Reprinted from JOURNAL OF APPROXIMATION THEORY, Vol. 3, No. 3, September 1970

Chebyshev Rational Approximations to Certain Entire Functions in $[0, +\infty)$*

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Communicated by G. Meinardus

Received October 28, 1969

1. INTRODUCTION

For any nonnegative integer m, let π_m denote the collection of all real polynomials of degree at most m, and for any nonnegative integers m and n, let $\pi_{m,n}$ denote the collection of all real rational functions $r_{m,n}(x)$ of the form

$$r_{m,n}(x) \equiv \frac{p_m(x)}{q_n(x)}, \qquad \text{where} \quad p_m \in \pi_m \quad \text{and} \quad q_n \in \pi_n. \tag{1.1}$$

Recently, it was shown that Chebyshev rational approximations in $\pi_{m,n}$ to e^{-x} in $[0, +\infty)$ for $m \leqslant n$ *converge geometrically*. More precisely, define

$$\lambda^*_{m,n} = \inf_{r_{m,n} \in \pi_{m,n}} \{ \sup_{0 \leqslant x < +\infty} | r_{m,n}(x) - e^{-x} | \}, \qquad m \leqslant n. \tag{1.2}$$

Then, for any sequence of nonnegative integers $\{m(n)\}_{n=0}^{\infty}$ with $m(n) \leqslant n$ for each $n \geqslant 0$, it was shown in [2] that

$$\overline{\lim_{n\to\infty}} (\lambda^*_{m(n),n})^{1/n} = \beta < 1, \qquad (\beta \leqslant 0.43501), \tag{1.3}$$

and that

$$\overline{\lim_{n\to\infty}} (\lambda^*_{0,n})^{1/n} = \gamma > 0, \quad (\gamma \geqslant \tfrac{1}{6}). \tag{1.4}$$

* This research was supported in part by AEC Grant AT(11-1)-2075.

† Dedicated to Professor Lothar Collatz on the occasion of his sixtieth birthday.

It is natural to ask if results analogous to (1.3) and (1.4) are valid for functions other than e^{-x}, and the purpose of this paper is to establish such analogs for reciprocals of entire functions of *perfectly regular growth with nonnegative coefficients.*

2. Entire Functions of Perfectly Regular Growth

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function, and let $M_f(r) = \max_{|z| \leqslant r} |f(z)|$ $(0 \leqslant r < \infty)$.

Definition. An entire function f is of perfectly regular growth (ρ, B) (cf. Valiron [4], p. 45) iff there exist two (finite) positive constants ρ and B such that

$$\lim_{r \to +\infty} \ln M_f(r)/r^\rho = B. \tag{2.1}$$

We remark that entire functions satisfying (2.1) are also entire functions of *order* ρ *and finite type* B (cf. Boas [1], p. 8).

Valiron [4], p. 44 has shown that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an entire function of perfectly regular growth (ρ, B) iff, given any $\epsilon > 0$, there exists an $n_0(\epsilon)$ such that

$$\frac{k \mid a_k \mid^{\rho/k}}{\rho e} < B + \epsilon \qquad \forall\ k \geqslant n_0(\epsilon), \tag{2.2}$$

and there exists a sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers with $n_p \to \infty$ as $p \to \infty$ and $\lim_{p\to\infty}(n_{p+1}/n_p) = 1$, such that

$$\lim_{p\to\infty} \frac{n_p \mid a_{n_p} \mid^{\rho/n_p}}{\rho e} = B. \tag{2.3}$$

For our purposes, it is somewhat more convenient to express (2.2) and (2.3) equivalently as

$$((k!) \mid a_k \mid^{\rho})^{1/k} < \rho B + \epsilon \qquad \forall\ k \geqslant n_0(\epsilon), \tag{2.4}$$

and

$$\lim_{p\to\infty} ((n_p!) \mid a_{n_p} \mid^{\rho})^{1/n_p} = \rho B. \tag{2.5}$$

In what is to follow, we shall assume that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an entire function of perfectly regular growth (ρ, B), and in addition that $a_k \geqslant 0$ for all $k \geqslant 0$.

