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The Effect of Quadrature Errors in the Numerical Solution of Boundary Value Problems by Variational Techniques*

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Dedicated to A. M. Ostrowski on his 75th Birthday

§ 1. Introduction

In [4], the approximate solutions of a class of real nonlinear two-point boundary value problems were obtained from the application of the classical Rayleigh-Ritz procedure to the variational formulation of these problems by minimizing over finite-dimensional subspaces. For certain sequences of approximating subspaces, such as the piecewise-polynomial Hermite and spline subspaces, upper bounds for the rates of convergence of these approximations can be theoretically determined. In practical computation on a digital computer, these approximate solutions are however not precisely obtained since certain integrals arising in the Rayleigh-Ritz formulation are replaced necessarily by quadrature formulas.

The object of this paper is to investigate the errors introduced in the approximate solutions by such quadrature formulas. In particular, we shall obtain bounds for the errors introduced by such quadrature schemes, as they apply to finite-dimensional subspaces of piecewise-polynomial functions, and we shall determine when these quadrature errors are *consistent* with (i.e. the same order as) the approximation errors of the Rayleigh-Ritz method. We shall also show how certain quadrature schemes coupled with particular finite dimensional subspaces give *well-known difference approximations* to such boundary value problems. In addition, numerical results based on such consistent quadrature schemes are also presented.

§ 2. Formulation of the Problem

As in [4], we consider the following real nonlinear two-point boundary value problem

$$\mathscr{L}[u(x)] = f(x, u(x)), \quad 0 < x < 1, \tag{2.1}$$

with Dirichlet boundary conditions

$$D^k u(0) = D^k u(1) = 0, \quad 0 \leqslant k \leqslant n-1, \quad D \equiv \frac{d}{dx}, \tag{2.2}$$

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where

$$\mathscr{L}[u(x)] = \sum_{j=0}^{n} (-1)^{j+1} D^j [p_j(x) D^j u(x)], \quad n \geqslant 1. \tag{2.3}$$

The coefficient functions $p_j(x)$ are assumed to be of class $C^j[0, 1]$, $0 \leqslant j \leqslant n$, although weaker assumptions are also possible (cf. [4, § 8]).

Let S denote the linear space of all real functions $w(x)$ satisfying the boundary conditions (2.2), such that $w(x) \in C^{n-1}[0, 1]$ with $D^{n-1} w(x)$ absolutely continuous in $[0, 1]$, and $D^n w(x) \in L^2[0, 1]$. As in [4], we assume that there exist two real constants β and $K > 0$ such that

$$\|w\|_{L^\infty} \equiv \sup_{x \in [0,1]} |w(x)| \leqslant K \left\{ \int_0^1 \left(\sum_{j=0}^{n} p_j(x) (D^j w(x))^2 + \beta (w(x))^2 \right) dx \right\}^{1/2} \tag{2.4}$$

for all $w \in S$. This assumption, as noted in [4], is implied by either of the following

$$\|D^l w\|_{L^\infty} \leqslant K \left\{ \int_0^1 \left(\sum_{j=0}^{n} p_j(x) (D^j w(x))^2 + \beta (w(x))^2 \right) dx \right\}^{1/2}, \tag{2.5}$$

$$\|D^{l+1} w\|_{L^2} \leqslant K \left\{ \int_0^1 \left(\sum_{j=0}^{n} p_j(x) (D^j w(x))^2 + \beta (w(x))^2 \right) dx \right\}^{1/2} \tag{2.5'}$$

for some l, $0 \leqslant l \leqslant n-1$.

Next, we introduce the finite quantity Λ (cf. [4, Lemma 1]), defined by

$$\Lambda = \inf_{\substack{w \in S \\ w \not\equiv 0}} \frac{\int_0^1 \left\{ \sum_{j=0}^{n} p_j(x) (D^j w(x))^2 \right\} dx}{\int_0^1 \{w(x)\}^2 \, dx}. \tag{2.6}$$

We then assume that $f(x, u) \in C^0([0, 1]) \times R)$, and that there exists a constant γ such that

$$\frac{f(x, u) - f(x, v)}{u - v} \geqslant \gamma > -\Lambda \tag{2.7}$$

for all $x \in [0, 1]$, and all $-\infty < u$, $v < +\infty$ with $u \neq v$ and for each $c > 0$, there exists a real number $M(c)$ such that $u \neq v$, $|u| \leqslant c$, $|v| \leqslant c$ implies that

$$\frac{f(x, u) - f(x, v)}{u - v} \leqslant M(c) < \infty \tag{2.7'}$$

for all $x \in [0, 1]$.

Inequality (2.4) implies that the quantity

$$\|w\|_\gamma \equiv \left\{ \int_0^1 \left(\sum_{j=0}^{n} p_j(x)\,(D^j w(x))^2 + \gamma (w(x))^2 \right) dx \right\}^{1/2} \tag{2.8}$$

is a norm on S, and this is the norm basically used in the sections to follow. By Corollary 2 of [4], if (2.4) is valid for some constants β and K, then (2.4) is in particular valid for $\beta = \gamma$. Hence, for all $w \in S$, we have

$$\|w\|_{L^\infty} \leqslant K \|w\|_\gamma . \tag{2.9}$$

From Section 2 of [4], we know that if $\varphi(x)$ is a classical solution of (2.1)–(2.2), then $\varphi(x)$ strictly minimizes the functional

$$F[w] = \int_0^1 \left\{ \tfrac{1}{2} \sum_{j=0}^{n} p_j(x)\,(D^j w(x))^2 + \int_0^{w(x)} f(x,\eta)\, d\eta \right\} dx \tag{2.10}$$

over the space S, and thus $\varphi(x)$ is the unique solution of (2.1)–(2.2). We shall assume in the following that (2.1)–(2.2) possesses a classical solution $\varphi(x)$.

For any finite-dimensional subspace S_M of S, it is known [4, Theorem 2] that there is a unique function $\hat{w}_M(x) \in S_M$ which minimizes $F[w]$ over S_M. In theory, determining the unique element $\hat{w}_M(x)$ which minimizes $F[w]$ over S_M can be accomplished as follows. Assuming that $\{w_i(x)\}_{i=1}^{M}$ are linearly independent functions which span S_M, then simply solving the nonlinear system of equations

$$\frac{\partial F\left[\sum_{j=1}^{M} u_j w_j\right]}{\partial u_i} = 0, \quad 1 \leqslant i \leqslant M,$$

for the unknowns $u_1, u_2, \ldots, u_M$ uniquely determines $\hat{w}_M(x)$ in S_M. By using the definition of F plus integration by parts, this system can also be expressed as

$$0 = \int_0^1 \left\{ \left(\sum_{j=0}^{n} p_j(x) \left(\sum_{k=1}^{M} u_k D^j w_k(x) \right) D^j w_i(x) \right) + f\left(x, \sum_{j=1}^{M} u_j w_j(x)\right) w_i(x) \right\} dx, \quad 1 \leqslant i \leqslant M, \tag{2.11}$$

where the $u_1, u_2, \ldots, u_M$ are unknowns. Defining

$$a_{i,j} = \int_0^1 \left\{ \sum_{k=0}^{n} p_k(x)\, D^k w_i(x)\, D^k w_j(x) \right\} dx, \quad 1 \leqslant i, j \leqslant M, \tag{2.12}$$

and

$$g_i(\mathbf{u}) = \int_0^1 f\left(x, \sum_{j=1}^{M} u_j w_j(x)\right) w_i(x)\, dx\,, \quad 1 \leqslant i \leqslant M\,, \tag{2.13}$$

system (2.11) may be written in the matrix form

$$\mathbf{0} = A\mathbf{u} + \mathbf{g}(\mathbf{u})\,, \tag{2.14}$$

where $A \equiv (a_{i,j})$ is an $M \times M$ real symmetric matrix.

Practically speaking, the entries $a_{i,j}$ of the matrix A can be computed exactly, since, in most cases, the evaluation of these entries involves the integration of piecewise-polynomials, which is easily automated on a digital computer. This is true in higher dimensions as well. The evaluation of the quantities $g_i(\mathbf{u})$ in (2.13) is more troublesome, since the given function f in (2.1) is not in general a piecewise-polynomial function. This prompts us to use a quadrature scheme to evaluate the quantities in (2.13), which then generates a new system of nonlinear equations

$$\mathbf{0} = A\mathbf{u} + \tilde{\mathbf{g}}(\mathbf{u})\,, \tag{2.15}$$

where $\tilde{g}_i(\mathbf{u})$ is obtained from applying a particular quadrature scheme to $g_i(\mathbf{u})$. The solution $\tilde{\mathbf{u}}$ of (2.15) in turn generates a new function

$$\tilde{w}_M(x) = \sum_{i=1}^{M} \tilde{u}_i w_i(x)$$

in S_M.

In the next section, we shall discuss the choice of quadrature schemes for a given sequence of piecewise-polynomial subspaces $\{S_{M_i}\}_{i=1}^{\infty}$ of S so that the theoretical approximations $\{\hat{w}_{M_i}(x)\}_{i=1}^{\infty}$, determined successively from (2.14), and the approximations $\{\tilde{w}_{M_i}(x)\}_{i=1}^{\infty}$, determined successively from (2.15), have the same general order of accuracy.

