)$ lying on $|\lambda|=r(T)$ be of the form (4.19), and let the remaining singularities λ of $R(\lambda, T)$ satisfy (4.21) or (4.22), and if equality holds, let them be simple poles of $R(\lambda, T)$. Then, the optimal shift for T is $\tau_0=r(T)$. Moreover $\tilde{\gamma}(p)\equiv r_{W^p(r(T))x}(W(r(T)))$ and $\tilde{\Gamma}(p)\equiv r^{W^p(r(T))x}(W(r(T)))$ denote the Collatz bounds for the operator $W(r(T))=T+r(T)I$ with spectral radius $2r(T)$ for any $x\in K$ with $x\neq 0$, then

$$\tilde{\gamma}(p)=\tilde{\Gamma}(p)=2r(T)\{1+\mathcal{O}(q_s^p)\} \quad \text{as} \quad p\to\infty, \tag{4.23}$$

where q_s is defined in (4.20).

We remark that the asymptotic expression in (4.23) follows from (4.14′), and (4.23) becomes $\mathcal{O}(p^{\tilde{g}}q_s^p)$ if the multiplicity of λ for which equality holds in (4.22) is $\tilde{g}$.

Example. Let $Y=R^m$, $m\geqq 4$, be real Euclidean space, let K be the cone of all elements in R^m with nonnegative components, and let the positive operator T be given by the matrix

$$T=\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 0 & \\ \hline & & 0 & & Q \end{array}\right],$$

where Q is an $(m-4)\times(m-4)$ matrix with nonnegative
Then, $r(T)=1$ and the singularities of $R(\lambda, T)$ on $|\lambda|$
with $\alpha_j=2\pi(j-1)/4$, $1\leqq j\leqq 4$. In this case, $q_4=1/\sqrt{}$
$r(Q)\leqq\sqrt{2}-1$, then $\tau_0=r(T)=1$, and for any $x\in K$ w
we have that

$$\tilde{\gamma}(p)=\tilde{\Gamma}(p)=2\left\{1+\mathcal{O}\left(\frac{1}{2^{p/2}}\right)\right\} \quad \text{as}$$

The problem of determining the optimal shift $\tau_0\geqq$
$T\in[X]$ with property S and $r(T)>0$ whose spectrum
symmetric operators in Hilbert spaces, is also easily
singularities $\lambda\neq r(T)$ of $R(\lambda, T)$ satisfy

$$-r(T)\leqq\alpha\leqq\lambda\leqq\beta<r(T).$$

Then, by direct calculation from the definition of (4
for T is given by

$$\tau_0=\max\left\{0;\ \frac{-(\alpha+\beta)}{2}\right\}. \tag{4.24}$$

§ 5. Shifts and Nested Bounds

The preceding considerations suggest using the fo
nested bounds for the spectral radius $r(T)$ of a posit
scheme is optimal (with respect to the positive shifts)
are such as described in Theorem 4.3. In particular, thi
cyclic matrices all of whose eigenvalues are of the san

Let $T\in[Y]$ be a positive operator, let $x\in K$ with

$$\varphi(1)\equiv r_x(T),\qquad T_{(1)}\equiv T+\varphi($$

Further, define

$$\varphi(n+1)=r_{T_{(n)}\cdots T_{(1)}x}(T) \quad\text{and}\quad T_{(n+1)}=T$$

Similarly, if $r^x(T)<+\infty$, let $T^{(n+1)}=T+\Phi(n+1)I$,

$$\Phi(1)=r^x(T) \quad\text{and}\quad \Phi(n+1)=r^{T^{(n)}}$$

Theorem 5.1. Let T be a positive operator having p
be a pole of order $g\geqq 1$ of the resolvent operator
that $B_{1,g}x\neq 0$, and such that $x\leqq\omega B_{1,g}x$ holds wit
has been defined in (2.4). Further, assume that $r_x(T$

Then, we have

$$\varphi(1)\leqq\cdots\leqq\varphi(n)\leqq\cdots\leqq r(T)\leqq\cdots\leqq\Phi(\tag{5.1}$$

and

$$\lim_{n\to\infty}\varphi(n)=\lim_{n\to\infty}\Phi(n)=r(T \tag{5.2}$$

Proof. From the fact that the operators

$$T_{(k)}=T+\varphi(k)I,\qquad T^{(k)}=T+\Phi(k)I$$

are positive operators which commute with T for all positive integers $k \geqq 1$, the validity of (5.1) is a consequence of Lemma 2.

