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Numerical Methods of High-Order Accuracy for Singular Nonlinear Boundary Value Problems*

P. G. CIARLET, F. NATTERER, and R. S. VARGA

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§ 1. Introduction

In [2] and [3], the numerical approximation of the solution of real nonlinear two-point boundary value problems of the form

$$(1.1)\qquad L[u(x)]+f(x,u(x))=0,\qquad 0<x<1,$$

with Dirichlet boundary conditions

$$(1.2)\qquad D^k u(0)=D^k u(1)=0,\qquad D\equiv\frac{d}{dx},\qquad 0\leqq k\leqq n-1,$$

was considered, where

$$(1.3)\qquad L[u(x)]\equiv\sum_{j=0}^{n}(-1)^j D^j\{p_j(x)\,D^j u(x)\},\qquad n\geqq 1,$$

is a $2n$-th order self-adjoint linear differential operator, and it was shown that the Rayleigh-Ritz-Galerkin method is an efficient scheme, both theoretically and numerically, for solving such problems.

Our aim here is to extend the results of [2] and [3] to the case of *nonself-adjoint* linear differential operators whose coefficients have a *singularity* at one or both end points of the interval [0, 1]. For ease of exposition, we shall restrict ourselves here to second order operators, as in the particular case of

$$(1.4)\qquad \mathscr{L}[u(x)]\equiv D^2u(x)+\frac{\sigma}{x}Du(x),\qquad 0<x<1,$$

where σ is a constant satisfying $0\leqq\sigma<1$, and we will consider the associated real *nonlinear* Dirichlet problem

$$(1.5)\qquad \mathscr{L}[u(x)]=g(x,u(x)),\qquad 0<x<1,$$

$$(1.6)\qquad u(0)=\alpha,\qquad u(1)=\beta.$$

We remark that the one-dimensional boundary value problem of (1.4)—(1.6) can in certain cases be obtained by a separation of variables from the two-dimensional degenerate elliptic problem, treated by Gusman and Oganesyan [5].

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In [4], we also considered nonself-adjoint problems such as $D^2 u(x) = f(x, u(x), Du(x))$, $0 < x < 1$, and $u(0) = \alpha$, $u(1) = \beta$. However, if we write (1.4) and (1.5) as $D^2 u(x) = g(x, u(x)) - (\sigma/x) Du(x) \equiv f(x, u(x), Du(x))$, the particular dependence of $f(x, u, v)$ on its third argument is such that the analysis of [4] is not directly applicable. Thus, we feel that the problem (1.4)—(1.6) has interest in itself.

The particular singular boundary value problem (1.4)—(1.6) has been recently considered by Jamet [6, 7], and Parter [9], in the linear case only. In particular, Jamet has studied the application of a standard three-point finite difference scheme associated with a uniform mesh of mesh size h, and he has shown that the error is $O(h^{1-\sigma})$ in the max-norm, the exponent $(1-\sigma)$ of h being best possible. Jamet's method is essentially equivalent to finding the Rayleigh-Ritz-Galerkin piecewise-linear approximation of the solution Φ on a uniform mesh. By using a more suitable subspace for Rayleigh-Ritz-Galerkin's method, we shall improve Jamet's result by proving in inequality (3.16) that the error in the uniform norm for our Galerkin approximation is $O(h^{2-\sigma})$, the exponent of h again being best possible.

In order that we may apply our techniques to solving (1.5)—(1.6), we first replace the boundary conditions of (1.6) by homogeneous ones. If we were to subtract off the linear function $\alpha + (\beta - \alpha)x$ from $u(x)$ in (1.5), a singularity might be added to the function $g(x, u)$ of (1.5) for the new problem. Instead, we put $v(x) \equiv u(x) - (\alpha + (\beta - \alpha)x^2)$, so that solving (1.5)—(1.6) is equivalent to solving

(1.7) $$\mathscr{L}[v(x)] = h(x, v(x)), \quad 0 < x < 1,$$

(1.8) $$v(0) = v(1) = 0,$$

where $h(x, v) \equiv g(x, v + \alpha + (\beta - \alpha)x^2) - 2(\beta - \alpha)(1 + \sigma)$.

Next, we put the boundary value problem of (1.7)—(1.8) into a self-adjoint form, using the fact that Φ is a solution of (1.7)—(1.8) if and only if it is also a solution of

(1.9) $$D\{x^\sigma D u(x)\} = f(x, u(x)), \quad 0 < x < 1,$$

(1.10) $$u(0) = u(1) = 0,$$

where

(1.11) $$f(x, u) \equiv x^\sigma h(x, u).$$

At this point, we generalize the differential equation of (1.9) to

(1.12) $$D\{p(x) D u(x)\} = f(x, u(x)), \quad 0 < x < 1,$$

where we assume that the function $p(x)$ satisfies

(1.13)
$$\begin{aligned} &\text{(i)} \quad p(x) > 0 \quad \text{in } (0, 1), \\ &\text{(ii)} \quad p \in C^1(0, 1), \quad \text{and} \\ &\text{(iii)} \quad \frac{1}{p} \in L^1[0, 1]. \end{aligned}$$

It is easy to verify that the particular choice $p(x) = x^\sigma$, $0 \leqq \sigma < 1$, does in fact satisfy all the conditions of (1.13).