§ 3. Linear Case

In this section, we begin with the assumption that the function f of (2.1) is *independent* of u. The integrals of (2.13) are then also independent of u, and in this case, we have

$$g_i = \int_0^1 f(x)\, w_i(x)\, dx \equiv L[f(x)\, w_i(x)]\,, \quad 1 \leqslant i \leqslant M\,, \tag{3.1}$$

where the integral of (3.1) is regarded as a bounded linear functional L, on $C^0[0, 1]$.

We now associate with the subspace S_M a linear functional L_M which is to approximate L, and we define

$$\tilde{g}_i \equiv L_M[f(x)\, w_i(x)], \quad 1 \leqslant i \leqslant M, \tag{3.2}$$

as the approximation of g_i in (3.1). Note that the matrix problem of (2.14) reduces now to

$$A\mathbf{u} + \mathbf{g} = \mathbf{0}, \tag{3.3}$$

and the use of the approximate linear functional L_M gives the associated matrix problem

$$A\mathbf{u} + \tilde{\mathbf{g}} = \mathbf{0}. \tag{3.4}$$

As previously noted, A is a real symmetric $M \times M$ matrix, but the assumption of (2.7) gives us in this case that γ is at most zero, and hence A must be positive. Since it can be verified that the quadratic form $\mathbf{y}^T A \mathbf{y}$ can be expressed in terms of the norm $\|\cdot\|_0$ by

$$\mathbf{y}^T A \mathbf{y} = \Big\| \sum_{i=1}^{M} y_i w_i(x) \Big\|_0^2, \tag{3.5}$$

then A is obviously positive definite. This implies that each of the matrix problems of (3.3) and (3.4) admits a unique solution, denoted by $\hat{\mathbf{u}}$ and $\tilde{\mathbf{u}}$, respectively. The associated functions in S_M are respectively denoted by

$$\hat{w}_M(x) \equiv \sum_{i=1}^{M} \hat{u}_i w_i(x) \quad \text{and} \quad \tilde{w}_M(x) \equiv \sum_{i=1}^{M} \tilde{u}_i w_i(x).$$

It then follows from (3.3) and (3.4) that $A(\hat{\mathbf{u}} - \tilde{\mathbf{u}}) = \tilde{\mathbf{g}} - \mathbf{g}$, and premultiplying by $(\hat{\mathbf{u}} - \tilde{\mathbf{u}})^T$ and using the identity of (3.5) then gives

$$(\hat{\mathbf{u}} - \tilde{\mathbf{u}})^T A (\hat{\mathbf{u}} - \tilde{\mathbf{u}}) = \|\hat{w}_M - \tilde{w}_M\|_0^2 = (\hat{\mathbf{u}} - \tilde{\mathbf{u}})^T (\tilde{\mathbf{g}} - \mathbf{g}).$$

Using the definitions of the functionals L and L_M, the last quantity above can be expressed as $(L_M - L)\,[f(x)\,(\hat{w}_M(x) - \tilde{w}_M(x)]$, and thus

$$\|\hat{w}_M - \tilde{w}_M\|_0^2 = (L_M - L)\,[f(x)\,(\hat{w}_M(x) - \tilde{w}_M(x)]. \tag{3.6}$$

This equation will be used repeatedly in this section.

Our object now is to bound $\|\hat{w}_M - \tilde{w}_M\|_0$ for certain types of quadratures L_M, after making particular assumptions on $f(x)$ and the subspace S_M. Because of (2.9) for the case $\gamma = 0$, such a bound for $\|\hat{w}_M - \tilde{w}_M\|_0$ will give a related bound for $\hat{w}_M(x) - \tilde{w}_M(x)$ in the uniform norm.

We now restrict our attention to subspaces $S_M(\pi)$ of S of *piecewise-polynomial functions*. More precisely, $\pi: 0 = x_0' < x_1' < \cdots < x_{N+1}' = 1$ is a partition of $[0, 1]$ such that for any $w(x) \in S_M(\pi)$, $w(x)$ is a polynomial of degree n_0 on each subinterval (x'_j, x'_{j+1}) of $[0, 1]$ defined by π. Such subspaces include the Hermite subspaces $H_0^{(m)}(\pi)$ and the spline subspaces $Sp_0^{(m)}(\pi)$ as special cases (cf. [4, § 6–7]). For non-trivial subspaces of S, we remark that n_0 necessarily satisfies $n_0 \geqslant n$. Next, we assume that the function f of (2.1) is such that $D^k f(x)$ is continuous on each subinterval $[x'_j, x'_{j+1}]$ defined by π for all $0 \leqslant k \leqslant m_0$. This latter hypothesis is of course valid if $f(x) \in C^{m_0}[0, 1]$, but it also holds for functions $f(x)$ whose m_0-th derivative is piecewise continuous on $[0, 1]$, with points of discontinuity a subset of the joints x'_j defined by the partition π. The important point is that since f is *given*, the quantity m_0 and the possible points of discontinuity of $D^{m_0} f$ can be determined directly.

As our first choice for the bounded linear functional L_M, consider a quadrature scheme of the form

$$\int_{y_0}^{y_m} \sigma(t)\, dt \doteq \sum_{i=0}^{m} \alpha_i \sigma(\tau_i), \tag{3.7}$$

where $y_0 \leqslant \tau_0 < \tau_1 < \cdots < \tau_m \leqslant y_m$ are selected points of $[y_0, y_m]$. Then, given any $\sigma(t) \in C^{m_0}[y_0, y_m]$, m_0 determined from f, it is always possible to select (cf. [6, p. 36 and p. 40]) a quadrature scheme of the form (3.7) such that the quadrature error of (3.7) satisfies

$$\left| \sum_{i=0}^{m} \alpha_i \sigma(\tau_i) - \int_{y_0}^{y_m} \sigma(t)\, dt \right| \leqslant K_1 (y_m - y_0)^{m_0+1} \|D^{m_0}\sigma\|_{L^\infty[y_0, y_m]}, \tag{3.8}$$

where K_1 is independent of the interval length. For example, if $m_0 = 2$, the trapezoidal rule with $m = 1$ can be selected in (3.7); if $m_0 = 10$, a five-point Gaussian quadrature scheme with $m = 4$ can be selected in (3.7). This being the case, the basic quadrature scheme of (3.7) can, after a linear change of scale, be applied on each subinterval (x'_j, x'_{j+1}) determined by π, and this in turn defines the linear functional L_M of (3.2), which takes the composite form

$$L_M[\sigma(x)] = \sum_{k=0}^{m(N+1)} \beta_k \sigma(x_k), \tag{3.9}$$

where $0 = x'_0 \leqslant x_1 < \cdots < x_m \leqslant x'_1 \leqslant x_{m+1} < \cdots \leqslant x'_{N+1} = 1$, and the coefficients β_k depend upon the coefficients α_i of (3.7) and the mesh lengths $x'_{j+1} - x'_j$. For additional notation, let $h_j \equiv x'_{j+1} - x'_j$, and let $\bar{\pi} \equiv \max_{0 \leqslant j \leqslant N} h_j$. This brings us to

THEOREM 1. *Assuming that f of (2.1) is independent of u, let $\pi: 0 = x'_0 < x'_1 < \cdots < x'_{N+1} = 1$ be any partition of $[0, 1]$ such that $D^k f$ is continuous on each subinterval $[x'_j, x'_{j+1}]$, $0 \leqslant j \leqslant N$, for all $0 \leqslant k \leqslant m_0$, and let $S_M(\pi)$ be any finite-dimensional subspace of S such that for any $w(x) \in S_M(\pi)$, $w(x)$ is a polynomial of degree n_0 on each subinterval* *defined by* π. *Then, for* $m_0 \geqslant n_0$, *the linear functional* L_M *defined in* (3.9) *is such that*

$$\|\hat{w}_M - \tilde{w}_M\|_0 \leqslant K_2 (\bar{\pi})^{m_0 - n_0}, \tag{3.10}$$

where K_2 *is a constant, independent of* π.

Proof. Expressing $(L_M - L)\,[f(\hat{w} - \tilde{w})]$ as a sum of terms and applying (3.8) to each of these terms gives

$$\begin{aligned}\|\hat{w}_M - \tilde{w}_M\|_0^2 &= (L_M - L)\,[f(x)\,(\hat{w}_M(x) - \tilde{w}_M(x))] \\ &\leqslant K_1 \sum_{j=0}^{N} (h_j)^{m_0+1} \|D^{m_0}\{f\,[\hat{w}_M - \tilde{w}_M]\}\|_{L^\infty[x'_j, x'_{j+1}]},\end{aligned} \tag{3.11}$$

where K_1 is independent of π. By hypothesis, there exists a constant C_1, independent of π, such that

$$\max_{0 \leqslant k \leqslant m_0} \max_{0 \leqslant j \leqslant N} \{\|D^k f\|_{L^\infty[x'_j, x'_{j+1}]}\} = C_1,$$

and consequently, using the Leibnitz formula for differentiating a product, the sum of (3.11) is bounded above by

$$C_1 K_1 \sum_{j=0}^{N} (h_j)^{m_0+1} \sum_{k=0}^{m_0} \binom{m_0}{k} \|D^k [\hat{w}_M - \tilde{w}_M]\|_{L^\infty[x'_j, x'_{j+1}]}. \tag{3.12}$$

