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ACCURATE NUMERICAL METHODS FOR NONLINEAR BOUNDARY VALUE PROBLEMS*

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1. Introduction. The use of variational or projectional methods to approximate solutions of nonlinear boundary value problems has received a great deal of attention lately, cf. [3], [9], [10], [23], [25], [26] and [39]. Of course, the idea of using the Ritz–Galerkin method to approximate these solutions is not new. What is new, however, is that effective error bounds for such approximations have been developed (cf. equation (2.5)) at roughly the same time that spline and Hermite piecewise-polynomial functions have independently grown into vogue. These spline and Hermite functions are particularly attractive for high-speed computers, since the proper choice of basis functions for these subspaces gives associated coefficient matrices which are *sparse* (cf. [12]). The net result is that the combination of using spline and Hermite functions in a Ritz–Galerkin setting with the new error bounds offers a highly effective tool for approximating the solutions of such nonlinear boundary value problems.

The purpose of this paper is to show how this combination does in fact lead to very accurate numerical approximations of solutions of nonlinear boundary value problems. Since most of the extensive numerical computations using spline and Hermite functions have been for one-dimensional problems, we shall confine our discussion and numerical results to such problems.

In § 2, we give a theoretical background for the special results to follow. Then, in § 3, we look specifically at two-point nonlinear boundary value problems, and § 4 contains sample numerical results of particular experiments. Because the techniques developed also apply quite easily to one-dimensional eigenvalue problems, we study such eigenvalue problems in § 5, and give related numerical results.

2. Theoretical background. The theoretical basis for the material presented here is contained in [14]. Let B be a reflexive Banach space over the real field, and let B^* be the dual of B. We denote respectively by $\|\cdot\|$ and $\|\cdot\|^*$ the norms in B and in B^*, and $(\cdot,\cdot)$ denotes the usual pairing between B and B^*, i.e., if $v^* \in B^*$ and $u \in B$, then the value of the functional v^* at u is (v^*, u).

Let T be a (possibly nonlinear) mapping from B into B^* satisfying the following two hypotheses:

(H_1) T is *strongly monotone* [9], [26], [39], i.e., there exists a continuous and strictly increasing function $c(r)$ on $[0, +\infty)$ with $c(0) = 0$ and $\lim_{r\to+\infty} c(r) = +\infty$

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such that

$$|(Tu - Tv, u - v)| \geqq c(\|u - v\|) \cdot \|u - v\| \quad \text{for all} \quad u, v \in B. \tag{2.1}$$

(H_2) T is *finitely continuous*, i.e., T is continuous from finite-dimensional subspaces of B into B^* with the weak-star topology of B^*. In other words, given any finite-dimensional subspace B^k of B and any sequence $\{u_n\}_{n=1}^{\infty}$ of elements of B^k which converges to an element $u \in B^k$, the sequence $\{(Tu_n, v)\}_{n=1}^{\infty}$ converges to (Tu, v) for any $v \in B$.

We consider the following problem, called *Problem* P: Determine $u \in B$ such that

$$Tu = 0, \tag{2.2}$$

or equivalently, determine $u \in B$ such that

$$(Tu, v) = 0 \quad \text{for all} \quad v \in B. \tag{2.3}$$

Similarly, given a finite-dimensional subspace B^k of B, we consider the following approximate problem, called *Problem* P_k: determine $u_k \in B^k$ such that

$$(Tu_k, v) = 0 \quad \text{for all} \quad v \in B^k. \tag{2.4}$$

We now state the following result, due to Browder [9].

LEMMA 2.1. *Let T satisfy* (H_1) *and* (H_2). *Then Problem* P *has a unique solution u. Similarly, given any finite-dimensional subspace B^k of B, the corresponding Problem* P_k *has a unique solution* u_k.

To have an estimate between the solution u of Problem P and the solution u_k of Problem P_k, we need additional hypotheses on the mapping T (cf. Theorem 2.1). These in turn will allow us to obtain sufficient conditions guaranteeing the convergence of the u_k's to the solution u (cf. Corollary 2.1). We begin with the following theorem.

THEOREM 2.1. *Let T satisfy* (H_1), (H_2) *and*

(H_3) *T is bounded, i.e., T maps bounded subsets of B into bounded subsets of* B^* *(with respect to the strong topology of* B^**). Then, given any finite-dimensional subspace B^k of B, there exists a constant K, independent of B^k, such that*

$$c(\|u_k - u\|) \cdot \|u_k - u\| \leqq K \inf\{\|w - u\|; w \in B^k\}. \tag{2.5}$$

Similarly, let T satisfy (H_2),

(H'_1) *condition* (H_1) *holds with* $c(r) \equiv \alpha r$, $\alpha > 0$, *i.e.*,

$$|(Tu - Tv, u - v)| \geqq \alpha(\|u - v\|)^2 \quad \textit{for all} \quad u, v \in B, \tag{2.6}$$

and

(H'_3) *T is Lipschitz continuous with respect to the strong topology of* B^* *for bounded arguments (a special case of hypothesis* (H_3)), *i.e., given* $M > 0$, *there exists a constant* $C(M)$, *depending only upon M, such that*

$$\|Tu - Tv\|^* \leqq C(M)\|u - v\| \quad \textit{for all} \quad u, v \in B \quad \textit{with} \quad \|u\|, \|v\| \leqq M. \tag{2.7}$$

Then, given any finite-dimensional subspace B^k of B, there exists a constant K′, *independent of* B^k, *such that*

$$\|u_k - u\| \leqq K' \inf\{\|w - u\|\,;\, w \in B^k\}. \tag{2.8}$$

Proof. We begin by showing that (H_1) implies that the same a priori bound holds for both the solution u and the "approximate" solutions u_k. We have, by using (2.1) and (2.4),

$$c(\|u_k\|)\|u_k\| \leqq |(Tu_k - T0, u_k)| = |(T0, u_k)| \leqq \|T0\|^*\|u_k\|,$$

and thus $c(\|u_k\|) \leqq M_0$, with $M_0 = \|T0\|^*$. Clearly, the same bound is valid for u.