To begin our discussion, we define S to be the linear space of all real-valued functions $w \in C^0[0, 1]$ satisfying the boundary conditions of (1.10), such that w is absolutely continuous on $[0, 1]$, and such that

$$\sqrt{p(x)}\, D w(x) \in L^2[0, 1]. \tag{1.14}$$

Note that $H_0^1[0, 1] \subset S$, where $H_0^1[0, 1]$ is the Sobolev space of absolutely continuous functions defined over $[0, 1]$ satisfying the boundary conditions of (1.10), and such that $Dw \in L^2[0, 1]$. That we are actually obliged to extend the definition of our space of admissible functions (which, in [2], was $S = H_0^1[0, 1]$ in the special case of second-order problems) to a space containing *strictly* $H_0^1[0, 1]$ may be seen from the following example: The boundary value problem $D\{x^\sigma D u(x)\} = -3(\sigma + 2)\, x^{1+\sigma}$, $0 < x < 1$, together with the homogeneous boundary conditions of (1.10), has the unique solution $\Phi(x) \equiv x^{1-\sigma} - x^3$ which belongs to S for $0 \leqq \sigma < 1$, but does not belong to $H_0^1[0, 1]$ whenever $\frac{1}{2} \leqq \sigma \leqq 1$. Also, the space S as defined above is a special case of the so-called *weighted Sobolev spaces*, such spaces being a useful tool in studying weak solutions of singular boundary value problems (cf. Nečas [8, Chapter 6] and Gusman and Oganesyan [5]).

Next, we introduce the positive quantity (see Lemma 1)

$$\Lambda = \inf_{\substack{w \in S \\ w \neq 0}} \frac{\int_0^1 p(x)\, [D w(x)]^2\, dx}{\int_0^1 [w(x)]^2\, dx}. \tag{1.15}$$

We assume that the real function $f(x, u)$ given in (1.12) is continuous in $[0, 1] \times R$, and continuously differentiable with respect to u for all $0 \leqq x \leqq 1$, and all real u, and that there exists a constant γ such that

$$f_u(x, u) \equiv \frac{\partial f}{\partial u}(x, u) \geqq \gamma > -\Lambda, \quad \text{for all} \quad 0 \leqq x \leqq 1, \quad \text{and all real } u. \tag{1.16}$$

We remark that the theoretical results to follow remain valid if we replace (1.16) by a weaker assumption, based on divided differences of f (cf. [2, § 8]).

To outline the subsequent material, § 2 briefly lists the basic results concerning the application of the Rayleigh-Ritz-Galerkin method to the problem (1.12)—(1.10), and since these results are identical to those given in §§ 2—4 of [2], the corresponding proofs will be omitted. However, an essential difference arises when particular subspaces are considered in § 3. This is due to the fact that the solution of (1.12)—(1.10) has in general unbounded derivatives at the end points of $[0, 1]$. Thus, the error bounds derived in [2] are no longer valid since they depended upon L^∞-bounds of some derivative of the solution, and this difficulty is circumvented by introducing an appropriate approximating subspace. In so doing, we employ a method introduced by Ciarlet [1] and then generalized by Perrin, Price, and Varga [10]. Finally in § 4, a numerical example is included.

We remark that by the classic change of variables,

$$z(x) = \int_0^x \frac{d\xi}{p(\xi)},$$

where $z(x)$ is a continuous strictly increasing function, the problem (1.10)—(1.12) can be reduced to the following nonsingular form:

$$(1.12') \qquad \frac{d^2 U(z)}{dz^2} = F(z, U(z)), \quad 0 < z < z_1 \equiv z(1),$$

$$(1.10') \qquad U(0) = U(z_1) = 0,$$

where, if $x(z)$ denotes the inverse function of $z(x)$, then

$$U(z) \equiv u(x(z)) \quad \text{and} \quad F(z, U) \equiv p(x(z))\, f(x(z), U).$$

Although this reduction is possible, we feel that it is desirable, as in [7], to directly consider the numerical approximation of the solution (1.10)—(1.12).

§ 2. Variational Formulation

We begin with the following result.

Lemma 1. The quantity

$$(2.1) \qquad \|w\|_0 \equiv \left\{ \int_0^1 p(x)\,[Dw(x)]^2\,dx \right\}^{\frac{1}{2}}$$

is a norm over the space S, and the following holds:

$$(2.2) \qquad \|w\|_{L^\infty[0,1]} \equiv \sup\{|w(x)| : 0 \leqq x \leqq 1\} \leqq \sqrt{r(1)}\,\|w\|_0 \quad \text{for all} \quad w \in S,$$

where $r(x)$ is defined by

$$(2.3) \qquad r(x) = \int_0^x \frac{dt}{p(t)} \quad \text{for all} \quad x \in [0, 1].$$

Finally, the quantity Λ defined in (1.15) is positive.