Because of the assumption that the elements of $S_M(\pi)$ are piecewise-polynomials of degree n_0 with $n_0 \leqslant m_0$, the sum on k in (3.12) can be reduced to $0 \leqslant k \leqslant n_0$. Now, by a theorem of Markov [11, p. 138], there exists a constant C_2, independent of π, such that

$$\|D^k(\hat{w}_M - \tilde{w}_M)\|_{L^\infty[x'_j, x'_{j+1}]} \leqslant \frac{C_2 \|\hat{w}_M - \tilde{w}_M\|_{L^\infty[x'_j, x'_{j+1}]}}{h_j^k} \tag{3.13}$$

for all $0 \leqslant j \leqslant N$, and all $0 \leqslant k \leqslant n_0$. Substituting (3.13) in (3.12), and using the fact that $\sum_{j=0}^{N} h_j = 1$, then we have from (3.11) that there exists a constant K_2, dependent on m_0 and n_0 but independent of π, such that

$$\|\hat{w}_M - \tilde{w}_M\|_0^2 \leqslant K_2 \|\hat{w}_M - \tilde{w}_M\|_{L^\infty} \sum_{j=0}^{N} h_j^{m_0 - n_0 + 1} \leqslant K_2 (\bar{\pi})^{m_0 - n_0} \|\hat{w}_M - \tilde{w}_M\|_{L^\infty[0,1]}. \tag{3.14}$$

But as $\|v\|_{L^\infty} \leqslant K \|v\|_0$ from (2.9) for any $v \in S$, we can cancel a term $\|\hat{w}_M - \tilde{w}_M\|_0$ in (3.14), which gives the desired result of (3.10). Q.E.D.

We see from (3.10) that if we have a sequence $\{S_{M_i}(\pi_i)\}_{i=1}^{\infty}$ of finite dimensional subspaces of S such that the elements of any $S_{M_i}(\pi_i)$ are piecewise-polynomials of *fixed* degree n_0, and if $\lim_{i \to \infty} \bar{\pi}_i = 0$, then if m_0, dependent only on f, satisfies $m_0 > n_0$, we evidently have

$$\lim_{i \to \infty} \|\hat{w}_{M_i} - \tilde{w}_{M_i}\|_0 = 0.$$

This means that the quadrature error, introduced by computing $\tilde{w}_{M_i}$ rather than $\hat{w}_{M_i}$, tends to zero with i. This error, however, may or may not be small relative to $\|\hat{w}_{M_i}-\varphi\|_0$. This brings us to

DEFINITION 1. Let C be a collection of partitions π of $[0, 1]$, and for each $\pi\in C$, let $S_M(\pi)$ be a finite dimensional subspace of S consisting of elements which are polynomials of fixed degree n_0 on subintervals of $[0, 1]$ defined by π, and let $\hat{w}_M(x)$, the function which minimizes $F[w]$ of (2.10) over $S_M(\pi)$, satisfy

$$\|\hat{w}_M - \varphi\|_N \leqslant K_3(\bar{\pi})^l \quad \text{for all} \quad \pi\in C, \tag{3.15}$$

where K_3 and l are positive constants independent of π, $\varphi(x)$ is the solution of (2.1)–(2.2), and $\|\cdot\|_N$ is some norm on the space S. Then, the choice of linear functionals in (3.9) is *consistent* in the norm $\|\cdot\|_N$ with the bounds of (3.15) if there exists a constant K_4, independent of π, such that

$$\|\tilde{w}_M - \hat{w}_M\|_N \leqslant K_4(\bar{\pi})^l \quad \text{for all} \quad \pi\in C. \tag{3.16}$$

We remark that with the triangle inequality, the bounds of (3.15) for the norm $\|\cdot\|_0$, and the result of Theorem 1, it follows that

$$\|\tilde{w}_M - \varphi\|_0 \leqslant \|\tilde{w}_M - \hat{w}_M\|_0 + \|\hat{w}_M - \varphi\|_0 \leqslant K_2(\bar{\pi})^{m_0-n_0} + K_3(\bar{\pi})^l, \pi\in C. \tag{3.17}$$

Thus, it follows that $m_0-n_0\geqslant l$ gives a consistent choice of functionals in (3.9) in the norm $\|\cdot\|_0$ which *preserves* the asymptotic accuracy of (3.15) in this norm. It is also true that even if this choice is *not* consistent in the norm $\|\cdot\|_0$, i.e., if $1<m_0-n_0<l$, it nonetheless follows that when the collection C is a *sequence* of partitions $\{\pi_i\}_{i=1}^{\infty}$, then the associated sequence $\{\tilde{w}_{M_i}(x)\}_{i=1}^{\infty}$ converges *uniformly* to $\varphi(x)$ as $i\to\infty$ when $\lim_{i\to\infty}\bar{\pi}_i=0$.

If we assume that the differential operator $\mathscr{L}$ in (2.3) is *strongly elliptic* [14, p. 175], i.e., $p_n(x)\geqslant\omega>0$ for all $x\in[0, 1]$, then the following improvement of Theorem 1 is possible.

THEOREM 2. *With the hypotheses of Theorem 1, assume that the operator* $\mathscr{L}$ *of* (2.3) *is strongly elliptic. Then, for the linear functional* L_M *defined in* (3.9), *there exists a constant* K_5, *independent of the partition* π, *such that*

$$\|\hat{w}_M - \tilde{w}_M\|_0 \leqslant K_5(\bar{\pi})^{m_0} \quad \textit{if} \quad m_0 < n, \tag{3.18}$$

and

$$\|\hat{w}_M - \tilde{w}_M\|_0 \leqslant K_5(\bar{\pi})^{m_0-\min(m_0,n_0)+n-1} \quad \textit{if} \min(m_0, n_0) \geqslant n, \tag{3.18'}$$

where n *is determined from* (2.3).

Proof. As in the proof of Theorem 1, we can write (cf. (3.12)) that

$$\|\hat{w}_M - \tilde{w}_M\|_0^2 \leqslant C_1 K_1 \sum_{j=0}^{N} h_j^{m_0+1} \sum_{k=0}^{m_0} \binom{m_0}{k} \|D^k(\hat{w}_M - \tilde{w}_M)\|_{L^\infty[x'_j, x'_{j+1}]} . \tag{3.19}$$

If $r \equiv \min(m_0, n_0)$, the sum on k above can be restricted to $0 \leqslant k \leqslant r$, and this sum is then divided into $0 \leqslant k \leqslant n-1$, and $n \leqslant k \leqslant r$:

$$\begin{aligned}\|\hat{w}_M - \tilde{w}_M\|_0^2 \leqslant C_1 K_1 \sum_{j=0}^{N} h_j^{m_0+1} \Bigg\{ &\sum_{k=0}^{n-1} \binom{m_0}{k} \|D^k(\hat{w}_M - \tilde{w}_M)\|_{L^\infty[x'_j, x'_{j+1}]} \\ &+ \sum_{k=n}^{r} \binom{m_0}{k} \|D^k(\hat{w}_M - \tilde{w}_M)\|_{L^\infty[x_j', x_j'+1]} \Bigg\} .\end{aligned} \tag{3.19'}$$

Of course, if $r < n$, this last sum is omitted. For any $v(x) \in S$, it follows from the boundary conditions of (2.2), the fundamental theorem of calculus, and Schwarz's inequality (cf. [13, § 3]) that

$$\|D^j v\|_{L^\infty[0,1]} \leqslant \tfrac{1}{2} \|D^{j+1} v\|_{L^2[0,1]} \quad \text{for all } 0 \leqslant j \leqslant n-1 .$$

Also, it is evident, for all $0 \leqslant j \leqslant n-1$, that

$$\|D^{j+1} v\|_{L^2[0,1]} \leqslant \|v\|_{n,2} \equiv \left\{ \int_0^1 \left[\sum_{k=0}^{n} (D^k v(t))^2 \right] dt \right\}^{1/2}$$

for any $v(x) \in S$. Thus, we see that the first sum on k in (3.19′) can be bounded above a constant, dependent now on n and m_0 but independent of π, times $\|\hat{w} - \tilde{w}\|_{n,2}$. For the second sum on k in (3.19′), we again use the Markov inequality, viz.

$$\|D^k(\hat{w}_M - \tilde{w}_M)\|_{L^\infty[x'_j, x'_{j+1}]} \leqslant C'_2 \frac{\|D^{n-1}(\hat{w}_M - \tilde{w}_M)\|_{L^\infty[x'_j, x'_{j+1}]}}{h_j^{k-n+1}}, \quad r \geqslant k \geqslant n .$$

Combining these inequalities with the inequality of (3.19′) gives

$$\|\hat{w}_M - \tilde{w}_M\|_0^2 \leqslant K_6 \{(\bar{\pi})^{m_0} \|\hat{w}_M - \tilde{w}_M\|_{n,2} + (\bar{\pi})^{m_0 - r + n - 1} \|\hat{w} - \tilde{w}\|_{n,2}\}, \tag{3.20}$$

where the last term is omitted if $r < n$. Finally, because the operator $\mathscr{L}$ of (2.3) in this case is strongly elliptic, it follows by Gårding's inequality [14, p. 175] that there exist positive constants c_1 and c_2 such that $\|v\|_{n,2}^2 \leqslant c_1 \|v\|_0^2 + c_2 \|v\|_{L^2[0,1]}^2$ for all $v(x) \in S$. Moreover, since Λ of (2.6) in the linear case is, by hypothesis (2.7), necessarily positive, it follows from (2.6) that $\Lambda \|v\|_{L^2[0,1]} \leqslant \|v\|_0$ for all $v(x) \in S$, whence

$$\|v\|_{n,2}^2 \leqslant \left(c_1 + \frac{c_2}{\Lambda} \right) \|v\|_0^2 .$$

With the above inequality applied to (3.20), the results of (3.18)–(3.18′) follow.