Let w be now an arbitrary element of B^k. Then by (2.3) and (2.4), we have $(Tu_k - Tu, u_k - w) = 0$ since $\{u_k - w\} \in B^k \subset B$. Thus from (2.1),

$$\begin{aligned} c(\|u_k - u\|)\|u_k - u\| &\leqq |(Tu_k - Tu, u_k - u)| = |(Tu_k - Tu, w - u)| \\ &\leqq \|Tu_k - Tu\|^*\|w - u\|. \end{aligned} \tag{2.9}$$

If T is bounded, then $\|Tu_k - Tu\|^*$ is bounded independently of B^k and the conclusion of (2.5) follows, since w is arbitrary. Similarly, if T satisfies (H'_1) and (H'_3), the conclusion of (2.8) follows with $K' = C(M_0)/\alpha$, by (2.9).

As an immediate consequence, we have the following corollary.

COROLLARY 2.1. *Let* $\{B^k\}_{k=1}^{\infty}$ *be a sequence of finite-dimensional subspaces of* B *with the property that*

$$\lim_{k \to +\infty} \{\inf\{\|w - u\|\,;\, w \in B^k\}\} = 0, \tag{2.10}$$

where u *is the unique solution of Problem* P. *If* T *satisfies* (H_1), (H_2), (H_3) (*including as a special case* (H'_1), (H'_2), (H'_3)), *then*

$$\lim_{k \to +\infty} \{\|u_k - u\|\} = 0, \tag{2.11}$$

where u_k, $k = 1, 2, \cdots$, *are the unique solutions of Problem* P_k.

We now introduce some standard notation for the following sections. For m a positive integer, the Sobolev space $W^{m,2}[a, b]$ consists of all real-valued functions $f(x)$ defined on $[a, b)$ such that f and its distributional derivatives D^jf with $0 \leqq j \leqq m$ all belong to $L^2[a, b]$. The Sobolev space $W^{m,2}[a, b]$ is a Hilbert space with respect to the inner product

$$(u, v)_m \equiv \int_a^b \left\{\sum_{j=0}^{m} D^ju(x) \cdot D^jv(x)\right\} dx, \quad u, v \in W^{m,2}[a, b], \tag{2.12}$$

and we denote the norm associated with this inner product by $\|\cdot\|_m$. The space $W_0^{m,2}[a, b]$ is then the closure in the norm $\|\cdot\|_m$ of all infinitely differentiable functions with compact support in $[a, b]$. Finally,

$$\|w\|_{L^\infty[a,b]} \equiv \sup_{x \in [a,b]} |w(x)| \tag{2.13}$$

denotes the uniform norm of any real-valued function $w(x)$ defined on $[a, b]$.

3. Two-point boundary value problems. As a particular application of the theory given in § 2, consider the approximate solution of the following two-point nonlinear boundary value problem:

$$M[u(x)] + f(x, u(x)) = 0, \quad a < x < b, \tag{3.1}$$

where

$$M[u(x)] \equiv \sum_{0 \leqq i,j \leqq n} (-1)^j D^j(\sigma_{i,j}(x) D^i u(x)), \qquad n \geqq 1, \qquad D \equiv \frac{d}{dx}, \tag{3.1'}$$

subject to the homogeneous boundary conditions of

$$D^j u(a) = D^j u(b) = 0, \qquad 0 \leqq j \leqq n-1. \tag{3.2}$$

For the coefficient functions $\sigma_{i,j}(x)$, $0 \leqq i, j \leqq n$, of (3.1′), we assume that:

(3.3a) the coefficient functions $\sigma_{i,j}(x)$, $0 \leqq i, j \leqq n$, are bounded, real-valued and measurable in x in $[a, b]$;

and

there exists a positive constant c such that

$$\int_a^b \left\{ \sum_{0 \leqq i,j \leqq n} \sigma_{i,j}(x) D^i w(x) \cdot D^j w(x) \right\} dx \geqq c \|w\|_n^2 \tag{3.3b}$$

for all $w(x) \in W_0^{n,2}[a, b]$.

It follows from (3.3b) that

$$\Lambda \equiv \inf_{\substack{w \in W_0^{n,2}[a,b] \\ w \not\equiv 0}} \frac{\int_a^b \left\{ \sum_{0 \leqq i,j \leqq n} \sigma_{i,j}(x) D^i w(x) \cdot D^j w(x) \right\} dx}{\int_a^b w^2(x)\, dx} \tag{3.4}$$

is *positive*. With respect to the function $f(x, u)$ of (3.1), we assume that:

(3.5a) $f(x, u)$ is a real-valued function on $[a, b] \times R$ such that $f(x, u_0(x)) \in L^2[a, b]$ for any $u_0(x) \in W_0^{n,2}[a, b]$;

there exists a real constant γ such that

$$\frac{f(x, u) - f(x, v)}{u - v} \geqq \gamma > -\Lambda \tag{3.5b}$$

for almost all $x \in [a, b]$ and all $-\infty < u, v < +\infty$ with $u \neq v$;

for each positive real number c, there exists a positive constant $M(c)$ such that

$$\frac{f(x, u) - f(x, v)}{u - v} \leqq M(c) \tag{3.5c}$$

for almost all $x \in [a, b]$ and all $-\infty < u, v < +\infty$ with $u \neq v$ and $|u| \leqq c$, $|v| \leqq c$.

With these assumptions, the following result is a slight extension of [14, Theorem 7.1] to the nonself-adjoint case.

THEOREM 3.1. *With the assumptions* (3.3a, b) *and* (3.5a, b, c), *the two-point nonlinear boundary value problem of* (3.1)–(3.2) *has a unique generalized solution* $u(x)$ *in* $W_0^{n,2}[a,b]$. *Moreover, if* B^k *is any finite-dimensional subspace of* $W_0^{n,2}[a,b]$, *then the approximate problem* P_k (cf. (2.4)) *has a unique solution* $u_k(x)$, *and there exist positive constants* K_1 *and* K_2, *independent of the choice of* B^k, *such that*

$$\|D^i(u_k - u)\|_{L^\infty[a,b]} \leqq K_1\|u_k - u\|_n \leqq K_2 \inf\{\|w_k - u\|_n; w_k \in B^k\} \tag{3.6}$$

for all $0 \leqq i \leqq n-1$.