Proof. Let $w \in S$. Since $\sqrt{p(x)}\,Dw(x) \in L^2[0, 1]$ and $p(x) > 0$ in $(0, 1)$ by (1.13) (i), it follows from $w(0) = w(1) = 0$ that $\|\cdot\|_0$ is a norm over the space S.

Next, since $w(x)$ is absolutely continuous and $w(0) = 0$,

$$w(x) = \int_0^x Dw(\xi)\,d\xi = \int_0^x \frac{1}{\sqrt{p(\xi)}} \left(\sqrt{p(\xi)} \cdot Dw(\xi)\right) d\xi \quad \text{for all} \quad x \in [0, 1],$$

so that, using the Cauchy-Schwarz inequality and the definition of $r(x)$ in (2.3), we have

$$|w(x)| \leqq \sqrt{r(1)} \cdot \|w\|_0 \quad \text{for all} \quad x \in [0, 1],$$

which proves (2.2).

Finally, it follows from the inequality of (2.2) that

$$\Lambda = \inf_{\substack{w \in S \\ w \neq 0}} \frac{\int_0^1 p(x)\,[Dw(x)]^2\,dx}{\int_0^1 [w(x)]^2\,dx} \geqq \frac{1}{r(1)} > 0,$$

which completes the proof. Q.E.D.

As in [2], we make the hypothesis that the boundary value problem (1.12)—(1—10) has a classical solution Φ (i.e., $\Phi \in C^0[0, 1] \cap C^2(0, 1)$). This implies that $\Phi \in S$, since it follows from (1.12) that

$$p(x) D\Phi(x) = \int_{x_0}^{x} f(\eta, \Phi(\eta))\, d\eta + p(x_0) D\Phi(x_0), \quad \text{for all } x, x_0 \in (0, 1).$$

Keeping x_0 fixed, we see that $p(x) D\Phi(x)$ can be extended to a continuous function over $[0, 1]$, say $q(x)$, and thus, from (1.13) (iii), $\sqrt{p(x)}\, D\Phi(x) \in L^2[0, 1]$:

$$\int_0^1 p(x) (D\Phi(x))^2 dx = \int_0^1 \frac{(q(x))^2 dx}{p(x)} < +\infty.$$

Next, we have as in Theorem 1 of [2],

Theorem 1. With the assumptions of (1.13) and (1.16), Φ *strictly* minimizes the following functional

$$F[w] = \int_0^1 \left\{ \tfrac{1}{2} p(x) [D w(x)^2] + \int_0^{w(x)} f(x, \eta)\, d\eta \right\} dx, \tag{2.4}$$

over the space S, and thus Φ is the unique solution of (1.12)—(1.10).

Proof. It is readily verified with the above assumptions that

$$F[w] \geqq F[\Phi] + \frac{(\Lambda + \gamma)}{2} \int_0^1 [w(x) - \Phi(x)]^2 dx, \quad \text{for all } w \in S,$$

proving that $F[w] > F[\Phi]$ unless $w \equiv \Phi$. Q.E.D.

We now briefly describe the approximation scheme. Let S_M be any finite-dimensional subspace (of dimension M) of S, and let w_i, $1 \leqq i \leqq M$, be M linearly independent functions in S_M. Then, the above inequality allows us to prove, exactly as in Theorem 2 of [2]:

Theorem 2. With the assumptions of (1.13) and (1.16), there exists a unique function $\hat{w}_M = \sum_{i=1}^{M} \hat{u}_i w_i$ in S_M which minimizes the functional $F[w]$ of (2.4) over S_M.

In what follows, we shall for brevity call $\hat{w}_M$ the *Galerkin* approximation of Φ on the subspace S_M.

As in [2, Lemma 2], we have

Lemma 2. Let g be a continuous function over $[0, 1]$ satisfying

$$-\Lambda < \gamma \leqq g(x) \leqq \Gamma, \quad \text{for all } x \in [0, 1], \tag{2.5}$$

for some constants γ, Γ. Then

$$\|w\|_g \equiv \left\{ \int_0^1 \{p(x) [D w(x)]^2 + g(x) [w(x)]^2\} dx \right\}^{\frac{1}{2}} \tag{2.6}$$

is a norm over the space S, and moreover this norm is equivalent to the norm $\|\cdot\|_0$ of (2.1), i.e., there exist two constants $m = m(\gamma, \Gamma)$ and $M = M(\gamma, \Gamma)$ such that

$$(2.7) \qquad m(\gamma, \Gamma)\,\|w\|_g \leqq \|w\|_0 \leqq M(\gamma, \Gamma)\,\|w\|_g, \quad \text{for all} \quad w \in S.$$

As in Theorem 3 of [2], we have the following fundamental result:

Theorem 3. Let Φ be the (classical) solution of (1.12)—(1.10) subject to the assumptions of (1.13) and (1.16), let S_M be any finite-dimensional subspace of S, and let $\hat{w}_M$ be the unique Galerkin approximation which minimizes $F[w]$ over S_M. Then, there exists a constant C, which is *independent* of S_M, such that the following error bound is valid:

$$(2.8) \qquad \|\hat{w}_M - \Phi\|_{L^\infty[0,1]} \leqq \sqrt{r(1)}\,\|\hat{w}_M - \Phi\|_0 \leqq C \inf\{\|w - \Phi\|_0;\ w \in S_M\}.$$

The following is then an immediate consequence of Theorem 3.