Q.E.D.

There is another convenient way to approximate the quantities g_i in (3.1). Suppose we substitute for $f(x)$ in $\int_0^1 f(x)\, w_i(x)dx$, a function $\tilde{f}(x)$ which is an interpolate of $f(x)$, and we evaluate $\int_0^1 f(x)\, w_i(x)dx$ *exactly*. If we again assume that we have a partition $\pi: 0 = x'_0 < x'_1 < < x \cdots'_{N+1} = 1$ of $[0, 1]$, then if $w_i(x)$ is a polynomial on each subinterval $[x'_i, x'_{i+1}]$, $0 \leqslant i \leqslant N$, defined by π, and if $\tilde{f}(x)$ is a polynomial on $[x'_i, x'_{i+1}]$, $0 \leqslant i \leqslant N$, this integral $\int_0^1 \tilde{f}(x)\, w_i(x)dx$ is simply the sum of integrals of polynomials over $[x'_i, x'_{i+1}]$, $0 \leqslant i \leqslant N$, and hence is easy to calculate on a digital computer. We determine how accurate an approximation $\tilde{f}(x)$ must be to $f(x)$ in order for this type of quadrature scheme to be useful in our variational technique.

As before, if we approximate $f(x)$ by $\tilde{f}_M(x)$ in (3.1), we generate a system of equations (3.2) where

$$\tilde{g}_j = L_M[f w_j] = L[\tilde{f}_M w_j] \equiv \int_0^1 \tilde{f}_M(x)\, w_j(x)\, dx, \quad 1 \leqslant j \leqslant M. \tag{3.21}$$

This in turn again serves to define $\tilde{w}_M(x) = \sum_{i=1}^M \tilde{u}_i w_i(x)$ from the solution of (3.4).

THEOREM 3. *Assuming that f of (2.1) is independent of u, let $\pi: 0 = x'_0 < x'_1 < \cdots < < x'_{N+1} = 1$ be any partition of $[0, 1]$ such that f is continuous on each subinterval $[x'_j, x'_{j+1}]$, $0 \leqslant j \leqslant N$, and let $S_M(\pi)$ be any finite dimensional subspace of S such that for any $v(x) \in S_M(\pi)$, $v(x)$ is a polynomial on each subinterval defined by π. If $\tilde{f}_M(x;\pi)$ is a continuous piecewise-polynomial interpolate of $f(x)$ such that $\tilde{f}_M(x;\pi)$ is a polynomial on each subinterval defined by π, let L_M be the associated linear functional of* (3.21). *Then, for the constant K of* (2.9),

$$\|\hat{w}_M - \tilde{w}_M\|_0 \leqslant K\, \|f - \tilde{f}_M\|_{L^r} \quad \textit{for any} \quad 1 \leqslant r \leqslant +\infty. \tag{3.22}$$

Proof. From (3.6), we have

$$\|\hat{w}_M - \tilde{w}_M\|_0^2 = \int_0^1 [\tilde{f}_M(x;\pi) - f(x)]\, (\hat{w}_M(x) - \tilde{w}_M(x))\, dx,$$

and applying Hölder's inequality gives

$$\|\hat{w}_M - \tilde{w}_M\|_0^2 \leqslant \|\tilde{f}_M - f\|_{L^r} \cdot \|\hat{w}_M - \tilde{w}_M\|_{L^{r'}}, \quad \text{where} \quad \frac{1}{r} + \frac{1}{r'} = 1. \tag{3.23}$$

But from (2.9),

$$\|\hat{w}_M - \tilde{w}_M\|_{L^{r'}} \leqslant \|\hat{w}_M - \tilde{w}_M\|_{L^\infty} \leqslant K\, \|\hat{w}_M - \tilde{w}_M\|_0,$$

and we can cancel a factor $\|\hat{w}_M - \tilde{w}_M\|_0$ in (3.23), which gives the desired result of (3.22). Q.E.D.

From the inequality of (3.22), it is now clear how the piecewise-polynomial interpolate is to be chosen so as to have a consistent quadrature scheme in some norm. If $f(x) \in C^{2m}[0, 1]$, we then have ([3, Theorem 2] and [10, Theorem 9]) that there is a continuous piecewise-polynomial interpolate $\tilde{f}_M(x)$ such that

$$\|f - \tilde{f}_M\|_{L^2} \leqslant K(\bar{\pi})^{2m}, \tag{3.24}$$

where K is independent of π. For example, if $f(x) \in C^2[0, 1]$, then the piecewise linear interpolate $\tilde{f}_M(x)$ of $f(x)$ satisfies $\|f - \tilde{f}_M\|_{L^2} \leqslant K(\bar{\pi})^2$.

To illustrate the results of these theorems, let us consider the particular boundary value problem

$$D^2 u(x) = f(x), \quad 0 < x < 1, \quad \text{with} \quad u(0) = u(1) = 0, \tag{3.25}$$

corresponding to the choice $\mathscr{L} = D^2$, $n = 1$, in (2.1)–(2.2). For this example, $\mathscr{L}$ is strongly elliptic, and we can apply the results of Theorem 2. Using the continuous piecewise linear functions of the Hermite space $H_0^{(1)}(\pi)$, then $n_0 = 1$, and if $f(x) \in C^{m_0}[0, 1]$, $m_0 \geqslant 0$, then the results of [4, Theorem 10] give us that the inequality of (3.15) is in fact valid for $l = 1$ in the norm $\|\cdot\|_0$, i.e.,

$$\|\hat{w}_i - \varphi\|_0 \leqslant K_3(\bar{\pi}), \quad i \geqslant 1.$$

Thus, in order to obtain a collection of linear functionals consistent with the above inequalities in the norm $\|\cdot\|_0$, it is clear from Theorem 2 that m_0 must satisfy $m_0 \geqslant 2$. As previously mentioned, the quadrature error of (3.8) for $m_0 = 2$ is valid in particular for the trapezoidal rule. Thus, quadrature based on the trapezoidal rule gives a collection of linear functionals consistent with the above error bounds in the norm $\|\cdot\|_0$. We remark that Theorem 1 also gives the same conclusions in this case.

We now show that the interpolation technique of Theorem 3 for the problem of (3.25), when used in conjunction with the elements of the continuous piecewise linear functions of the Hermite space $H_0^{(1)}(\pi)$, can give rise to *standard finite difference methods* for (3.25). For a *uniform* partition $\pi_N: 0 = x_0' < x_1' < \cdots < x_{N+1}' = 1$ where $x_i' \equiv ih$, $h \equiv 1/N+1$, the following functions $\{t_i(x)\}_{i=1}^N$ form a basis for $H_0^{(1)}(\pi_N)$ (cf. A of Fig. 1 of [4]):

$$t_i(x) = \left\{ \begin{array}{ll} (x - x_{i-1}')/h, & x_{i-1}' \leqslant x \leqslant x_i' \\ (x_{i+1}' - x)/h, & x_i' \leqslant x \leqslant x_{i+1}' \\ 0, & x \in [0, 1], \quad x \notin [x_{i-1}', x_{i+1}'] \end{array} \right\}, \quad 1 \leqslant i \leqslant N.$$

The associated $N \times N$ matrix $A = (a_{i,j})$ of (3.3) has entries given by

$$a_{i,j} = \int_0^1 t_i'(x)\, t_j'(x)\, dx = \begin{cases} 2/h, & i = j \\ -1/h, & |i - j| = 1 \\ 0, & \text{otherwise}. \end{cases} \tag{3.26}$$

Now in (3.3),

$$g_i = \int_0^1 f(x)\, t_i(x)\, dx = \int_{x'_{i-1}}^{x'_i} f(x) \left(\frac{x - x'_{i-1}}{h} \right) dx + \int_{x'_i}^{x'_{i+1}} f(x) \left(\frac{x'_{i+1} - x}{h} \right) dx \tag{3.27}$$

for $1 \leqslant i \leqslant N$. Suppose we wish to use the trapezoidal rule to approximate the last two integrals of (3.27). We thus obtain

$$\tilde{g}_i = \frac{h}{2} \left[f(x'_i) \left(\frac{x'_i - x'_{i-1}}{h} \right) \right] + \frac{h}{2} \left[f(x'_i) \left(\frac{x'_{i+1} - x'_i}{h} \right) \right] = h f(x'_i), \quad 1 \leqslant i \leqslant N.$$

Hence, the system (3.4) can be written as

$$B\tilde{\mathbf{u}} = \mathbf{f}, \tag{3.28}$$

where

$$B = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix} \quad \text{and } \underline{f} = \begin{bmatrix} -f(x'_1) \\ -f(x'_2) \\ \vdots \\ -f(x'_N) \end{bmatrix}.$$

The system (3.28) then yields a solution $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N)^T$ and $\tilde{w}(x) = \sum_{j=1}^N \tilde{u}_j t_j(x)$ is our approximation to the solution of (3.25). Note that $\tilde{u}_i = \tilde{w}(x'_i)$ is our approximation to the solution of (3.25) at x'_i. In fact, the system (3.28) is *exactly* the same system one obtains when one approximates the solution of (3.25) at the points $x'_1, x'_2, \ldots, x'_N$ by the standard three-point finite difference technique described in [8, p. 63] and [12, p. 61]. *Hence, the three-point finite difference scheme may be thought of as applying the variational technique to the subspace $H_0^{(1)}(\pi)$ of continuous piecewise linear functions, followed by an application of the trapezoidal rule.*