Proof. For any $u, v \in W_0^{n,2}[a,b]$, we formally define the "quasi-bilinear" form from (3.1):

$$a(u,v) \equiv \int_a^b \left\{\sum_{0 \leqq i,j \leqq n} \sigma_{i,j}(x)D^iu(x)\cdot D^jv(x) + f(x,u(x))\cdot v(x)\right\} dx. \tag{3.7}$$

From the assumptions (3.3a, b) and (3.5a, b, c), it is easily seen that for any fixed $u \in W_0^{n,2}[a,b]$, there exists a constant $K = K_u$, depending only on u, such that

$$|a(u,v)| \leqq K_u\|v\|_n \quad \text{for all} \quad v \in W_0^{n,2}[a,b]. \tag{3.8}$$

Consequently, $a(u,v)$ is for each $u \in W_0^{n,2}[a,b]$ a continuous linear functional in $v \in W_0^{n,2}[a,b]$, and we can thus write

$$a(u,v) = (Tu,v)_n \quad \text{for all} \quad u,v \in W_0^{n,2}[a,b], \tag{3.9}$$

where T defines a mapping of $W_0^{n,2}[a,b]$ into $W_0^{n,2}[a,b]$. That T so defined is bounded and finitely continuous also follows easily.

To show that T is strongly monotone, we have from (3.7) and (3.9) that

$$\begin{aligned}(Tu - Tv, u - v)_n &= a(u, u-v) - a(v, u-v)\\ &= \int_a^b \left\{\sum_{0 \leqq i,j \leqq n} \sigma_{i,j}D^i(u-v)\cdot D^j(u-v)\right.\\ &\qquad \left. + \left(\frac{f(x,u) - f(x,v)}{u-v}\right)(u-v)^2\right\} dx.\end{aligned}$$

Using hypotheses (3.3b) and (3.5b) and the positivity of Λ, it then follows that

$$(Tu - Tv, u - v)_n \geqq c\left(\frac{\Lambda + \min(\gamma,0)}{\Lambda}\right)\|u - v\|_n^2 \quad \text{for all} \quad u, v \in W_0^{n,2}[a,b], \tag{3.10}$$

and hence, T is strongly monotone.

We now show that T is Lipschitz continuous for bounded arguments. For any u, v, $w \in W_0^{n,2}[a,b]$, we have, using Schwarz's inequality and hypothesis

(3.5c), that

$$
\begin{aligned}
|(Tu - Tv, w)_n| &= |a(u, w) - a(v, w)| \\
&= \int_a^b \left\{ \sum_{0 \leqq i,j \leqq n} \sigma_{i,j} D^i(u - v) \cdot D^j w \right. \\
&\quad \left. + \left(\frac{f(x, u) - f(x, v)}{u - v} \right) (u - v) \cdot w \right\} dx \\
&\leqq (\tau + M(c)) \|u - v\|_n \cdot \|w\|_n,
\end{aligned}
$$

where we have assumed that $\tau \equiv \sum_{0 \leqq i,j \leqq n} \|\sigma_{i,j}\|_{L^\infty[a,b]}$, and that $\|u\|_\infty \leqq c$, $\|v\|_\infty \leqq c$. Thus,

$$
(3.11) \qquad \|Tu - Tv\|_n = \sup_{w \in W^{n,2}[a,b]} \frac{|(Tu - Tv, w)|}{\|w\|_n} \leqq (\tau + M(c)) \|u - v\|_n,
$$

which establishes that T is Lipschitz continuous for bounded arguments. Finally, as a consequence of the Sobolev imbedding theorem in one dimension (cf. [38, p. 174]), we know that there exists a positive constant K_1 such that

$$
\|D^i w\|_{L^\infty[a,b]} \leqq K_1 \|w\|_n \quad \text{for all} \quad w \in W_0^{n,2}[a, b], \quad \text{all} \quad 0 \leqq i \leqq n - 1.
$$

The remainder of Theorem 3.1 then follows immediately from Theorem 2.1.

Our objective now is to specialize the general finite-dimensional subspaces of $W_0^{n,2}[a, b]$ to subspaces of L-splines, which were considered in [2] and [34]. To explain briefly the nature of L-splines, let L be any rth order linear differential operator of the form

$$
(3.12) \qquad L[v(x)] = \sum_{j=0}^{r} c_j(x) D^j v(x), \quad r \geqq 1, \quad v \in C^r[a, b],
$$

where we assume that the coefficient function $c_j(x)$ is in $C^j[a, b]$ for all $0 \leqq j \leqq r$, and that, in addition, there exists a positive constant ω such that

$$
(3.13) \qquad c_r(x) \geqq \omega > 0 \quad \text{for all} \quad x \in [a, b].
$$

Next, let $\Delta : a = x_0 < x_1 < \cdots < x_{N+1} = b$ denote any partition of the interval $[a, b]$, and let $\mathbf{z} = (z_1, z_2, \cdots, z_N)$, the *incidence vector* associated with Δ, be any vector with positive integer components z_i satisfying $1 \leqq z_i \leqq r$ for all $1 \leqq i \leqq N$. Then, $\mathrm{Sp}(L, \Delta, \mathbf{z})$ is defined [34] as the collection of all real-valued functions $s(x)$, called *L-splines*, defined on $[a, b]$, such that

$$
(3.14) \qquad
\begin{aligned}
&L^*L[s(x)] = 0 \quad \text{for} \quad x \in (x_i, x_{i+1}) \quad \text{for each } i, && 0 \leqq i \leqq N, \\
&D^k s(x_i-) = D^k s(x_i+) \quad \text{for all} \quad 0 \leqq k \leqq 2r - 1 - z_i, && 1 \leqq i \leqq N,
\end{aligned}
$$

where L^* denotes the formal adjoint of L, i.e., for any $v(x) \in C^r[a, b]$, $L^*[v(x)] \equiv \sum_{j=0}^{r} (-1)^j D^j(a_j(x) v(x))$. As an important special case, if $L[u(x)] \equiv D^r u(x)$, and $\hat{z}_1 = \hat{z}_2 = \cdots = \hat{z}_N = 1$, then the elements of $\mathrm{Sp}(D^r, \Delta, \hat{\mathbf{z}})$ are then simply *the natural spline functions* [32] and Sp $(D^r, \Delta, \hat{\mathbf{z}})$ becomes $\mathrm{Sp}^{(r)}(\Delta)$ in the notation of [36]. Similarly, when $L[u(x)] = D^r u(x)$ and $\tilde{z}_1 = \tilde{z}_2 = \cdots = \tilde{z}_N = r$, the elements of Sp $(D^r, \Delta, \tilde{\mathbf{z}})$ are then simply the *Hermite piecewise-polynomial functions*, and Sp $(D^r, \Delta, \mathbf{z})$ becomes $H^{(r)}(\Delta)$ in the notation of [12] and [36].