3. Upper Bounds for $\lambda_{m,n}$

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be of perfectly regular growth (ρ, B) with nonnegative coefficients a_k and set $s_n(z) \equiv \sum_{k=0}^{n} a_k z^k$ $(n = 0, 1, \ldots)$. The first few partial sums $s_n(z)$ may be identically zero, but as the coefficients a_k are nonnegative and not all zero, it follows that there exists a positive integer n^* such that $0 < s_n(x) \leqslant f(x)$ for all $x > 0$ and all $n \geqslant n^*$. Thus

$$0 \leqslant \frac{1}{s_n(x)} - \frac{1}{f(x)} = \frac{f(x) - s_n(x)}{f(x) \cdot s_n(x)} \leqslant \frac{\sum_{k=n+1}^{\infty} a_k x^k}{s_n^2(x)} \qquad \forall\, x > 0, \qquad \forall\, n \geqslant n^*.$$

Given any ϵ with $0 < \epsilon < \rho B$, it follows from (2.4) that there exists an $\tilde{n}(\epsilon) \geqslant n^*$ such that

$$0 \leqslant a_k < \left(\frac{(\rho B + \epsilon)^k}{k!}\right)^{1/\rho} \qquad \forall\, k \geqslant \tilde{n}(\epsilon).$$

Then, a simple calculation shows that

$$\begin{aligned} 0 \leqslant \frac{1}{s_n(x)} - \frac{1}{f(x)} &\leqslant \frac{\sum_{k=n+1}^{\infty} \left[\frac{(\rho B + \epsilon)^k}{k!}\right]^{1/\rho} x^k}{s_n^2(x)} \\ &\leqslant \left[\frac{(\rho B + \epsilon)^{n+1}}{(n+1)!}\right]^{1/\rho} \left(\frac{x^{n+1}}{s_n^2(x)}\right) \sum_{k=0}^{\infty} \frac{(\rho B + \epsilon)^{k/\rho} x^k}{(n+2)^{k/\rho}} \end{aligned}$$

for all $0 < x < \left(\frac{n+2}{\rho B + \epsilon}\right)^{1/\rho}$ and for all $n \geqslant \tilde{n}(\epsilon)$. Summing the above geometric series gives

$$0 \leqslant \frac{1}{s_n(x)} - \frac{1}{f(x)} \leqslant \left[\frac{(\rho B + \epsilon)^{n+1}}{(n+1)!}\right]^{1/\rho} \cdot \left(\frac{x^{n+1}}{s_n^2(x)}\right) \cdot \left\{\frac{(n+2)^{1/\rho}}{(n+2)^{1/\rho} - (n+1)^{1/\rho}}\right\}$$

$$\forall\, n \geqslant \tilde{n}(\epsilon), \quad \forall\, 0 < x \leqslant \left(\frac{n+1}{\rho B + \epsilon}\right)^{1/\rho}. \tag{3.1}$$

We now seek an inequality of the form

$$K_n x^{n+1} \leqslant (s_n(x))^2 \qquad \forall\, x \geqslant 0, \tag{3.2}$$

holding for every n of the form $2n_p - 1$. With the same ϵ as before, it follows from (2.4) and (2.5) that there exists a $p_1(\epsilon) \geqslant n^*$ such that

$$\left[\frac{(\rho B - \epsilon)^{n_p}}{(n_p)!}\right]^{1/\rho} < a_{n_p} \quad \text{and} \quad a_n < \left[\frac{(\rho B + \epsilon)^n}{n!}\right]^{1/\rho}$$

$$\text{for} \quad n = 2n_p - 1, \quad \forall\, p \geqslant p_1(\epsilon). \tag{3.3}$$