To push this observation further, consider the same problem, subspace and mesh, but now let us approximate g_i in (3.27) by a different method. For $f(x)$ in

$$g_i = \int_{x'_{i-1}}^{x'_{i+1}} f(x)\, t_i(x)\, dx,$$

suppose we substitute the *quadratic* interpolation polynomial

$$\tilde{f}_i(x) = \frac{1}{2h^2} [(x - x'_{i-1})(x - x'_i) f(x'_{i-1}) - (x - x'_{i-1})(x - x'_{i+1}) f(x'_i) + (x - x'_{i-1})(x - x'_i) f(x'_{i+1})]$$

and integrate exactly; here, $\tilde{f}_i(x)$ is simply the Lagrange interpolation of $f(x)$ on $[x'_{i-1}, x'_{i+1}]$ at x'_{i-1}, x_i, and x'_{i+1}. Hence, our system (3.4) is

$$B\mathbf{u} = \mathbf{v} \tag{3.29}$$

where B is the same as B in (3.28) and

$$v_i = \frac{1}{h} \int_{x'_{i-1}}^{x'_{i+1}} \tilde{f}_i(x)\, t_i(x)\, dx .$$

It can be verified that

$$v_i = \frac{10 f(x'_i) + f(x'_{i+1}) + f(x'_{i-1})}{12} .$$

The system (3.29) yields a solution vector $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_N)^T$ and we note that

$$\tilde{w}(x_i) = \sum_{j=1}^{N} \tilde{u}_j t_j(x_i) = \tilde{u}_i$$

is the approximation to the solution of (3.25) at x'_i. But system (3.29) is exactly the same system one obtains from approximating the solution of (3.25) at the points $x'_1, x'_2, \ldots, x'_N$ by Collatz's *Mehrstellenverfahren*, described in [5, p. 164] and also in [12, p. 180]. Collatz's method is $0(h^4)$ at the mesh points, assuming that the solution is in $C^6[0, 1]$. In other words, *this known finite difference method is a special case of the variational approach to the problem* (3.25), *coupled with an appropriate quadrature.*

§ 4. Nonlinear Case

In this section, we assume that the function f of (2.1) depends on u. Furthermore, we will assume that the differential operator $\mathscr{L}$ in (2.3) is strongly elliptic, i.e., by Gårding's inequality [14, p. 175], (2.5′) holds for $l = n - 1$. Hence, as shown in Corollary 2 of [4], it is easy to verify that the norm $\|\cdot\|_\gamma$ is equivalent to the Sobolev norm.

For the general problem (2.1)–(2.2), if we approximate the integrals $g_i(\mathbf{u})$, $1 \leqslant i \leqslant M$, in (2.13) by a quadrature scheme and denote these approximations by $\tilde{g}_i(\mathbf{u})$, $1 \leqslant i \leqslant M$, this gives us the new system of equations

$$\mathbf{0} = A\mathbf{u} + \tilde{\mathbf{g}}(\mathbf{u}) . \tag{4.1}$$

Unlike the case in which f is independent of u, we are not assured that the system (4.1) has unique solution. As in the last section, we will denote by $\hat{w}_M(x)$ the approximation generated by the system (2.14) using subspace S_M.

Let $\pi: 0 = x'_0 < x'_1 < \ldots < x'_{N+1} = 1$ be a partition of the interval $[0, 1]$. Writing

$g_i(\mathbf{u})$ as the sum

$$\sum_{k=0}^{n} \int_{x'_k}^{x'_{k+1}} f\left(x, \sum_{j=1}^{M} u_j w_j(x)\right) w_i(x)\, dx$$

and applying (3.7) to the $N+1$ integrals in this sum, we obtain the approximation, which we write in a simplified notation, as

$$\tilde{g}_i(\mathbf{u}) = \sum_{j=0}^{M_0} \beta_j f\left(x_j, \sum_{k=1}^{M} u_k w_k(x_j)\right) w_i(x_j), \quad M_0 \equiv m(N+1). \tag{4.2}$$

Putting $\tilde{g}_i(\mathbf{u})$ in (2.14) for $g_i(\mathbf{u})$, we obtain the system (4.1). We will now prove that the new system (4.1), when proper restrictions are put on the quadrature scheme (3.7), has a unique solution $\tilde{\mathbf{u}}$.

THEOREM 4. *Given any finite-dimensional subspace S_M of S spanned by the linearly independent set $\{w_i(x)\}_{i=1}^{M}$ and given the quadrature scheme*

$$\int_{y_0}^{y_m} \sigma(y)\, dy \doteq \sum_{i=0}^{M} \alpha_i \sigma(\tau_i),$$

where $y_0 \leqslant \tau_0 < \tau_1 < \cdots < \tau_m \leqslant y_m$, if $\alpha_i \geqslant 0$ for $0 \leqslant i \leqslant m$, $\sum_{i=0}^{m} \alpha_i = y_m - y_0$, and if, in the notation (4.2),

$$\sum_{k=0}^{M_0} \beta_k w_i(x_k)\, w_j(x_k) = \int_0^1 w_i(x)\, w_j(x)\, dx \tag{4.3}$$

for $1 \leqslant i, j \leqslant M$ (note that $\alpha_i \geqslant 0$ implies $\beta_k \geqslant 0$ and $\sum_{i=0}^{m} \alpha_i = y_m - y_0$ implies $\sum_{k=0}^{M_0} \beta_k = 1$), then there exists a unique solution $\tilde{\mathbf{u}}$ to the system (4.1) *which may be written, using the notation of* (4.2), *as*

$$\begin{aligned} 0 = \int_0^1 &\left\{ \sum_{k=0}^{n} p_k(x)\, D^k \left(\sum_{j=1}^{M} u_j w_j(x) \right) D^k w_i(x) \right\} dx \\ &+ \sum_{k=0}^{M_0} \beta_k f\left(x_k, \sum_{j=1}^{M} u_j w_j(x_k)\right) w_i(x_k), \quad 1 \leqslant i \leqslant M \end{aligned} \tag{4.4}$$

Proof. Because of the similarities with the steps of [4, § 3], we shall only sketch the proof here. Complete details are given in [7].

We first define the following functional on S_M:

$$H[w] = \int_0^1 \left\{ \tfrac{1}{2} \sum_{j=0}^{n} p_j(x) \left(D^j \left(\sum_{i=1}^{M} u_i w_i(x) \right) \right)^2 \right\} dx + \sum_{k=0}^{M_0} \beta_k \int_0^{\sum_{i=1}^{M} u_i w_i(x_k)} f(x_k, \eta)\, d\eta,$$

where $w(x)=\sum_{i=1}^{M} u_i w_i(x)$. Note that if we take the partial derivatives of H with respect to u_i, $1 \leqslant i \leqslant M$, and set them equal to zero, we get exactly system (4.4). Hence, if we can show that the functional $H[w]$ has a unique stationary value over S_M, then this would imply that (4.4) has a unique solution. $H[w]$ is a functional over R^M and we write $H[w]$ as $H[\mathbf{u}]$ where $\mathbf{u}=(u_1, u_2, \ldots, u_M)^T$. In order to show $H[\mathbf{u}]$ has a unique stationary value, we first show that $H[\mathbf{u}]$, $\mathbf{u} \in R^M$, is bounded below. Next, we verify that the set $\{\mathbf{u} \in R^M;\ H[\mathbf{u}] \leqslant H[\mathbf{0}]=0\}$ is compact. Then, we prove that $H[\mathbf{u}]$ represents a strictly convex surface. These facts then imply that $H[\mathbf{u}]$ has a unique stationary value, which is a minimum, and therefore imply Theorem 4. Q.E.D.

THEOREM 5. *Let C be any collection of quasi-uniform partitions $\pi: 0=x_0' < \cdots < x_{N+1}'=1$ of $[0,1]$, i.e., if $\underline{\pi} \equiv \min_{0 \leqslant j \leqslant N}(x_{j+1}'-x_j')$, then there exists a constant $\sigma>0$ such that $\sigma\underline{\pi} \geqslant \bar{\pi}$ for all $\pi \in C$, and for each $\pi \in C$, let $S_M(\pi)$ be a finite-dimensional subspace of S consisting of polynomial L-spline* [10] *functions such that for any $v(x) \in S_M(\pi)$, $v(x)$ is a polynomial of degree at most n_0 on each subinterval defined by π. If $\partial^k f/\partial x^k(x, v(x))$, $0 \leqslant k \leqslant m_0$, is continuous in each subinterval defined by π, for all $v(x) \in S_M(\pi)$, for all $\pi \in C$, the solution φ of* (2.1)–(2.2) *is in $C^{n_0+1}[0,1]$, the quadrature scheme* (3.7), *used to approximate the $g_j(\mathbf{u})$ in* (2.13), *satisfies all the hypotheses of Theorem 4 for each subspace $S_M(\pi)$, and $m_0 \geqslant n_0$, then there exists a positive constant K_7 such that*

$$\|\hat{w}_M - \tilde{w}_M\|_\gamma \leqslant K_7 (\bar{\pi})^s \quad \text{for all} \quad \pi \in C, \tag{4.5}$$

where $s \equiv \min(m_0 - n_0 + n - 1,\ n_0 + 1 - n)$.