Given a function $f(x) \in C^{r-1}[a, b]$, where r is the order of the differential operator L of (3.12), there are various ways in which one might interpolate f in Sp $(L, \Delta, \mathbf{z})$. As a particular case, it is shown in [34] that there exists a unique element $s(x) \in$ Sp $(L, \Delta, \mathbf{z})$ such that

$$\text{(3.15)}\quad \begin{aligned} D^k s(x_i) &= D^k f(x_i), \quad 0 \leqq k \leqq z_i - 1, \quad 1 \leqq i \leqq N, \\ D^k s(x_i) &= D^k f(x_i), \quad 0 \leqq k \leqq r - 1 \quad \text{for} \quad i = 0 \quad \text{and} \quad i = N + 1. \end{aligned}$$

This element $s(x)$ is called the Sp $(L, \Delta, \mathbf{z})$-*interpolate of* $f(x)$ *of Type* I. For example, if $f(x) \in C^1[a, b]$, if $L[u(x)] \equiv D^2 u(x)$, and if $\tilde{z}_1 = \tilde{z}_2 = \cdots = \tilde{z}_N = 1$, then the piecewise-cubic function $s(x) \in$ Sp $(D^2, \Delta, \hat{\mathbf{z}})$ which satisfies (3.15) with $r = 2$, is just the natural cubic spline interpolation (of Type I) of $f(x)$. It is clear that, given the parameters $\alpha_i^{(k)}$, $0 \leqq k \leqq z_i - 1$, $0 \leqq i \leqq N + 1$ (where we define for convenience $z_0 = z_{N+1} = r$), there exists a unique function $s(x)$ in Sp $(L, \Delta, \mathbf{z})$ such that

$$D^k s(x_i) = \alpha_i^{(k)}, \quad 0 \leqq k \leqq z_i - 1, \quad 0 \leqq i \leqq N + 1,$$

and we denote by $\mathrm{Sp}^I(L, \Delta, \mathbf{z})$ the finite-dimensional subspace of Sp $(L, \Delta, \mathbf{z})$ of all such functions.

We now give some error bounds for interpolation in $\mathrm{Sp}^I(L, \Delta, \mathbf{z})$. For any partition $\Delta : a = x_0 < x_1 < \cdots < x_{N+1} = b$ of $[a, b]$, let $\bar{\Delta} \equiv \max_{0 \leqq i \leqq N}(x_{i+1} - x_i)$, and let $\mathbf{z} = (z_1, \cdots, z_N)$ be any associated incidence vector. Based on an extension of results of [12, Theorems 7 and 9], it was shown in [28] that if $f(x) \in W^{r,2}[a, b]$, then there exists a positive constant M such that for *any* partition Δ of $[a, b]$ and *any* associated incidence vector $\mathbf{z}$,

$$\text{(3.16)}\qquad \|D^j(f - s)\|_{L^2[a,b]} \leqq M(\bar{\Delta})^{r-j}\|Lf\|_{L^2[a,b]}, \qquad 0 \leqq j \leqq r,$$

where $s(x)$ is the unique $\mathrm{Sp}^I(L, \Delta, \mathbf{z})$-interpolate of $f(x)$. Similarly, if $f(x) \in W^{2r,2}[a, b]$, there exists a positive constant M' such that for *any* partition Δ and *any* associated incidence vector $\mathbf{z}$,

$$\text{(3.17)}\qquad \|D^j(f - s)\|_{L^2[a,b]} \leqq M'(\bar{\Delta})^{2r-j}\|L^*Lf\|_{L^2[a,b]}, \qquad 0 \leqq j \leqq r.$$

With these error bounds for interpolation in $\mathrm{Sp}^I(L, \Delta, \mathbf{z})$, we can apply the results of Theorem 3.1 as follows. For $r \geqq n$, let $\mathrm{Sp}_0^I(L, \Delta, \mathbf{z})$ denote the subspace of $\mathrm{Sp}^I(L, \Delta, \mathbf{z})$ of elements which satisfy the homogeneous boundary conditions of (3.2). Then, it follows by construction that $\mathrm{Sp}_0^I(L, \Delta, \mathbf{z})$ is a finite-dimensional subspace of $W_0^{n,2}[a, b]$. Applying Theorem 3.1 with $B^k \equiv \mathrm{Sp}_0^I(L, \Delta, \mathbf{z})$ gives us the following theorem (cf. [14, Theorems 7.2]).

THEOREM 3.2. *With the assumptions* (3.3a, b) *and* (3.5a, b, c), *let* $u(x)$ *be the unique generalized solution of* (3.1)–(3.2) *in* $W_0^n[a, b]$ *and for any partition* Δ *of* $[a, b]$, *and any associated incidence vector* $\mathbf{z}$, *let* $\hat{u}$ *be the unique solution of approximate problem* P_k *for the subspace* $B^k \equiv \mathrm{Sp}_0^I(L, \Delta, \mathbf{z})$, *where the order* r *of* L

satisfies $r \geqq n$. *Then there exist positive constants* K_1 *and* K_2, *independent of* Δ *and* z, *such that if* $u(x) \in W^{t,2}[a, b]$ *with* $t \geqq r$, *then*

$$\|D^i(\hat{u} - u)\|_{L^\infty[a,b]} \leqq K_1 \|\hat{u} - u\|_n \leqq K_2(\bar{\Delta})^{r-n} \|Lu\|_{L^2[a,b]} \tag{3.18}$$

for all $0 \leqq i \leqq n - 1$. *Similarly, if* $u(x) \in W^{2t,2}[a, b]$ *with* $t \geqq r$, *then there exist positive constants* K_1 *and* K_2', *independent of* Δ *and* $\mathbf{z}$, *such that*

$$\|D^i(\hat{u} - u)\|_{L^\infty[a,b]} \leqq K_1 \|\hat{u} - u\|_n \leqq K_2'(\bar{\Delta})^{2r-n} \|L^*Lu\|_{L^2[a,b]} \tag{3.19}$$

for all $0 \leqq i \leqq n - 1$.