Now, writing $(s_n(x))^2 = \sum_{j=0}^{2n} \beta_{j,n} x^j$ where $\beta_{j,n} \equiv \sum_{k=0}^{j} a_k a_{j-k}$, we have that $(s_n(x))^2 \geqslant \beta_{n+1,n} x^{n+1}\ \forall x \geqslant 0$. With $n = 2n_p - 1$ where $p \geqslant p_1(\epsilon)$, it is clear that

$$\beta_{n+1,n} = \sum_{k=0}^{n+1} a_k a_{n+1-k} \geqslant a_{n_p}^2 > \left[\frac{(\rho B - \epsilon)^{n_p}}{(n_p)!}\right]^{2/\rho} \geqslant \left[\frac{(\rho B - \epsilon)^{n+1}}{(n+1)!}\right]^{1/\rho} . \{2n/n+1\}^{1/\rho},$$

the last inequality following from $(2k)!/(k!)^2 \geqslant 2^{2k}/2k$ for all $k \geqslant 1$. If we set

$$K_n = \left[\frac{(\rho B - \epsilon)^{n+1}}{(n+1)!}\right]^{1/\rho} \cdot \left[\frac{2^n}{n+1}\right]^{1/\rho} \tag{3.4}$$

then the inequality (3.2) is valid for all $n = 2n_p - 1$ where $p \geqslant p_1(\epsilon)$. Replacing $(s_n(x))^2$ in (3.1) by the lower bound of (3.2) thus gives

$$0 \leqslant \frac{1}{s_n(x)} - \frac{1}{f(x)}$$
$$\leqslant \left[\left(\frac{\rho B + \epsilon}{\rho B - \epsilon}\right)\frac{1}{2}\right]^{n/\rho} \left[\left(\frac{\rho B + \epsilon}{\rho B - \epsilon}\right)^{1/\rho} \cdot \left[\frac{(n+2)^{1/\rho}}{(n+2)^{1/\rho} - (n+1)^{1/\rho}}\right](n+1)^{1/\rho}\right\} \tag{3.5}$$

$\forall n = 2n_p - 1$ with $p \geqslant p_1(\epsilon)$, $\forall 0 < x \leqslant (n + 1/\rho B + \epsilon)^{1/\rho}$.

Let $x \geqslant (n + 1/\rho B + \epsilon)^{1/\rho}$. Since $n = 2n_p - 1$, $n \geqslant n_p$, and consequently

$$0 \leqslant \frac{1}{s_n(x)} - \frac{1}{f(x)} \leqslant \frac{1}{s_n(x)} \leqslant \frac{1}{a_{n_p} x^{n_p}} \leqslant \frac{1}{a_{n_p}\left(\dfrac{n+1}{\rho B + \epsilon}\right)^{n_p/\rho}} .$$

Using the first inequality of (3.3), we have

$$0 \leqslant \frac{1}{s_n(x)} - \frac{1}{f(x)} \leqslant \left(\frac{\rho B + \epsilon}{\rho B - \epsilon}\right)^{n_p/\rho} \left(\frac{(n_p!)}{(n+1)^{n_p}}\right)^{1/\rho} .$$

By Stirling's inequality $k! \leqslant k^k e^{-k} \sqrt{2\pi k}(1 + 1/4k)$ and the fact that $n + 1 = 2n_p$, we obtain

$$0 \leqslant \frac{1}{s_n(x)} - \frac{1}{f(x)}$$
$$\leqslant \left[\left(\frac{\rho B + \epsilon}{\rho B - \epsilon}\right) \cdot \frac{1}{2e}\right]^{n/2\rho} \cdot \left\{\left[\left(\frac{\rho B + \epsilon}{\rho B - \epsilon}\right) \cdot \frac{1}{2e}\right]^{1/2} \sqrt{2\pi n_p}\left(1 + \frac{1}{4n_p}\right)\right\}^{1/\rho}$$
$$\forall x \geqslant \left(\frac{n+1}{\rho B + \epsilon}\right)^{1/\rho}. \tag{3.6}$$

A simple comparison of the upper bounds in (3.5) and (3.6) show that the first is the larger for large p. Therefore, if

$$g_n \equiv \sup_{0<x<\infty} \left| \frac{1}{s_n(x)} - \frac{1}{f(x)} \right|, \qquad \forall\, n \geqslant \tilde{n},$$

then it follows, using (3.5), that

$$\overline{\lim_{p\to\infty}} (g_{2n_p-1})^{1/(2n_p-1)} \leqslant \frac{1}{2^{1/\rho}}. \tag{3.7}$$

To extend the result of (3.7), observe that

$$0 \leqslant \frac{1}{s_m(x)} - \frac{1}{f(x)} \leqslant \frac{1}{s_n(x)} - \frac{1}{f(x)} \qquad \forall\, x > 0, \quad \forall m \geqslant n \geqslant n^*.$$