Theorem 4. Let Φ be the (classical) solution of (1.12)—(1.10) subject to the assumptions of (1.13) and (1.16), let $\{S_{M_i}\}_{i=1}^{\infty}$ be any sequence of (not necessarily nested) finite-dimensional subspaces of S, and let $\{\hat{w}_{M_i}\}_{i=1}^{\infty}$ be the corresponding sequence of Galerkin approximations obtained by minimizing $F[w]$ over the subspaces S_{M_i}. If

$$(2.9) \qquad \lim_{i\to\infty} \{\inf\{\|w - \Phi\|_0;\ w \in S_{M_i}\}\} = 0,$$

then $\{\hat{w}_{M_i}\}_{i=1}^{\infty}$ converges uniformly to Φ.

§ 3. Approximating Subspaces

Let Π: $0 = x_0 < x_1 < x_2 < \cdots < x_{N+1} = 1$ denote *any* partition Π of $[0, 1]$. Then, with Π, we define the space S^{Π} as being the subspace of S whose functions w satisfy

$$(3.1) \qquad D\{p(x)\,D w(x)\} = 0, \quad x_j < x < x_{j+1} \quad \text{for all} \quad 0 \leqq j \leqq N.$$

For computational purposes, a convenient basis for S^{Π} can be obtained in terms of the function $r(x)$ of (2.3) as follows. Let $h_i(x) \equiv r(x) - r(x_i)$ and let w_i, $1 \leqq i \leqq N$, be defined by:

$$(3.2) \qquad w_i(x) \equiv \begin{cases} 0, & 0 \leqq x \leqq x_{i-1}, \\ h_{i-1}(x)/h_{i-1}(x_i), & x_{i-1} \leqq x \leqq x_i, \\ 1 - [h_i(x)/h_i(x_{i+1})], & x_i \leqq x \leqq x_{i+1}, \\ 0, & x_{i+1} \leqq x \leqq 1. \end{cases}$$

It is readily verified that each w_i, $1 \leqq i \leqq N$, belongs to the space S, and satisfies (3.1), as well as $w_i(0) = w_i(1) = 0$. Since in addition

$$w_i(x_j) = \delta_{i,j}, \quad 1 \leqq i \leqq N, \quad 0 \leqq j \leqq N+1,$$

any function $g \in S^{\Pi}$ can be expanded with respect to the basis $\{w_i\}_{i=1}^{N}$ as

$$g(x) = \sum_{i=1}^{N} g(x_i)\, w_i(x).$$

Given $\Phi \in S$, we define its S^{Π}-*interpolate* $\tilde{w}$ to be the unique element in S^{Π} which satisfies

$$(3.3)\qquad \tilde{w}(x_i) = \Phi(x_i), \quad 0 \leqq i \leqq N+1.$$

We then prove

Lemma 3. Let Φ be the solution of (1.12)—(1.10), and let $\tilde{w}$ be its unique S^{Π}-interpolate. Then,

$$(3.4)\qquad \|\tilde{w} - \Phi\|_{L^\infty[0,1]} \leqq M \cdot \ell(\Pi),$$

where $M \equiv \sup\{|f(x, \Phi(x))|;\ 0 \leqq x \leqq 1\}$, and

$$(3.5)\qquad \ell(\Pi) \equiv \max_{0 \leqq i \leqq N} \left\{ (x_{i+1} - x_i) \int_{x_i}^{x_{i+1}} \frac{dt}{p(t)} \right\}.$$

Proof. Consider any interval $[x_i, x_{i+1}]$, $0 \leqq i \leqq N$. Since $\tilde{w} - \Phi$ vanishes at the two end points of this interval, an integration by parts gives us that

$$\int_{x_i}^{x_{i+1}} p(t)\{D(\tilde{w}(t) - \Phi(t))\}^2\,dt = -\int_{x_i}^{x_{i+1}} D\{p(t)\,D(\tilde{w}(t) - \Phi(t))\}\,(\tilde{w}(t) - \Phi(t))\,dt.$$

We remark that this integration by parts is valid even in the cases $i = 0, N$. Next, as $D\{p(t)\,D\,\tilde{w}(t)\} = 0$ in $[x_i, x_{i+1}]$, and $D\{p(t)\,D\Phi(t)\} \equiv f(t, \Phi(t))$, the integral can be expressed as

$$\int_{x_i}^{x_{i+1}} p(t)\{D(\tilde{w}(t) - \Phi(t))\}^2\,dt = \int_{x_i}^{x_{i+1}} f(t, \Phi(t))\,(\tilde{w}(t) - \Phi(t))\,dt,$$

and thus,

$$(3.6)\qquad \int_{x_i}^{x_{i+1}} p(t)\{D(\tilde{w}(t) - \Phi(t))\}^2\,dt \leqq M(x_{i+1} - x_i)\,\|\tilde{w} - \Phi\|_{L^\infty[x_i, x_{i+1}]}.$$