Proof. For any partition $\pi \in C$, let $\{w_i(x)\}_{i=1}^{M}$, be a basis for $S_M(\pi)$. We recall from (2.14) that $\hat{\mathbf{u}}$ satisfies the system

$$(A\mathbf{u})_i = -\int_0^1 f\left(x, \sum_{j=1}^{M} u_j w_j(x)\right) w_i(x)\, dx, \quad 1 \leqslant i \leqslant M, \tag{4.6}$$

and we have from (4.4) that $\tilde{\mathbf{u}}$ satisfies

$$(A\tilde{\mathbf{u}})_i = -\sum_{k=0}^{M_0} \beta_k f\left(x_k, \sum_{j=1}^{M} \tilde{\mathbf{u}}_j w_j(x_k)\right) w_i(x_k), \quad 1 \leqslant i \leqslant M. \tag{4.7}$$

Letting $\bar{w}(x)=\sum_{j=1}^{M} \bar{u}_j w_j(x)$ be the *interpolate* in S_M of the unique solution $\varphi(x)$ of problem (2.1)–(2.2), we define

$$\varepsilon_i = (A\bar{\mathbf{u}})_i + \int_0^1 f(x, \bar{w}(x))\, w_i(x)\, dx, \quad 1 \leqslant i \leqslant M,$$

which can be written as

$$(A\bar{\mathbf{u}})_i = -\sum_{k=0}^{M_0} \beta_k f(x_k, \bar{w}(x_k))\, w_i(x_k) + \varepsilon_i - \int_0^1 f(x, \bar{w}(x))\, w_i(x)\, dx + \sum_{k=0}^{M_0} \beta_k f(x_k, \bar{w}(x_k))\, w_i(x_k), \tag{4.8}$$

for $1 \leqslant i \leqslant M$. Subtracting (4.7) from (4.8) and premultiplying by $(\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T$ we obtain

$$\begin{aligned}(\bar{\mathbf{u}} - \tilde{\mathbf{u}})^T A(\bar{\mathbf{u}} - \tilde{\mathbf{u}}) = \sum_{k=0}^{M_0} \beta_k [f(x_k, \tilde{w}(x_k)) - f(x_k, \bar{w}(x_k))]\,(\bar{w}(x_k) - \tilde{w}(x_k)) \\ + (\bar{\mathbf{u}} - \tilde{\mathbf{u}})^T \boldsymbol{\varepsilon} + \sum_{k=0}^{M_0} \beta_k f(x_k, \bar{w}(x_k))\,(\bar{w}(x_k) - \tilde{w}(x_k)) \\ - \int_0^1 f(x, \bar{w}(x))\,(\bar{w}(x) - \tilde{w}(x))\, dx,\end{aligned}$$

where $\tilde{w}(x) = \sum_{j=1}^{M} \tilde{u}_j w_j(x)$. From (2.7), we know that

$$[f(x, \bar{w}(x)) - f(x, \tilde{w}(x))]\,(\bar{w}(x) - \tilde{w}(x)) \geqslant \gamma(\bar{w}(x) - \tilde{w}(x))^2. \tag{4.10}$$

Since we are assuming $\beta_k \geqslant 0$, $0 \leqslant k \leqslant M_0$, then (4.9) and (4.10) imply

$$\begin{aligned}(\bar{\mathbf{u}} - \tilde{\mathbf{u}})^T A(\bar{\mathbf{u}} - \tilde{\mathbf{u}}) + \gamma \sum_{k=0}^{M_0} \beta_k (\bar{w}(x_k) - \tilde{w}(x_k))^2 \leqslant (\bar{\mathbf{u}} - \tilde{\mathbf{u}})^T \boldsymbol{\varepsilon} \\ + \sum_{k=0}^{M_0} \beta_k f(x_k, \bar{w}(x_k))\,(\bar{w}(x_k) - \tilde{w}(x_k)) \\ - \int_0^1 f(x, \bar{w}(x))\,(\bar{w}(x) - \tilde{w}(x))\, dx.\end{aligned} \tag{4.11}$$

Now by assumption (4.3) and the definition of $\|\cdot\|_\gamma$, the quantity on the left of the inequality above is just $\|\bar{w} - \tilde{w}\|_\gamma^2$, and hence

$$\begin{aligned}\|\bar{w} - \tilde{w}\|_\gamma^2 \leqslant (\bar{\mathbf{u}} - \tilde{\mathbf{u}})^T \boldsymbol{\varepsilon} + \sum_{k=0}^{M_0} \beta_k f(x_k, \bar{w}(x_k))\,(\bar{w}(x_k) - \tilde{w}(x_k)) \\ - \int_{M_0}^1 f(x, \bar{w}(x))\,(\bar{w}(x) - \tilde{w}(x))\, dx.\end{aligned} \tag{4.12}$$

The quantity

$$\sum_{k=0}^{M_0} \beta_k f(x_k, \bar{w}(x_k))\,(\bar{w}(x_k) - \tilde{w}(x_k)) - \int_0^1 f(x, \bar{w}(x))\,(\bar{w}(x) - \tilde{w}(x))\, dx$$

on the right hand side of (4.12) is simply the error in applying our quadrature scheme (3.7) on the intervals $[x_i', x_{i+1}']$, $1 \leqslant i \leqslant N$, to the function $f(x, \bar{w}(x))(\bar{w}(x) - \tilde{w}(x))$. We see from (3.8) that this error is bounded above by

$$K_1 \sum_{j=0}^{N} (h_j)^{m_0+1} \|D^{m_0}\{f(\cdot, \bar{w})(\bar{w} - \tilde{w})\}\|_{L^\infty[x'_j, x'_{j+1}]}. \tag{4.13}$$

In an argument similar (cf. [7, pp. 60–62]) to that used in the proofs of Theorems 1 and 2, using the assumed smoothness of f and the boundedness of derivatives of the L-spline interpolate $\bar{w}(x)$ from Theorem 10 of [10], we can bound the term (4.13) above by

$$K_8 \|\bar{w} - \tilde{w}\|_\gamma (\bar{\pi})^{m_0 - n_0 + n - 1} \tag{4.14}$$

where K_8 is a positive constant. It can be verified (Section 5, Chapter I of [7]) that there is a constant K_9 such that

$$(\bar{\mathbf{u}} - \tilde{\mathbf{u}})^T \boldsymbol{\varepsilon} \leqslant K_9 \|\bar{w} - \hat{w}\|_\gamma \|\bar{w} - \tilde{w}\|_\gamma. \tag{4.15}$$

Hence, using (4.12), (4.13), (4.14), and (4.15), we have

$$\|\bar{w} - \tilde{w}\|_\gamma \leqslant K_9 \|\hat{w} - \bar{w}\|_\gamma + K_8 (\bar{\pi})^{m_0 - n_0 + n - 1}. \tag{4.16}$$

Hence,

$$\begin{aligned} \|\hat{w} - \tilde{w}\|_\gamma &\leqslant \|\hat{w} - \bar{w}\|_\gamma + \|\bar{w} - \tilde{w}\|_\gamma \\ &\leqslant (1 + K_9)\|\hat{w} - \bar{w}\|_\gamma + K_8 (\bar{\pi})^{m_0 - n_0 + n - 1} \\ &\leqslant (1 + K_9) K_{10} (\bar{\pi})^{n_0 + 1 - n} + K_8 (\bar{\pi})^{m_0 - n_0 + n - 1}, \end{aligned}$$

where the inequality $\|\hat{w} - \bar{w}\|_\gamma \leqslant K_{10}(\bar{\pi})^{n_0 + 1 - n}$ is provided by [10, Theorem 24]. Q.E.D.

We remark that, for *particular* polynomial L-spline subspaces of S, it can be shown (cf. [9]) that $\|\hat{w} - \bar{w}\|_\gamma \leqslant K_{10}(\bar{\pi})^{n_0+1}$. From (4.16), this means for such subspaces that the result of (4.15) of Theorem 5 is valid for $s \equiv \min(m_0 - n_0 + n - 1, n_0 + 1)$. This will be useful in § 5.

We now discuss the analogue of Definition 1 for the nonlinear case.

DEFINITION 2. Let C be any collection of quasi-uniform partitions of $[0, 1]$, and for each $\pi \in C$, let $S_M(\pi)$ be a finite dimensional subspace of S consisting of polynomial L-spline functions such that for any $v(x) \in S_M(\pi)$, $v(x)$ is a polynomial of degree at most n_0 on each subinterval defined by π, and let $\hat{w}_M(x)$, the function which minimizes $F[w]$ of (2.10) over $S_M(\pi)$, satisfy

$$\|\hat{w}_M - \varphi\|_N \leqslant K_{11}(\bar{\pi})^l, \quad \text{for all} \quad \pi \in C, \tag{4.17}$$

where K_{11} and l are positive constant independent of π, φ is the solution of (2.1)–(2.2), and $\|\cdot\|_N$ is some norm on the space S. Then, the choice of the quadrature scheme in (3.7) is *consistent* in the norm $\|\cdot\|_N$ with the bounds (4.17) if there exists a positive constant K_{12}, independent of π, such that

$$\|\hat{w}_M - \tilde{w}_M\|_N \leqslant K_{12}(\bar{\pi})^l \quad \text{for all} \quad \pi \in C. \tag{4.18}$$

COROLLARY. *If the hypotheses of Theorem 5 hold, then the quadrature scheme in* (3.7) *is consistent in the norm* $\|\cdot\|_\gamma$ *with the bound* $\|\hat{w}_M - \varphi\|_\gamma \leqslant K_{11}(\bar{\pi})^{no+1-n}$ *provided by* [9, Theorem 24], *if* $m_0 \geqslant 2 + 2n_0 - 2n$.