Thus, from the definition of g_n, it follows that

$$g_m \leqslant g_n \qquad \forall\, m \geqslant n \geqslant n^*. \tag{3.8}$$

For any positive integer n sufficiently large, choose an n_p so that $2n_p - 1 \leqslant n < 2n_{p+1} - 1$. From (3.8), we have that

$$g_n^{1/n} \leqslant g_{2n_p-1}^{1/n} = [g_{2n_p-1}^{1/(2n_p-1)}]^{(2n_p-1)/n}$$

Since g_{2n_p-1} is, from (3.7), less than unity for p sufficiently large, replacing n in the exponent of the above expression by $2n_{p+1} - 1$ gives

$$g_n^{1/n} \leqslant [g_{2n_p-1}^{1/(2n_p-1)}]^{(2n_p-1)/(2n_{p+1}-1)},$$

but as $\lim_{p\to\infty}(n_{p+1}/n_p) = 1$, it easily follows from (3.7) that

$$\overline{\lim_{n\to\infty}}\, g_n^{1/n} \leqslant \frac{1}{2^{1/\rho}}. \tag{3.9}$$

To establish a stronger result than (3.9), we have

$$\frac{1}{s_n(x)} - \frac{1}{f(x)} = \frac{\sum_{k=n+1}^{\infty} a_k x^k}{f(x) \cdot s_n(x)} \geqslant \frac{a_{n+1} x^{n+1}}{f^2(x)} \qquad \forall\, x > 0, \quad \forall\, n \geqslant n^*.$$

With $n + 1 = n_p$, we have from (3.3) that

$$\frac{1}{s_n(x)} - \frac{1}{f(x)} > \left\{ \frac{(\rho B - \epsilon)^{n+1}}{(n+1)!} \right\}^{1/\rho} \cdot \frac{x^{n+1}}{f^2(x)} \qquad \forall\, x > 0, \quad p \geqslant p_1(\epsilon),$$

and it is clear from (2.1) that there exists an $R_1(\epsilon) > 0$ such that

$$f(x) < e^{(B+\epsilon/\rho)x^\rho} \qquad \forall\, x > R_1(\epsilon).$$

Hence,

$$\frac{1}{s_n(x)} - \frac{1}{f(x)} > \left\{\frac{(\rho B - \epsilon)^{n+1}}{(n+1)!}\right\}^{1/\rho} \cdot \frac{x^{n+1}}{e^{2(B+\epsilon/\rho)x^\rho}},$$

$$n+1 = n_p, \quad p \geqslant p_1(\epsilon), \quad x > R_1(\epsilon).$$

If we evaluate the right side of the last inequality at $x = \{(n+1)/2(\rho B + \epsilon)\}^{1/\rho}$, which is compatible with $x > R_1(\epsilon)$ if n is sufficiently large, we obtain

$$g_n > \left\{\frac{(\rho B - \epsilon)^{n+1}}{(n+1)!}\right\}^{1/\rho} \cdot \left\{\frac{n+1}{2(\rho B + \epsilon)}\right\}^{(n+1)/\rho} e^{(n+1)/\rho}.$$

Hence, it readily follows that

$$\underline{\lim}_{p\to\infty} (g_{n_p-1})^{1/(n_p-1)} \geqslant \frac{1}{2^{1/\rho}}.$$

Then, using the same method which established (3.9) from (3.7), one proves that

$$\underline{\lim}_{n\to\infty} g_n^{1/n} \geqslant \frac{1}{2^{1/\rho}}. \tag{3.10}$$

Thus, combining with (3.9) gives

THEOREM 1. *Let $f(z)$ be an entire function of perfectly regular growth (ρ, B) with nonnegative coefficients. Then,*

$$\lim_{n\to\infty} \left(\sup_{0<x<\infty} \left|\frac{1}{s_n(x)} - \frac{1}{f(x)}\right|\right)^{1/n} = \frac{1}{2^{1/\rho}}. \tag{3.11}$$