Next, we can write for any $x \in [x_i, x_{i+1}]$, $0 \leqq i \leqq N$, that

$$\tilde{w}(x) - \Phi(x) = \int_{x_i}^{x} D\{\tilde{w}(t) - \Phi(t)\}\,dt = \int_{x_i}^{x} \frac{1}{\sqrt{p(t)}}\left(\sqrt{p(t)}\,D\{\tilde{w}(t) - \Phi(t)\}\right)dt,$$

and hence by applying the Cauchy-Schwarz inequality, we obtain

$$|\tilde{w}(x) - \Phi(x)| \leqq \left\{\int_{x_i}^{x_{i+1}} \frac{dt}{p(t)}\right\}^{\frac{1}{2}} \left\{\int_{x_i}^{x_{i+1}} p(t)\{D(\tilde{w}(t) - \Phi(t))\}^2\,dt.\right\}^{\frac{1}{2}}.$$

As this holds for all $x \in [x_i, x_{i+1}]$, we have

$$\|\tilde{w} - \Phi\|_{L^\infty[x_i, x_{i+1}]} \leqq \left\{\int_{x_i}^{x_{i+1}} \frac{dt}{p(t)}\right\}^{\frac{1}{2}} \left\{\int_{x_i}^{x_{i+1}} p(t)\{D(\tilde{w}(t) - \Phi(t))\}^2\,dt\right\}^{\frac{1}{2}}.$$

Combining this with the inequality of (3.6) then gives

$$\|\tilde{w}-\Phi\|_{L^\infty[x_i,x_{i+1}]} \leqq M(x_{i+1}-x_i)\int_{x_i}^{x_{i+1}}\frac{dt}{p(t)},$$

and thus, with the definition of $\ell(\Pi)$ of (3.5), the desired inequality of (3.4) follows. Q.E.D.

We can now prove:

Theorem 5. Let Φ be the solution of (1.12)—(1.10), subject to the assumptions of (1.13) and (1.16), and let $\hat{w}$ be the unique Galerkin approximation which minimizes $F[w]$ over the subspace S^{Π}. Then, there exists a constant C, independent of the partition Π such that

$$\|\hat{w}-\Phi\|_{L^\infty[0,1]} \leqq C\cdot\ell(\Pi). \tag{3.7}$$

Proof. Following Ciarlet [1] and Perrin, Price, and Varga [10], the basic idea is to compare $\hat{w}$ with the S^{Π}-interpolate $\tilde{w}$ of the solution Φ (cf. inequality (3.13)), using the fact that the functions of S^{Π} satisfy in each open interval (x_i, x_{i+1}), $0\leqq i\leqq N$, the differential equation $D\{p(x)\,Dw(x)\}=0$ of (3.1).

Let

$$k_i=\int_0^1\{p(x)\,D\tilde{w}(x)\,Dw_i(x)+f(x,\tilde{w}(x))\,w_i(x)\}\,dx, \qquad 1\leqq i\leqq N.$$

Since

$$\int_0^1\{p(x)\,D\Phi(x)\,Dw_i(x)+f(x,\Phi(x))\,w_i(x)\}\,dx=0, \qquad 1\leqq i\leqq N,$$

as an integration by parts shows, we may rewrite k_i as

$$k_i=\int_0^1\{p(x)\,(D\tilde{w}(x)-D\Phi(x))\,Dw_i(x)+\tilde{g}(x)\,(\tilde{w}(x)-\Phi(x))\,w_i(x)\}\,dx,$$

where $\tilde{g}(x)\equiv f_u(x,\Theta(x)\,\tilde{w}(x)+(1-\Theta(x))\,\Phi(x))$ with $0<\Theta(x)<1$. By hypothesis, we know that $\tilde{g}(x)$ is a continuous function on $[0, 1]$, and moreover, from (1.16) and the fact that a priori bounds, independent of h, can be found for $\Phi(x)$ and $\tilde{w}(x)$ (cf. [2, Lemma 4]), then

$$-\Lambda<\gamma\leqq\tilde{g}(x)\leqq\tilde{\Gamma}, \qquad \text{for all} \quad 0\leqq x\leqq 1,$$

where $\tilde{\Gamma}$ is some constant independent of h.

Next,

$$\int_0^1 p(x)\,(D\tilde{w}(x)-D\Phi(x))\,Dw_i(x)\,dx=0, \qquad 1\leqq i\leqq N,$$

as an integration by parts shows, using (3.1) and (3.3), so that

$$k_i=\int_0^1\tilde{g}(x)\,(\tilde{w}(x)-\Phi(x))\,w_i(x)\,dx, \qquad 1\leqq i\leqq N. \tag{3.8}$$

As in [2], it is easy to see that the unique function $\hat{w}$ which minimizes $F[w]$ over the subspace S^{Π} satisfies

$$\int_0^1 \{p(x)\, D\hat{w}(x)\, Dw_i(x) + f(x, \hat{w}(x))\, w_i(x)\}\, dx = 0, \qquad 1 \leqq i \leqq N,$$

so that we may also express k_i as

$$k_i = \int_0^1 \{p(x)\,(D\tilde{w}(x) - D\hat{w}(x))\, Dw_i(x) + \hat{g}(x)\,(\tilde{w}(x) - \hat{w}(x))\, w_i(x)\}\, dx, \qquad 1 \leqq i \leqq N, \tag{3.9}$$

where $\hat{g}(x)$ similarly satisfies the bounds

$$-\Lambda < \gamma \leqq \hat{g}(x) \leqq \hat{\Gamma}, \qquad \text{for all} \quad 0 \leqq x \leqq 1,$$

$\hat{\Gamma}$ being some constant independent of h.