The error bounds of (3.19) of Theorem 3.2 can be improved (cf. [11] and [31]) if

(i) the generalized solution $u(x)$ of (3.1)–(3.2) is smoother, say of class $W^{2m,2}[a, b]$, where $m = n + q$ and q is a nonnegative integer, and

(ii) appropriate L-spline subspaces are selected.

Specifically, suppose that we can express the differential operator M of (3.1′) as

$$M[v(x)] = l^*l[v(x)] + \sum_{0 \leqq i,j \leqq k} (-1)^j D^j(\tilde{\sigma}_{i,j}(x) D^i v(x)), \tag{3.20}$$

where $\tilde{\sigma}_{i,j}(x) \in C^i[a, b]$ for all $0 \leqq i, j \leqq k$, where $0 \leqq k \leqq n$, and

$$l[v(x)] \equiv \sum_{j=0}^{n} \beta_j(x) D^j v(x), \tag{3.21}$$

where we assume that $\beta_j(x) \in C^j[a, b]$ for all $0 \leqq j \leqq n$, and that

$$\beta_n(x) \geqq \omega > 0 \quad \text{for all} \quad x \in [a, b]$$

for some positive constant ω. In this case, we select the finite-dimensional subspaces $H_q(l, \Delta, \mathbf{z})$ of $W_0^{n,2}[a, b]$, which are described in detail in [30] of this volume (see also [22] and [28]). The improved error bounds are then given by the following theorem (cf. [28, Theorem 5]).

THEOREM 3.3. *With the assumptions* (3.3a, b), (3.5a, b, c) *and* (3.20), *assume that* $u(x)$, *the unique generalized solution of* (3.1)–(3.2) *in* $W_0^n[a, b]$, *is of class* $W^{2m,2}[a, b]$, *where* $m = n + q$, $q \geqq 0$, *and for any partition* $\Delta: a = x_0 < x_1 < \cdots < x_{N+1}$ *of* $[a, b]$ *and any associated incidence vector* $\mathbf{z} = (z_0, z_1, \cdots, z_{N+1})$ *with* $z_0 = z_{N+1} = m + q$ *and* $1 \leqq z_i \leqq m + q$ *for* $1 \leqq i \leqq N$, *let* $\hat{u}(x)$ *be the unique solution of the approximate problem* P_k *for the subspace* $B^k \equiv H_q(l, \Delta, \mathbf{z})$. *Then there exist positive constants* K_1 *and* K_2, *independent of* Δ *and* $\mathbf{z}$, *such that*

$$\|D^i(\hat{u} - u)\|_{L^2[a,b]} \leqq K_1(\bar{\Delta})^{2m - \max(\delta, i)}, \qquad 0 \leqq i \leqq n, \tag{3.22}$$

where $\delta \equiv \max\{2k - n; 0\}$, *and*

$$\|D^i(\hat{u} - u)\|_{L^\infty[a,b]} \leqq K_2(\bar{\Delta})^{2m - \max(\delta, i) - 1/2}, \qquad 0 \leqq i \leqq n - 1. \tag{3.23}$$

To check the assumption in Theorem 3.3 that the generalized solution $u(x)$ of (3.1)–(3.2) is of class $W^{2m,2}[a, b]$, $m \geqq n$, one can use known *regularity theorems* (cf. [27, Chap. 4]). For example, if the coefficient function $\sigma_{i,j}(x)$ of (3.1′) is more than just bounded and measurable (cf. (3.3a)), say of class $C^j[a, b]$ for all $0 \leqq i, j \leqq n$, then the solution $u(x)$ is in $W^{2n,2}[a, b]$.

Improved error bounds in the uniform norm can be similarly obtained with somewhat stronger hypotheses (cf. [28, Theorem 5]).

THEOREM 3.4. *With the assumptions of Theorem* 3.3, *let* $\mathscr{F}$ *be a collection of subspaces* $H_q(l, \Delta, \mathbf{z})$ *of* $W^{n,2}[a, b]$ *such that* $\tilde{u}(x)$, *the unique* $H_q(l, \Delta, \mathbf{z})$-*interpolate of* $u(x)$, *in the sense that*

$$D^j\tilde{u}(x_i) = D^j u(x_i), \quad 0 \leqq j \leqq z_i - 1 - 2q, \tag{3.24}$$

for $z_i \geqq 1 + 2q$, *satisfies for some positive constant* K',

$$\|D^i(\tilde{u} - u)\|_{L^\infty[a,b]} \leqq K'(\bar{\Delta})^{2m-i} \quad \text{for all} \quad 0 \leqq i \leqq n - 1, \quad \text{all } H_q(l, \Delta, \mathbf{z}) \in \mathscr{F}. \tag{3.25}$$

If $u(x)$ *is in* $C^{2m}[a, b]$, *then there exists a positive constant* K_3 *such that*

$$\|D^i(\hat{u} - u)\|_{L^\infty[a,b]} \leqq K_3(\bar{\Delta})^{2m - \max(\delta, i)}, \qquad 0 \leqq i \leqq n - 1, \tag{3.26}$$

and all $H_q(l, \Delta, \mathbf{z}) \in \mathscr{F}$.

Other finite-dimensional subspaces of $W_0^{n,2}[a, b]$ can of course be used in the Ritz–Galerkin approximation of the solution of (3.1)–(3.2). Thus, g-splines are sometimes useful in this regard (cf. [1], [33] and [34]).

However, because the use of polynomial subspaces of $W_0^{n,2}[a, b]$ in Ritz–Galerkin methods is classic, and because the associated theory is particularly well suited to the numerical result of the next section, we now summarize the application of the classical results of D. Jackson and S. Bernstein on polynomial approximation to the problem (3.1)–(3.2). Details can be found in [12].