It is interesting to consider what happens if we take the quantities $g_i(\mathbf{u})$ in (2.13) and instead of applying a quadrature scheme such as (3.7) to $f(x, \sum_{j=1}^M u_j w_j(x))\, w_i(x)$, we interpolate f by $\tilde{f}$ and then evaluate

$$\tilde{g}_i(\mathbf{u}) = \int_0^1 \tilde{f}\left(x, \sum_{j=1}^M u_j w_j(x)\right) w_i(x)\, dx \tag{4.19}$$

exactly. The new nonlinear system that we would generate is

$$0 = d_i(\mathbf{u}) = (A\mathbf{u})_i + \tilde{g}_i(\mathbf{u}), \quad 1 \leqslant i \leqslant M, \tag{4.20}$$

where A is defined in (2.12) and $\tilde{g}_i(\mathbf{u})$ in (4.19). The first thing we must do is to try to make assumptions on $\tilde{f}$ so that we are assured that (4.20) has a unique solution. As we saw earlier in this section, a convenient way to do this is to find a strictly convex functional such that its gradient set to zero is exactly the system you wish to solve.

From Theorem 10.45 of [1], we know that in order for there to exist a functional whose gradient set to zero is our system $d_i(\mathbf{u}) = 0$, $1 \leqslant i \leqslant M$, described in (4.20), we must have

$$\frac{\partial d_i(\mathbf{u})}{\partial u_j} = \frac{\partial d_j(\mathbf{u})}{\partial u_i}, \quad 1 \leqslant i, j \leqslant M.$$

When we would want to obtain a very accurate approximation to the solution of (2.1)–(2.2) using this interpolation method, we would probably want to use an interpolation scheme, such as piecewise Hermite interpolation or spline interpolation, which would use derivatives of $f(x, \sum_{j=1}^M u_j w_j(x))$ with respect to x at certain points. Notice that these derivatives depend upon the derivatives of the basis functions with respect to x. For example,

$$\frac{d}{dx} f\left(x, \sum_{j=1}^M u_j w_j(x)\right) = \frac{\partial f}{\partial x}\left(x, \sum_{j=1}^M u_j w_j(x)\right) + \frac{\partial f}{\partial u}\left(x, \sum_{j=1}^M u_j w_j(x)\right)\left(\sum_{k=1}^M u_k \frac{dw_k(x)}{dx}\right).$$

Suppose we assume that the interpolation $\tilde{f}$ in our system (4.20) depends on the values of

$$f\left(x, \sum_{j=1}^M u_j w_j(x)\right) \quad \text{and} \quad \frac{d}{dx} f\left(x, \sum_{j=1}^M u_j w_j(x)\right)$$

at certain points. Then it is clear that the equality of

$$\frac{\partial d_i(\mathbf{u})}{\partial u_j} \quad \text{and} \quad \frac{\partial d_j(\mathbf{u})}{\partial u_i}$$

would depend on the equality of $w_j'(x)\, w_i(x)$ and $w_j(x)\, w_i'(x)$, which, in general, does not hold. Therefore, we see that we are not assured that there is a functional whose gradient is the system (4.20). This implies that we may not be able to use the computationally attractive minimizing algorithms which the existence of a strictly convex functional allows us to use. Therefore, although in some cases it may be applicable, in general, generating schemes to approximate $g_i(\mathbf{u})$ in (2.13) by interpolating f is not useful when f is a function of u as well as x.

It should be pointed out that the quadrature methods discussed so far in this section and in the previous section are not always applicable since it may be the case that either the function f does not satisfy the necessary differentiability assumptions or the degree of the polynomials in the subspace S_{M_i} of the sequence $\{S_{M_i}\}_{i=1}^{\infty}$ may be increasing with i, as is the case with the polynomial subspaces described in [4]. One quadrature scheme which can be used in these two cases, and, in fact which can *always* be considered consistent is Romberg integration. The computational disadvantage of Romberg integration is the large number of integrand evaluations necessary, as compared with, say, Gaussian quadrature.

§ 5. Numerical Examples

Let us now cite some particular examples of the use of consistent quadrature schemes.

Consider the special case of problem (2.1)–(2.2) given by

$$-D^4u(x) = -(x^4 + 14x^3 + 49x^2 + 32x - 12)\, e^x, \quad 0 < x < 1, \tag{5.1}$$

where

$$u(0) = u(1) = Du(0) = Du(1) = 0, \tag{5.2}$$

which corresponds to the bending of a thin beam, clamped at both ends. For this problem, (2.5′) is valid with $l=0$, $K=1/\pi$, and $\beta=0$. Also, it is easy to see in this case from (2.6) that Λ is positive, and γ in (2.7) can be chosen to be zero, since f is independent of u. The unique solution of (5.1)–(5.2) is $u(x)=x^2(1-x)^2e^x$, $0\leqslant x\leqslant 1$, which strictly minimizes the functional

$$F[w] = \int_0^1 \{\tfrac{1}{2}(D^2w(x))^2 - (x^4 + 14x^3 + 49x^2 + 32x - 12)\, e^x w(x)\}\, dx, \quad w \in S. \tag{5.3}$$

Let $S_{M_N} = H_0^{(2)}(\pi_N)$, the subspace of piecewise cubic Hermite polynomials [4, § 6], and assume π_N is the *uniform* partition on $[0, 1]$ with mesh size $h_N = 1/(N+1)$. If we denote by $\hat{w}_N(x)$ the unique element which minimizes $F[w]$ in (5.3) over $H_0^{(2)}(\pi_N)$, then from [9] we know that there exists constants K_1 and K_2 (independent of N) such that

$$\|\varphi - \hat{w}_N\|_{L^\infty} \leqslant K_1 h_N^4, \tag{5.4}$$

and

$$\|D(\varphi - \hat{w}_N)\|_{L^\infty} \leqslant K_2 h_N^3, \tag{5.5}$$

where φ is the unique solution of (5.1)–(5.2). The three-point Gaussian quadrature scheme with weight function unity was used to approximate the integrals involving the function $(x^4 + 14x^3 + 49x^2 + 32x - 12)\, e^x$ in the system of equations generated in minimizing $F[w]$ in (5.3) over $H_0^{(2)}(\pi_N)$, and the approximation obtained in $H_0^{(2)}(\pi_N)$ is denoted by $\tilde{w}_N(x)$. Noting that the differential operator D^4 in (5.1) is strongly elliptic and also that $(x^4 + 14x^3 + 49x^2 + 32x - 12)\, e^x \in C^\infty[0, 1]$, we see from Theorem 2 with $m_0 = 6$, $n_0 = 3$, and $n = 2$ that there exists a constant K_3 such that

$$\|\hat{w}_N - \tilde{w}_N\|_0 \leqslant K_3 h_N^4. \tag{5.6}$$

Therefore, the three-point Gaussian quadrature scheme with weight function unity is consistent by Definition 1 in the norm $\|\cdot\|_{L^\infty}$ with the bounds (5.4). Now from (2.9) and (5.6), we have

$$\|\hat{w}_N - \tilde{w}_N\|_{L^\infty} \leqslant K K_3 h_N^4, \tag{5.7}$$

and by applying the Markov theorem cited in § 3 to (5.7) there is a constant K_4 such that

$$\|D(\hat{w}_N - \tilde{w}_N)\|_{L^\infty} \leqslant K_4 h_N^3. \tag{5.8}$$

Hence, from (5.4), (5.5), (5.7), and (5.8),

$$\|\varphi - \tilde{w}_N\|_{L^\infty} \leqslant (K_1 + K K_3)\, h_N^4, \tag{5.9}$$

and

$$\|D(\varphi - \tilde{w}_N)\|_{L^\infty} \leqslant (K_2 + K_4)\, h_N^3. \tag{5.10}$$