If we define

$$\lambda_{m,n} \equiv \inf_{r_{m,n}\in\pi_{m,n}} \left\{\sup_{0<x<\infty} \left|\frac{1}{f(x)} - r_{m,n}(x)\right|\right\}, \tag{3.12}$$

the error for the best Chebyshev rational approximation of $1/f(x)$ in $[0, +\infty)$, then it is clear that

$$0 < \lambda_{n,n} \leqslant \lambda_{n-1,n} \leqslant \cdots \leqslant \lambda_{0,n} \leqslant g_n \qquad \forall\, n \geqslant n^*. \tag{3.13}$$

Thus, from (3.11) and (3.13), we have the following generalization of (1.3):

THEOREM 2. *Let $f(z)$ be an entire function of perfectly regular growth (ρ, B) with nonnegative coefficients. Then, for any sequence $\{m(n)\}_{n=0}^{\infty}$ of nonnegative integers with $m(n) \leqslant n$ for all $n \geqslant 0$,*

$$\overline{\lim_{n\to\infty}}(\lambda_{m(n),n})^{1/n} \leqslant \frac{1}{2^{1/\rho}} < 1. \tag{3.14}$$

It is not likely that the constant $2^{-1/\rho}$ appearing in (3.14) is best possible for the class of entire functions of perfectly regular growth (ρ, B) with nonnegative coefficients, since the rational functions $1/s_n(x)$ used to establish (3.11) obviously do not have the equi-oscillation of error property of best Chebyshev rational approximations. In particular, for the special case $f(z) = e^z$, we know from (1.3) that strict inequality holds in (3.14).

Since the case where $f(0) = 0$ has not been ruled out, it is also worth noting that the above theorems are applicable to entire functions $f(z)$ for which $1/f(x)$ is unbounded on $(0, \infty)$, such as $f(z) = z^m e^{z^n}$, $m > 0$, $f(z) = \sinh(z^n)$, and $f(z) = J_n(iz)$, $n > 0$, the n-th order Bessel function.

4. LOWER BOUNDS FOR $\lambda_{m,n}$

For entire functions of perfectly regular growth with nonnegative coefficients, we now establish the existence of a positive lower bound (cf. (4.1)) for the quantity $\lim_{n\to\infty}(\lambda_{0,n})^{1/n}$, thereby generalizing (1.4).

THEOREM 3. *Let $f(z)$ be an entire function of perfectly regular growth (ρ, B) with nonnegative coefficients. Then,*

$$\overline{\lim_{n\to\infty}}(\lambda_{0,n})^{1/n} \geqslant \frac{1}{2^{2+1/\rho}}. \tag{4.1}$$

Proof. For any $\epsilon > 0$, there exists, from (2.1). an $R(\epsilon) > 0$ such that

$$M_f(r) \leqslant e^{r^\rho B(1+\epsilon)} \qquad \forall\, r \geqslant R(\epsilon).$$

Since the coefficients of $f(z)$ are nonnegative,

$$0 \leqslant f(x) \leqslant f(r) = M_f(r) \leqslant e^{r^\rho B(1+\epsilon)} \qquad 0 \leqslant x \leqslant r, \quad \forall\, r \geqslant R(\epsilon). \tag{4.2}$$

Next, associated with the positive number

$$\alpha \equiv (2B\rho)^{-1/\rho},$$

there is a positive integer $n^*(\epsilon)$ such that $\alpha n^{1/\rho} \geqslant R(\epsilon)$ for all $n \geqslant n^*(\epsilon)$. Thus, from (4.2) with $r = \alpha n^{1/\rho}$, we have from the definition of α that

$$0 \leqslant f(x) \leqslant f(\alpha n^{1/\rho}) \leqslant e^{n(1+\epsilon)/2\rho} \qquad 0 \leqslant x \leqslant \alpha n^{1/\rho}, \quad \forall n \geqslant n^*(\epsilon). \tag{4.3}$$

Next, let q be any positive number such that

$$\overline{\lim_{n\to\infty}}(\lambda_{0,n})^{1/n} < 1/q. \tag{4.4}$$

Then there exists a positive integer $\tilde{n}$ such that $\lambda_{0,n} \leqslant 1/q^n$ for all $n \geqslant \tilde{n}$. This implies that there exists a sequence of polynomials $\{p_n(x)\}_{n=\tilde{n}}^{\infty}$, with $p_n \in \pi_n$, for which

$$\sup_{0<x<+\infty} \left| \frac{1}{p_n(x)} - \frac{1}{f(x)} \right| \leqslant \frac{1}{q^n} \qquad \forall\, n \geqslant \tilde{n}. \tag{4.5}$$