Writing $\tilde{w} = \sum_{i=1}^{N} \tilde{u}_i w_i$, and $\hat{w} = \sum_{i=1}^{N} \hat{u}_i w_i$, we obtain from (3.8)

$$\sum_{i=1}^{N} (\tilde{u}_i - \hat{u}_i)\, k_i = \int_0^1 \tilde{g}(x)\,(\tilde{w}(x) - \Phi(x))\,(\tilde{w}(x) - \hat{w}(x))\, dx,$$

and similarly we obtain from (3.9)

$$\sum_{i=1}^{N} (\tilde{u}_i - \hat{u}_i)\, k_i = \int_0^1 \{p(x)\,[D\tilde{w}(x) - D\hat{w}(x)]^2 + \hat{g}(x)\,[\tilde{w}(x) - \hat{w}(x)]^2\}\, dx,$$

so that, using the norm of (2.6),

$$(\|\tilde{w} - \hat{w}\|_{\hat{g}})^2 = \int_0^1 \tilde{g}(x)\,(\tilde{w}(x) - \Phi(x))\,(\tilde{w}(x) - \hat{w}(x))\, dx, \tag{3.10}$$

and hence from (2.7) and (3.10),

$$\begin{aligned}(\|\tilde{w} - \hat{w}\|_0)^2 &\leqq \{M(\gamma, \hat{\Gamma})\}^2\,(\|\tilde{w} - \hat{w}\|_{\hat{g}})^2 \\ &\leqq C_1 \|\tilde{w} - \Phi\|_{L^2[0,1]}\, \|\tilde{w} - \hat{w}\|_{L^2[0,1]},\end{aligned} \tag{3.11}$$

with $C_1 = \{M(\gamma, \hat{\Gamma})\}^2 (\max\{|\gamma|, |\tilde{\Gamma}|\})$. Since by (2.2),

$$\|\tilde{w} - \hat{w}\|_{L^2[0,1]} \leqq \|\tilde{w} - \hat{w}\|_{L^\infty[0,1]} \leqq \sqrt{r(1)}\, \|\tilde{w} - \hat{w}\|_0, \tag{3.12}$$

from (3.11), we obtain $\|\tilde{w} - \hat{w}\|_0 \leqq C_2 \|\tilde{w} - \Phi\|_{L^2[0,1]}$. Consequently, using (2.2),

$$\begin{aligned}\|\tilde{w} - \hat{w}\|_{L^\infty[0,1]} &\leqq \sqrt{r(1)}\, \|\tilde{w} - \hat{w}\|_0 \\ &\leqq C_3 \|\tilde{w} - \Phi\|_{L^2[0,1]} \leqq C_3 \|\tilde{w} - \Phi\|_{L^\infty[0,1]}.\end{aligned} \tag{3.13}$$

Thus, by combining the inequalities (3.12), (3.13), and (3.4) of Lemma 3, we finally have:

$$\begin{aligned}\|\hat{w} - \Phi\|_{L^\infty[0,1]} &\leqq \|\hat{w} - \tilde{w}\|_{L^\infty[0,1]} + \|\tilde{w} - \Phi\|_{L^\infty[0,1]} \\ &\leqq C_4 \|\tilde{w} - \Phi\|_{L^\infty[0,1]} \leqq C \cdot \ell(\Pi),\end{aligned}$$

which completes the proof. Q.E.D.

Several consequences of Theorem 5 can now be deduced. First, if the function $f(x, u)$ of (1.12) is *independent* of u, then the function $\tilde{g}(x)$ in (3.8) is necessarily zero, and it follows from (3.10) that $\tilde{w}(x) \equiv \hat{w}(x)$, and consequently, $\hat{w}(x)$ interpolates $\Phi(x)$ in the points x_i, $0 \leqq i \leqq N+1$. This gives us