Let $P_0^{(N)}$ be the collection of all real polynomials $p_N(x)$ of degree at most N which satisfy the boundary conditions of (3.2), where $N > 2n - 1$. Then, $P_0^{(N)}$ is a finite-dimensional subspace of $W_0^{n,2}[a, b]$, having dimension $N + 1 - 2n$. Thus, using the inequalities of (3.6) of Theorem 3.1 in conjunction with the results of D. Jackson (cf. [24, p. 66]) and S. Bernstein (cf. [24, p. 76]) directly gives us the following theorem (cf. [12]).

THEOREM 3.5. *With the assumptions* (3.3a, b) *and* (3.5a, b, c), *let* $u(x)$ *be the unique generalized solution of* (3.1)–(3.2) *in* $W_0^n[a, b]$, *and let* $\hat{p}_N(x)$ *be the unique solution of the approximate problem* P_k *for the subspace* $B^k \equiv P_0^{(N)}$. *If* $u(x) \in W^t[a, b]$ *with* $t \geqq n$ *and* $N \geqq \max(t, 2n - 1)$, *then there exist positive constants* K_1 *and* K_2, *independent of* N, *such that*

$$\|D^i(\hat{p}_N - u)\|_{L^\infty[a,b]} \leqq K_1\|\hat{p}_N - u\|_n \leqq \frac{K_2}{(N - n)^{t-n}}\,\omega\left(D^t u; \frac{1}{N - n}\right) \tag{3.27}$$

for all $0 \leqq i \leqq n - 1$. *Moreover, if* $u(x)$ *can be extended to an analytic function in some domain which contains the real interval* $[a, b]$, *then there exists a constant* μ *with* $0 \leqq \mu < 1$ *such that*

$$\limsup_{N\to\infty} (\|D^i(\hat{p}_N - u)\|_{L^\infty[a,b]})^{1/N} \leqq \mu \quad \text{for all} \quad 0 \leqq i \leqq n - 1. \tag{3.28}$$

The constant μ of (3.28) can be given a precise geometrical interpretation when the interval $[a, b]$ is such that $a = -1$ and $b = +1$. Let $\mathscr{E}_\rho$ be the largest ellipse in the complex plane with foci at $z = -1$ and $z = +1$ such that $u(z)$ is analytic in $\mathscr{E}_\rho$. If A and B are respectively the semimajor and semiminor axes of $\mathscr{E}_\rho$, then Bernstein has shown (cf. [24, p. 75]) that

$$\mu = \frac{1}{A + B}. \tag{3.29}$$

In particular, if $u(z)$ is an entire function, then $\mu = 0$.

4. Numerical results. To show how the error estimates of the previous section compare with actual numerical results, we consider the particular special case of (3.1)–(3.2) (cf. [12], [21]):

$$-D^2u(x) + e^{u(x)} = 0, \qquad 0 < x < 1, \tag{4.1}$$

subject to

$$u(0) = u(1) = 0. \tag{4.2}$$

For this problem, $\sigma_{1,1}(x) \equiv 1$ and $\sigma_{i,j}(x) \equiv 0$, $0 \leqq i + j < 2$, in (3.1′), and the assumptions (3.3a, b) and (3.5a, b, c) are all valid. Specifically, using the Rayleigh–Ritz inequality [19, p. 184], (3.3b) is valid with $c = \pi^2/(1 + \pi^2)$, and Λ of (3.4) is π^2. Choosing any $u_0(x)$ in $W_0^{1,2}[0, 1]$ shows that (3.5a) is satisfied, and γ can be chosen to be zero in (3.5b). Similarly, (3.5c) is easily seen to be valid.

A classical solution of (4.1)–(4.2) is known, viz.,

$$u(x) = -\log 2 + 2 \log \{c \sec [c(x - 1/2)/2]\}, \quad c \doteq 1.3360557, \tag{4.3}$$

which can be extended to a function which is analytic in the region in the complex domain defined by an ellipse with foci at $z = 0$ and $z = 1$, and semi-axes 4.7 and 4.6. In this case, μ of (3.29) is approximately 0.107.

To give an application of Theorem 3.4, we choose $l = D$ in (3.21) with all $\tilde{\sigma}_{i,j}(x) \equiv 0$, and $k = 0$, and we choose $m = 2$, so that $n = q = 1$. Using a uniform partition $\Delta(h)$ of $[0, 1]$, i.e., $\Delta(h): 0 = x_0(h) < x_1(h) < \cdots < x_{N+1}(h) = 1$, where $x_j(h) \equiv j/(N + 1)$, the choice of the incidence vector $\mathbf{z} = (3, 2, 2, \cdots, 2, 3)^T$ is such that the finite-dimensional subspace $H_1(D, \Delta(h), \mathbf{z})$ of $W_0^{1,2}[0, 1]$, described in § 3, is in fact the *Hermite space* $H_0^{(2)}(\Delta(h))$ *of piecewise cubic polynomials*. For this subspace, it is known (see [4], [7] and [35]) that the inequality of (3.25) is *valid* for any collection $\mathscr{F}$, and thus the inequality of (3.26) is valid, i.e., in this case,

$$\|\hat{u}(h) - u\|_{L^\infty[0,1]} \leqq K_3(\bar{\Delta}(h))^4. \tag{4.4}$$

Table 1 below gives the associated numerical results for this case.

TABLE 1

N	$\dim H_0^{(2)}(\Delta(h))$	$\|\hat{u}(h) - u\|_{L^\infty[0,1]}$	α
1	4	$5.10 \cdot 10^{-5}$	—
2	6	$1.21 \cdot 10^{-5}$	3.54
3	8	$4.24 \cdot 10^{-6}$	3.65
5	12	$9.58 \cdot 10^{-7}$	3.65
7	16	$3.10 \cdot 10^{-7}$	3.93
9	20	$1.28 \cdot 10^{-7}$	3.96

More accurate numerical results were obtained for the polynomial subspaces $P_0^{(N)}$, and these are given below in Table 2. In this case, as previously remarked, the semimajor and semiminor axes are respectively 4.7 and 4.6, so that $\mu = 0.107$. This means from Theorem 3.4 that for N large, we expect $\|\hat{p}_{N+1} - u\|_{L^\infty[0,1]}$ to be roughly 0.107 times $\|\hat{p}_N - u\|_{L^\infty[0,1]}$, which is already the case numerically from Table 2 for N quite small.