To prove (5.2), let us consider the operator function $g_n = g_n(T)$, where

$$g_n(\lambda) = \frac{(g-1)!\, f_n(\lambda)}{f_n^{(g-1)}(r(T))},$$

and where

$$f_n(\lambda) = \prod_{j=1}^{n} (\lambda + \varphi(j)).$$

Hence,

$$f_n^{(g-1)}(r(T)) = \sum_{k_1=1}^{n} \cdots \sum_{k_{g-1}=1}^{n} \prod_{\substack{j \neq k_1 \\ \cdots \\ j \neq k_{g-1} \\ j=1}}^{n} [r(T) + \varphi(j)].$$

We evidently have that $\lim_{n\to\infty} g_n(\lambda) = 0$ if either $|\lambda| < r(T)$ or $\lambda = r(T)\exp\{i\varphi\}$ with $0 < \varphi < 2\pi$. Since T is a positive operator having property S, and $\lambda_1 = r(T)$ is a pole of order g of the resolvent operator $R(\lambda, T)$, then all the elements $\lambda_2, \ldots, \lambda_s$ lying in $\sigma(T)$ and on the circumference $|\lambda| = r(T)$ are poles of $R(\lambda, T)$ of at most order g, according to Schaefer's theorem [20].

Thus, by (2.5')

$$g_n(T) = \sum_{j=1}^{s} \sum_{k=1}^{g} \frac{g_n^{(k-1)}(\lambda_j)}{(k-1)!} B_{j,k} + Z_n,$$

where $\lim_{n\to\infty} \|Z_n\|_{[X]} = 0$. Since $\lim_{n\to\infty} g_n^{(k)}(\lambda_j) = 0$ for $k = 0, 1, \ldots, g-1$ for $\lambda_j \in \sigma(T)$, $\lambda_j \neq r(T)$, and $\lim_{n\to\infty} g_n^{(k)}(r(T)) = 0$ for $k = 0, 1, \ldots, g-2$, we have in the $[X]$-norm that

$$\lim_{n\to\infty} g_n(T) = B_{1,g}.$$

Consequently, by Lemma 5,

$$\lim_{n\to\infty} \varphi(n) = \lim_{n\to\infty} \inf_{\substack{x' \in H' \\ \langle g_n(T)x,\, x'\rangle \neq 0}} \frac{\langle T g_n(T)\, x, x'\rangle}{g_n(T)\, x, x'\rangle}$$

$$= \inf_{\substack{x' \in H' \\ \langle B_{1,g} x,\, x'\rangle \neq 0}} \frac{\langle T B_{1,g} x, x'\rangle}{\langle B_{1,g} x, x'\rangle} = r(T),$$

which proves one part of (5.2). The remainder is similarly proved. Q.E.D.

In fact, we have just proved a slightly more general assertion, which is useful in cases for which the optimal shifts are not known.

Theorem 5.2. Let $T \in [Y]$ be a positive operator having property S and $r(T) > 0$. Let $x \in K$ be such that the relations $x \leqq \omega B_{1,g} x \neq 0$ hold with some positive integer g and $0 < \omega < +\infty$. Let $\{r_n\}$ and $\{R_n\}$ be sequences of positive numbers bounded below and above respectively. Then we have

$$\psi(1) \leqq \cdots \leqq \psi(n) \leqq \cdots \leqq r(T) \leqq \cdots \leqq \Psi(n) \leqq \cdots \leqq \Psi(1),$$

where

$$\psi(k) \equiv r_{(T+r_k I)\ldots(T+r_1 I)x}(T)$$

and

$$\Psi(k) \equiv r^{(T+R_k I)\ldots(T+R_1 I)x}(T).$$

Furthermore

$$\lim_{n\to\infty} \psi(n) = \lim_{n\to\infty} \Psi(n) = r(T).$$

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