Corollary 1. Let f of (1.12) be independent of u, and let $\hat{w}(x)$ be the unique Galerkin approximation in S^{Π} which minimizes the functional $F[w]$ over S^{Π}. Then, $\hat{w}(x) \equiv \tilde{w}(x)$ where $\tilde{w}(x)$ is the unique interpolate in S^{Π} of the solution $\Phi(x)$ of (1.12)–(1.10). Thus,

$$\hat{w}(ih) = \Phi(ih), \quad 0 \leqq i \leqq N+1. \tag{3.14}$$

The results of Theorem 5 and Corollary 1 make no assumptions on the partition Π of $[0, 1]$. In particular, the error bound of (3.7) of Theorem 5 does not resemble typical error bounds, in that no explicit dependence of $\ell(\Pi)$ on a mesh size h appears. For this reason, we investigate several types of partitions of $[0, 1]$. First, consider a *uniform* partition Π^h: $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ where $x_j = jh$, $0 \leqq j \leqq N+1 = h^{-1}$ of the interval $[0, 1]$. In this case,

$$\ell(\Pi^h) = h\left\{\max_{0 \leqq i \leqq N} \int_{ih}^{(i+1)h} \frac{dt}{p(t)}\right\} \equiv h \cdot \tau(h). \tag{3.15}$$

Note that from (1.13) (iii), $\tau(h) = o(1)$ as $h \to 0$. Hence, for a uniform partition Π^h, the error bound of (3.7) of Theorem 5 is at least of order h. In the case that $p(x) \geqq \omega > 0$ for all $x \in [0, 1]$, then $\tau(h) = O(h)$ as $h \to 0$, so that $\ell(\Pi^h) = O(h^2)$, which agrees with the error bounds of [1] and [10]. On the other hand, if p vanishes at one end point of $[0, 1]$, say $x = 0$, then it is clear that for h sufficiently small,

$$\tau(h) = \int_0^h \frac{dt}{p(t)}.$$

Hence, $\ell(\Pi^h)$ is ultimately determined by the behavior of p in the neighborhood of $x = 0$. For example, if $p(x) = x^{\sigma}$, $0 \leqq \sigma < 1$, then $\tau(h) = \frac{h^{1-\sigma}}{1-\sigma}$ for all $h > 0$, and thus

$$\|\hat{w} - \Phi\|_{L^{\infty}[0,1]} \leqq C \cdot \ell(\Pi^h) = C_1 h^{2-\sigma}. \tag{3.16}$$

The result of (3.16) is an improvement of the recent results of Jamet [6, 7], who established by means of difference methods for *linear* problems a discrete result like that of (3.16), however, with an exponent of h reduced to $1-\sigma$. We further remark that, in the linear case, the determination of Jamet's discrete solution and the determination of the Galerkin approximation $\hat{w}(x)$ for our variational approach both depend upon the solutions of tridiagonal matrix equations, so that computationally, the methods are comparable. The numerical results of § 4 do confirm the rate of convergence of (3.16), so that the subspaces of S which correspond to the above uniform partitions $\Pi^{(h)}$ of $[0, 1]$ are attractive for such singular problems.

It is natural to ask if an error bound such as that of (3.16) is *sharp*. To show this, consider the function $\Phi(x) \equiv x^{1-\sigma} - x^{2-\sigma}$, which satisfies the boundary conditions of (1.10).

Then,

$$D\{x^{\sigma} D\,\Phi(x)\} = \sigma - 2, \qquad 0 < x < 1,$$

and thus $\Phi(x)$ is the unique solution of (1.12)—(1.10) with $f(x, u) \equiv \sigma - 2$. Because f is independent of u in this case, then $\tilde{w}(x) = \hat{w}(x)$ from Corollary 1. Hence,

$$\|\hat{w} - \Phi\|_{L^{\infty}[0,1]} = \|\tilde{w} - \Phi\|_{L^{\infty}[0,1]} \geqq \|\tilde{w} - \Phi\|_{L^{\infty}[0,h]}.$$

By direct calculation, we find that

$$\|\tilde{w} - \Phi\|_{L^{\infty}[0,h]} = C_2 h^{2-\sigma},$$

where

$$C_2 \equiv \left\{\left(\frac{1-\sigma}{2-\sigma}\right)^{1-\sigma} - \left(\frac{1-\sigma}{2-\sigma}\right)^{2-\sigma}\right\}.$$

Hence, $\|\hat{w} - \Phi\|_{L^{\infty}[0,1]} \geqq C_2 h^{2-\sigma}$, showing that the error bound of (3.16) is sharp in this case.

Another form of partition of the interval $[0, 1]$ is suggested by the following considerations. Given a *fixed* number N of interior mesh points x_i, $1 \leqq i \leqq N$, of $(0, 1)$, then the function $\ell(\Pi)$ of (3.5), considered as a function only of the N variables $0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_N \leqq 1$, clearly assumes its minimum for at least one *optimal* partition $\hat{\Pi}$. In addition, it is not hard to show that the associated N points $\hat{x}_i$ are distinct points in $(0, 1)$, i.e., $\hat{\Pi}$: $0 < \hat{x}_1 < \hat{x}_2 < \cdots < \hat{x}_N < 1$, and that $\ell(\hat{\Pi}) = \inf_{\Pi}\{\ell(\Pi)\}$, for a fixed N, if and only if

$$\ell(\hat{\Pi}) = (\hat{x}_{i+1} - \hat{x}_i) \int_{\hat{x}}^{\hat{x}_{i+1}} \frac{dt}{p(t)}, \qquad \text{for each } i, \qquad 0 \leqq i \leqq N.$$