TABLE 2

N	$\dim P_0^{(N)}$	$\|\hat{p}_N - u\|_{L^\infty[0,1]}$
3	2	$4.23 \cdot 10^{-4}$
5	4	$3.12 \cdot 10^{-6}$
7	6	$5.03 \cdot 10^{-8}$

5. Eigenvalue problems. We next consider the eigenvalue problem

$$\mathscr{L}[u(x)] = \lambda \mathscr{M}[u(x)], \qquad 0 < x < 1, \tag{5.1}$$

where

$$\begin{aligned} \mathscr{L}[u(x)] &\equiv \sum_{j=0}^{n} (-1)^j D^j(p_j(x) D^j u(x)), \\ \mathscr{M}[u(x)] &\equiv \sum_{j=0}^{r} (-1)^j D^j(q_j(x) D^j u(x)), \end{aligned} \tag{5.1'}$$

subject to the homogeneous boundary conditions of

$$D^j u(0) = D^j u(1) = 0, \qquad 0 \leqq j \leqq n-1. \tag{5.2}$$

We assume that $0 \leqq r < n$, and that the coefficient functions $p_j(x)$ and $q_k(x)$ are real-valued functions of class $C^j[0,1]$, $0 \leqq j \leqq n$, and class $C^k[0,1]$, $0 \leqq k \leqq r$, respectively, and in addition, we require that

$$p_n(x) \quad \text{and} \quad q_r(x) \quad \text{do not vanish on} \quad [0,1]. \tag{5.3}$$

Letting $\mathscr{D}$ denote the set of real-valued functions in $C^{2n}[0,1]$ which satisfy (5.2), we assume that

$$\begin{aligned} (\mathscr{L}[u], v)_0 &= (u, \mathscr{L}[v])_0 \quad \text{for all} \quad u, v \in \mathscr{D}, \\ (\mathscr{M}[u], v)_0 &= (u, \mathscr{M}[v])_0 \quad \text{for all} \quad u, v \in \mathscr{D}, \end{aligned} \tag{5.4}$$

and that there exist positive constants K and d such that

$$(\mathscr{L}[u], u)_0 \geqq K(\mathscr{M}[u], u)_0 \geqq d(u, u)_0 \quad \text{for all} \quad u \in \mathscr{D}. \tag{5.5}$$

Defining the following inner products on $\mathscr{D}$,

$$\begin{aligned} (u, v)_D &\equiv (\mathscr{M}[u], v)_0 \quad \text{for all} \quad u, v \in \mathscr{D}, \\ (u, v)_N &\equiv (\mathscr{L}[u], v)_0 \quad \text{for all} \quad u, v \in \mathscr{D}, \end{aligned} \tag{5.6}$$

denote by H_D and H_N the Hilbert space completions of $\mathscr{D}$ with respect to the norms $\|\cdot\|_D$ and $\|\cdot\|_N$, respectively. It is then well known [15], [16], [17] that solving the eigenvalue problems (5.1)–(5.2) is equivalent to finding the extreme values and critical points of the *Rayleigh quotient*:

$$R[w] \equiv \frac{\|w\|_N^2}{\|w\|_D^2}, \quad w(x) \in H_N. \tag{5.7}$$

With the above assumptions, it is well known [8] that the eigenvalue problem of (5.1)–(5.2) has countably many eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ which are real, have no finite limit point, and can be arranged as

$$0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_k \leqq \lambda_{k+1} \leqq \cdots. \tag{5.8}$$

Moreover, there is a corresponding sequence of eigenfunctions $\{\varphi_j(x)\}_{j=0}^{\infty}$ of (5.1)–(5.2) with $\varphi_j(x) \in \mathscr{D}$, for which $\mathscr{L}[\varphi_j] = \lambda_j \mathscr{M}[\varphi_j]$. These eigenfunctions are orthonormal in the sense that

$$(\varphi_i, \varphi_j)_D = \delta_{i,j} \quad \text{for all} \quad i, j = 1, 2, \cdots, \tag{5.9}$$

and the sequence $\{\varphi_j(x)\}_{j=1}^{\infty}$ is complete in H_D.

Employing the Rayleigh–Ritz method, i.e., finding the extremal values of $R[w]$ of (5.7) over particular finite-dimensional subspace of H_N, the following results have been proved (cf. [13]). These results extend the results of Birkhoff, de Boor, Swartz and Wendroff [6] for cubic spline functions, which correspond to the special case $m = 2$ and $n = 1$ of (5.10) and (5.11). We now state these results.

THEOREM 5.1. *With the assumptions of* (5.4)–(5.6), *let* $\{\Delta_j\}_{j=1}^{\infty}$ *be a sequence of partitions of* $[0, 1]$, *let* $\{\mathbf{z}_j\}_{j=1}^{\infty}$ *be a corresponding sequence of incidence vectors associated with* $\{\Delta_j\}_{j=1}^{\infty}$, *and let* $\tilde{\lambda}_{k,j}$ *and* $\hat{\varphi}_{k,j}(x)$ *be the* kth *approximate eigenvalue and the* kth *approximate eigenfunction of* (5.1)–(5.2), *obtained by applying the Rayleigh–Ritz method to the subspace* $\mathrm{Sp}_0(L, \Delta_j, \mathbf{z}_j)$ *of* H_N. *If the eigenfunctions* $\{\varphi_i(x)\}_{i=1}^{k}$ *of* (5.1)–(5.2) *are of class* $W^{t,2}[0, 1]$, *with* $t \geqq 2m \geqq 2n$, *there exists a positive constant* K_1, *independent of* j, *and a positive integer* j_0 *such that*

$$\lambda_k \leqq \tilde{\lambda}_{k,j} \leqq \lambda_k + K_1(\bar{\Delta}_j)^{2(2m-n)} \quad \textit{for all} \quad j \geqq j_0. \tag{5.10}$$

Moreover, if the first eigenvalues are simple, i.e., $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$, *there exist a positive constant* K_2, *independent of* j, *and a positive integer* j_0 *such that*

$$\|\hat{\varphi}_{k,j} - \varphi_k\|_{L^\infty[a,b]} \leqq K\|\hat{\varphi}_{k,j} - \varphi_k\|_N \leqq K_2(\bar{\Delta}_j)^{2m-n} \quad \textit{for all} \quad j \geqq j_0. \tag{5.11}$$

Explicit calculations of eigenvalues by Birkhoff and de Boor [5], show the exponent of $\bar{\Delta}$ in (5.10) is *best possible*. The analogue of this for the inequality of (5.11) is similarly true for the eigenfunction approximation in the norm $\|\cdot\|_N$. However, in the norm $\|\cdot\|_{L^\infty[0,1]}$, the exponent of $\bar{\Delta}$ in (5.11) is *not* in general best possible, and can in fact be improved using particular l-spline techniques. Specifically, it is shown in [29] for particular cases that the exponent of Δ_j in (5.11) can be *increased* to $2m$.