The numerical results are given in Table 1, and in this table, we include the quantity

$$\alpha \equiv \log\left(\frac{\|\varphi - \tilde{w}_{n_1}\|_{L^\infty}}{\|\varphi - \tilde{w}_{n_2}\|_{L^\infty}}\right) \Big/ \log\left(\frac{h_{n_1}}{h_{n_2}}\right), \tag{5.11}$$

defined in terms of successive values of the mesh spacing h. The motivation for (5.11) is the fact that asymptotically, as $h_N \to 0$, we have

$$\|\varphi - \tilde{w}_N\|_{L^\infty} \sim K(h_N)^\alpha$$

Table 1

Subspace $H_0^{(2)}(\pi_N)$

h_N	Dim of $H_0^2(\pi_N)$	$\|\varphi - \tilde{w}_N\|_{L^\infty}$	α	$\|D(\varphi - \tilde{w}_N)\|_{L^\infty}$	α'
1/4	6	$1.56 \cdot 10^{-3}$	–	$1.98 \cdot 10^{-2}$	–
1/5	8	$6.94 \cdot 10^{-4}$	3.67	$1.09 \cdot 10^{-2}$	2.70
1/6	10	$3.53 \cdot 10^{-4}$	3.71	$6.63 \cdot 10^{-3}$	2.73
1/8	14	$1.19 \cdot 10^{-4}$	3.81	$2.91 \cdot 10^{-3}$	2.88
1/10	18	$5.05 \cdot 10^{-5}$	3.86	$1.55 \cdot 10^{-3}$	2.83
1/12	22	$2.51 \cdot 10^{-5}$	3.88	$4.27 \cdot 10^{-4}$	2.84
1/16	30	$8.21 \cdot 10^{-6}$	3.90	$4.06 \cdot 10^{-4}$	2.88
1/20	38	$3.44 \cdot 10^{-6}$	3.90	$2.15 \cdot 10^{-4}$	2.88
1/24	46	$1.70 \cdot 10^{-6}$	3.91	$1.27 \cdot 10^{-4}$	2.92

for some constants α and K which are independent of h_N. Then for two successive values of h, $h_{n_1} > h_{n_2}$, we have, asymptotically

$$\frac{\|\varphi - \tilde{w}_{n_1}\|_{L^\infty}}{\|\varphi - \tilde{w}_{n_2}\|_{L^\infty}} \sim \left(\frac{h_{n_1}}{h_{n_2}}\right)^{\alpha} \tag{5.12}$$

and (5.11) follows from (5.12). In the table, enough values of h are given to show that the computed exponent of (5.11) agrees quite well with the asymptotic value of $\alpha = 4$ from (5.9). The quantity α' in this table is defined similarly; and computationally agrees well with the exponent $\alpha = 3$ from (5.10).

As our second example, we consider

$$D^2 u(x) = e^{u(x)}, \quad 0 < x < 1 \quad \text{with} \quad u(0) = u(1) = 0. \tag{5.13}$$

In this sample, we verify that (2.4) is valid for $K = \frac{1}{2}$, $\beta = 0$, and Λ in (2.6) is π^2. We can choose γ in (2.7) to be zero. The unique solution of (5.11) is [2, p. 41]:

$$\varphi(x) = -\ln 2 + 2 \ln\{c \sec[c(x - \tfrac{1}{2})/2]\}, \quad c = 1.3360557,$$

which minimizes the functional

$$F[w] = \int_0^1 \{\tfrac{1}{2}(Dw(x))^2 + e^{w(x)} - 1\}\, dx, \quad w \in S. \tag{5.14}$$

The solution to problem (5.13) was approximated by minimizing $F[w]$ in (5.14) over the cubic Hermite subspace $H_0^{(2)}(\pi_N)$ and also the cubic spline subspace $Sp_0^{(2)}(\pi_N)$, where in each case the partition π_N is the uniform mesh on $[0, 1]$ with mesh size $h_N = 1/(N+1)$. When the functional $F[w]$ in (5.14) is minimized over either of these subspaces, the resulting approximations $\hat{w}_N(x)$ satisfy [9]

$$\|\varphi - \hat{w}_N\|_{L^\infty} \leqslant K_5 h_N^4. \tag{5.15}$$

For the two subspaces considered, the four-point Gaussian quadrature scheme with weight function unity was used to approximate the integrals resulting from the term $\int_0^1 (e^{w(x)} - 1)\,dx$ in the functional. Denoting the resulting approximations by $\tilde{w}_N(x)$, it follows from the remarks after Theorem 5 that, for these subspaces, the result of (4.5) of Theorem 5 is valid for $s = \min(m_0 - n_0 + n - 1, n_0 + 1) = 4$ where $m_0 = 8$ and $n_0 = 3$. Thus, with (2.9), we have

$$\|\hat{w}_N - \tilde{w}_N\|_{L^\infty} \leqslant K\,\|\hat{w}_N - \tilde{w}_N\|_0 \leqslant K_6 h_N^4. \tag{5.16}$$

Hence, from (5.15), and (5.16), the four-point Gaussian quadrature scheme is consistent in the norm $\|\cdot\|_{L^\infty}$ by Definition 2 for either of the subspaces considered, and we thus have

$$\|\varphi - \tilde{w}_N\|_{L^\infty} \leqslant (K_5 + K_6)\, h_N^4.$$

The numerical results in the norm $\|\cdot\|_{L^\infty}$ for the cubuc Hermite subspaces $H_0^{(2)}(\pi_N)$ and the cubic spline subspace $Sp_0^{(2)}(\pi_N)$ are given in Tables 2 and 3 respectively. In both cases, we include the quantity α, as defined in the previous example, to indicate the ratio of convergence computationally obtained in the norm $\|\cdot\|_{L^\infty}$. Note that the *observed* accuracy in this norm is Kh_N^4.

The last example we consider is the second order problem

$$D^2 u(x) = f_0(x, u), \quad 0 < x < 1 \quad \text{with} \quad u(0) = u(1) = 0, \tag{5.17}$$

where

$$f_0(x,u) = \begin{cases} -9.5u(x) + 6x^2 - 5x + (12/9.5), \\ \qquad 0 \leqslant x < \tfrac{1}{2} \\ -9.5u(x) + 6x^2 - 5x + (12/9.5) - 15(2x-1)^{1/2}/9.5 \\ \qquad -(2x-1)^{5/2}, \qquad \tfrac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

Table 2

Subspace $H_0^{(2)}(\pi_N)$

h_N	Dim of $H_0^{(2)}(\pi_N)$	$\|\varphi(x) - \tilde{w}_N(x)\|_{L^\infty}$	α
1/2	2	$5.10 \cdot 10^{-5}$	–
1/3	6	$1.21 \cdot 10^{-5}$	3.54
1/4	8	$4.24 \cdot 10^{-6}$	3.65
1/6	12	$9.58 \cdot 10^{-7}$	3.65
1/8	16	$3.10 \cdot 10^{-7}$	3.93
1/10	20	$1.28 \cdot 10^{-7}$	3.96
1/12	24	$6.28 \cdot 10^{-8}$	3.91

Table 3
Subspace $Sp_0^{(2)}(\pi_N)$

h_N	Dim of $Sp_0^{(2)}(\pi_N)$	$\|\varphi(x)-\tilde{w}_N(x)\|_{L^\infty}$	a
1/4	5	$5.70\cdot10^{-6}$	–
1/5	6	$2.39\cdot10^{-6}$	2.47
1/6	7	$1.19\cdot10^{-6}$	3.90
1/7	8	$6.44\cdot10^{-7}$	3.97
1/8	9	$3.63\cdot10^{-7}$	4.18

As in the last example, we verify that (2.4) is valid for $K=\frac{1}{2}$, $\beta=0$, $\Lambda=\pi^2$, and that we can choose γ in (2.7) to be zero. The unique solution of (5.17) is

$$\varphi(x)=\begin{cases}(6x^2-5x)/9.5, & 0\leqslant x<\frac{1}{2}\\ (6x^2-5x-(2x-1)^{5/2})/9.5, & \frac{1}{2}\leqslant x\leqslant 1,\end{cases} \tag{5.18}$$

which minimizes

$$F[w]=\int_0^1\left\{\tfrac{1}{2}(Dw(x))^2+\int_0^{w(x)} f_0(x,\eta)\,d\eta\right\}dx,\quad w\in S. \tag{5.19}$$

The subspace of piecewise linear function $H_0^{(1)}(\pi_N)$, where again π_N is the uniform mesh with mesh size $h_N=1/(N+1)$, was used to obtain approximations to $u(x)$ in (5.18). Denoting by $\hat{w}_N(x)$ the element which minimizes $F[w]$ over $H_0^{(1)}(\pi_N)$, we know from [8] that $\|\varphi-\hat{w}_N\|_{L^\infty}\leqslant K_7 h_N^2$. Because $\varphi(x)\in C^2[0,1]$, but $\varphi(x)\notin C^3[0,1]$, Romberg integration was used to evaluate the integrals involving the function $f_0(x,u)$ in the system of equations generated by minimizing $F[w]$ in (5.19) over $H_0^{(1)}(\pi_N)$ and $\tilde{w}_N(x)$ denotes the resulting approximation. The numerical results are given in Table 4 and again we include the quantity α, as defined in (5.11). We should

Table 4
Subspace $H_0^{(1)}(\pi_N)$

h_N	Dim of $H_0^{(1)}(\pi_N)$	$\|\varphi(x)-\tilde{w}_N(x)\|_{L^\infty}$	α
1/4	3	$6.27\cdot10^{-2}$	–
1/8	7	$2.67\cdot10^{-2}$	1.18
1/10	9	$1.94\cdot10^{-2}$	1.58
1/16	15	$8.54\cdot10^{-3}$	1.75
1/20	19	$5.62\cdot10^{-3}$	1.88
1/24	23	$3.97\cdot10^{-3}$	1.90
1/32	31	$2.26\cdot10^{-3}$	1.96
1/40	39	$1.46\cdot10^{-3}$	1.97
1/48	47	$1.02\cdot10^{-3}$	1.98

mention here that the standard three-point finite difference techniques [5, p. 63], as applied to this problem (5.15), in contrast only give that

$$\max_{0 \leqslant i \leqslant N+1} \left| \varphi\left(\frac{i}{N+1}\right) - v_i \right| \leqslant K_8 \omega\left(D^2\varphi; \frac{1}{N+1}\right) \leqslant \frac{K_9}{\sqrt{N+1}},$$

where v_i is the associated discrete approximation to $\varphi(i/N+1)$, and ω is the modulus of continuity.

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