Hence, the problem of finding an optimal partition amounts to finding N distinct points $\hat{x}_i \in (0, 1)$ with the property that all the quantities $(x_{i+1} - x_i) \int_{x_i}^{x_{i+1}} \frac{dt}{p(t)}$, $0 \leqq i \leqq N$, which occur in $\ell(\Pi)$ are *equal*. The advantage of an optimal partition is of course that for a given computational work (the solution of an $N \times N$ tridiagonal system), it is the one that minimizes the upper bound (3.7) of the error $\|\hat{w} - \Phi\|_{L^{\infty}[0,1]}$. However, searching for such an optimal partition may be time-consuming. Instead, the above characterization of optimal partitions may be used as follows: *Fix* the quantity $\tilde{\ell}(\Pi)$ in (3.7) so that the error $\|\hat{w} - \Phi\|_{L^{\infty}[0,1]}$ is less than or equal to a *given* accuracy. Then, define recursively $0 = \tilde{x}_0 < \tilde{x}_1 < \tilde{x}_2 < \cdots$, in such a way that

$$\tilde{\ell}(\Pi) = (\tilde{x}_{i+1} - \tilde{x}_i) \int_{\tilde{x}_i}^{\tilde{x}_{i+1}} \frac{dt}{p(t)}, \qquad i = 0, 1, 2, \ldots.$$

If no $\tilde{x}_i$ is unity, set $\tilde{x}_{N+1} = 1$, where N is the last point of the above sequence in $(0, 1)$. Otherwise, if $\tilde{x}_j = 1$ for some j, set $\tilde{x}_{N+1} = 1$. In view of the above remarks, such partitions $\tilde{\Pi}$: $0<\tilde{x}_1<\tilde{x}_2<\cdots<\tilde{x}_N<1$ can be called *quasi-optimal*, in that only the final quantity $(1-\tilde{x}_N)\int_{\tilde{x}_N}^{1}\frac{dt}{p(t)}$ may be smaller than $\tilde{\ell}(\Pi)$. Such quasi-optimal partitions have the advantage that for a fixed $\tilde{\ell}(\Pi)$, they lead to systems with the smallest possible number of unknowns, i.e., they minimize the computational work.

Finally, other finite-dimensional subspaces of S can also be considered. Specifically, if the subspace $H_0^1(\Pi)$ (cf. [2]) of continuous piecewise-linear functions of S is used with a uniform partition Π^h of mesh size h, then the analogue of (3.16) for the case $p(x)=x^\sigma$ is

$$\|\hat{w}-\Phi\|_{L^\infty[0,1]}\leqq C' h^{1-\sigma}, \tag{3.17}$$

which is essentially the same as Jamet's result. Moreover, the exponent of h in (3.17) can be shown to be best possible.

§ 4. Numerical Example

Consider the particular singular nonlinear boundary value problem:

$$D\{\sqrt{x}\,D\,u(x)\}=u^2-\left(\tfrac{3}{2}+x(1-x)^2\right),\qquad 0<x<1, \tag{4.1}$$

subject to the homogeneous boundary conditions

$$u(0)=u(1)=0, \tag{4.2}$$

which corresponds to the choice $p(x)=x^\sigma$ with $\sigma=\frac{1}{2}$ in (1.12). By restricting our attention to nonnegative solutions, it is known that there exists a unique nonnegative solution $\Phi(x)$ of (4.1)—(4.2) (cf. [2, p. 419]), which is given explicitly by

$$\Phi(x)=x^{\frac{1}{2}}-x^{\frac{3}{2}},\qquad 0\leqq x\leqq 1. \tag{4.3}$$

For our numerical example, we chose a *uniform* partition Π^h: $0=x_0<x_1<\cdots<x_{N+1}=1$ of $[0,1]$ with $x_j=jh$ and $h^{-1}=N+1$. Calling the resulting subspace of § 3 S^h, the nonnegative Galerkin approximation $\hat{w}_h(x)$ were determined for various values of the mesh spacing $h=1/(N+1)$. The results are given in the table below.

Table

h	dim S^h	$\|\Phi-\hat{w}_h\|_{L^\infty[0,1]}$	Exponent α
0.5	1	$1.3142\cdot 10^{-1}$	—
0.25	3	$4.6716\cdot 10^{-2}$	1.492
0.125	7	$1.6713\cdot 10^{-2}$	1.483
0.0625	15	$5.9575\cdot 10^{-3}$	1.488
0.03125	31	$2.1159\cdot 10^{-3}$	1.493
0.015625	63	$7.4962\cdot 10^{-4}$	1.497
0.0078125	127	$2.6507\cdot 10^{-4}$	1.500

The last column in the table gives the experimentally determined exponent α of h in $\|\widehat{w}_h - \Phi\|_{L^\infty[0,1]} \doteq K h^\alpha$ for each halving of the mesh h. The theoretical value of α is, from (3.16) given by $2 - \sigma = \frac{3}{2}$, which agrees well with the computed values of α.

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Prof. R. S. Varga
Department of Mathematics
Kent State University
Kent, Ohio 44240, U.S.A.

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