The choice of the polynomial subspace $P_0^{(m)}$ of H_N, where $m + 1 \geqq 2n + k$, similarly gives from the Rayleigh–Ritz a kth eigenvalue approximation $\tilde{\lambda}_{k,m}$ to λ_k and a kth eigenfunction approximation $\hat{\varphi}_{k,m}(x)$ to $\varphi_k(x)$. For such subspaces, we again state the following result of [13].

THEOREM 5.2. *With the assumptions of* (5.4)–(5.6), *assume that the eigenfunctions* $\{\varphi_i(x)\}_{i=1}^k$ *of* (5.1)–(5.2) *are of class* $C^t[0, 1]$, *with* $t \geqq 2n$. *Then, there exist constants* M_1 *and* M_2 *such that*

$$(5.12) \qquad \lambda_k \leqq \tilde{\lambda}_{k,m} \leqq \lambda_k + M_1 \left\{ \frac{1}{(m-n)^{t-n}} \left[\max_{1 \leqq i \leqq k} \omega\left(D^t \varphi_i, \frac{1}{m-n} \right) \right] \right\}^2$$

for all $m \geqq M_2$. *Moreover, if* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$, *there exist constants* M_3 *and* M_4 *such that*

$$(5.13) \qquad \begin{aligned} \|\hat{\varphi}_{k,m} - \varphi_k\|_{L^\infty[0,1]} &\leqq K \|\hat{\varphi}_{k,m} - \varphi_k\|_N \\ &\leqq M_3 \left\{ \frac{1}{(m-n)^{t-n}} \left[\max_{1 \leqq i \leqq k} \omega\left(D^t \varphi_i, \frac{1}{m-n} \right) \right] \right\} \end{aligned}$$

for all $m \geqq M_4$. *If the eigenfunctions* $\{\varphi_i(x)\}_{i=1}^k$ *can be extended to analytic functions in some domain in the complex plane which contains the interval* $[0, 1]$, *then there exist two constants* μ_1 *and* μ_2 *with* $0 \leqq \mu_1 < 1$ *and* $0 \leqq \mu_2 < 1$, *such that*

$$(5.14) \qquad \limsup_{m \to \infty} (\tilde{\lambda}_{k,m} - \lambda_k)^{1/m} = \mu_1,$$

and

$$(5.15) \qquad \limsup_{m \to \infty} (\|\hat{\varphi}_{k,m} - \varphi_k\|_{L^\infty[0,1]})^{1/m} = \mu_2.$$

There are extensive numerical results in [6] for cubic splines and cubic Hermite subspaces as applied to the Mathieu equation. However, we shall now give complementary numerical results here for a simpler eigenvalue problem (cf. [13], [20]), namely,

$$(5.16) \qquad -D^2 u(x) = \lambda u(x), \qquad 0 < x < 1,$$

subject to the boundary conditions of

$$(5.17) \qquad u(0) = u(1) = 0.$$

TABLE 3.
Quintic Hermite subspaces $H_0^{(3)}(\Delta(h))$

h	dim $H_0^{(3)}(\Delta(h))$	$\hat{\lambda}_1(h) - \pi^2$	$\hat{\lambda}_2(h) - 4\pi^2$	$\hat{\lambda}_3(h) - 9\pi^2$	$\hat{\lambda}_4(h) - 16\pi^2$
1/2	7	$1.27 \cdot 10^{-7}$	$1.65 \cdot 10^{-3}$	$3.51 \cdot 10^{-2}$	$3.83 \cdot 10^{-1}$
1/3	10	$3.66 \cdot 10^{-9}$	$1.18 \cdot 10^{-5}$	$5.98 \cdot 10^{-3}$	$3.59 \cdot 10^{-2}$
1/4	13	$2.42 \cdot 10^{-10}$	$9.96 \cdot 10^{-7}$	$1.18 \cdot 10^{-4}$	$1.32 \cdot 10^{-2}$
1/5	16	$7.41 \cdot 10^{-11}$	$9.53 \cdot 10^{-8}$	$1.62 \cdot 10^{-5}$	$5.06 \cdot 10^{-4}$

If the quintic Hermite subspace $H_0^{(3)}(\Delta(h))$ is applied to (5.16)–(5.17), then the inequality of (5.10) of Theorem 5.1 is valid with $m = 3$, $n = 1$, i.e., the exponent of Δ_j in (5.10) is 10. The numerical results are given in Table 3. On the other hand, since the eigenfunctions of (5.16)–(5.17) are entire functions, i.e., analytic in the entire complex plane, then (5.14) of Theorem 5.2 is valid with $\mu_1 = 0$. The exceedingly rapid convergence of the approximate eigenvalues in this case for the polynomial subspaces $P_0^{(m)}$ is given in Table 4.

TABLE 4.
Polynomial subspaces $P_0^{(m)}$

m	dim $P_0^{(m)}$	$\hat{\lambda}_{1,m} - \pi^2$	$\hat{\lambda}_{2,m} - 4\pi^2$	$\hat{\lambda}_{3,m} - 9\pi^2$	$\hat{\lambda}_{4,m} - 16\pi^2$
4	3	$1.45 \cdot 10^{-4}$	2.52	13.3	—
6	5	$8.66 \cdot 10^{-8}$	$2.31 \cdot 10^{-2}$	$3.47 \cdot 10^{-1}$	42.6
8	7	$2.60 \cdot 10^{-12}$	$5.56 \cdot 10^{-5}$	$3.03 \cdot 10^{-3}$	2.08

For further numerical results for Rayleigh–Ritz methods applied to piecewise-polynomial subspaces, see also [6], [18] and [37].

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