But, from (3.14), it is clear that we can restrict our attention to those q which are $\geqslant 2^{1/\rho}$. Because of this and the fact that $e^{1/2} < 2$, it is possible to choose $\epsilon > 0$ so small that

$$e^{n(1+\epsilon)/2\rho} < q^n \qquad \forall\, n \geqslant 1.$$

Hence, from (4.3), we have that

$$f(x) < q^n \qquad 0 \leqslant x \leqslant \alpha n^{1/\rho}, \qquad \forall\, n \geqslant n^*(\epsilon). \tag{4.6}$$

Next, using (4.5), it follows that

$$\frac{-f^2(x)}{q^n - f(x)} \leqslant p_n(x) - f(x) \leqslant \frac{f^2(x)}{q^n - f(x)}, \qquad 0 \leqslant x \leqslant \alpha n^{1/\rho}, \quad \forall\, n \geqslant \hat{n},$$

where $\hat{n} \equiv \max(\tilde{n}, n^*(\epsilon))$, and thus, from (4.6),

$$| p_n(x) - f(x)| \leqslant \frac{f^2(x)}{q^n - f(x)} \qquad \forall 0 \leqslant x \leqslant \alpha n^{1/\rho}, \quad \forall n \geqslant \hat{n}.$$

Because the right side of the above inequality is monotone increasing with x, we can write, from (4.3),

$$| p_n(x) - f(x)| \leqslant \frac{e^{n(1+\epsilon)/\rho}}{q^n - e^{n(1+\epsilon)/2\rho}} \qquad 0 \leqslant x \leqslant \alpha n^{1/\rho}, \quad \forall\, n \geqslant \hat{n}. \tag{4.7}$$

Now, let

$$K_n \equiv \inf_{r_n \in \pi_n} \{ \max_{0 \leqslant x \leqslant \alpha n^{1/\rho}} | r_n(x) - f(x)|\}, \qquad \forall\, n \geqslant 0. \tag{4.8}$$

According to (4.7), we evidently have

$$K_n \leqslant \frac{e^{n(1+\epsilon)/\rho}}{q^n - e^{n(1+\epsilon)/2\rho}} \qquad \forall\, n \geqslant \hat{n}. \tag{4.9}$$

In order to get a lower bound for K_n, we transform the interval $[0, \alpha n^{1/\rho}]$ into the interval $[-1, +1]$ by means of the linear transformation

$$x = \frac{\alpha n^{1/\rho}}{2}(t+1), \qquad -1 \leqslant t \leqslant 1.$$

The function

$$g(t) \equiv f\left\{\frac{\alpha n^{1/\rho}}{2}(t+1)\right\}$$

is also an entire function of t. All derivatives of $g(t)$ are monotone increasing for $t \geqslant -1$ because of the assumption that the coefficients of $f(z)$ are nonnegative. Using a theorem of S. Bernstein (cf. [3], p. 78), we can assert that

$$K_n \geqslant \frac{g^{(n+1)}(-1)}{2^n(n+1)!} \qquad \forall\, n \geqslant 0,$$

or equivalently,

$$K_n \geqslant \frac{\alpha^{n+1} n^{(n+1)/\rho} f^{(n+1)}(0)}{2^{2n+1}(n+1)!} = \frac{\alpha^{n+1} n^{(n+1)/\rho} \cdot a_{n+1}}{2^{2n+1}} \qquad \forall\, n \geqslant 0. \tag{4.10}$$

Comparing (4.9) with (4.10), we have

$$\frac{\alpha^{n+1} n^{(n+1)/\rho} a_{n+1}}{2^{2n+1}} \leqslant \frac{e^{n(1+\epsilon)/\rho}}{q^n - e^{n(1+\epsilon)/2\rho}} \qquad \forall\, n \geqslant \hat{n}. \tag{4.11}$$

In order to make the left side of the above inequality as large as possible, we make use of (2.3). A simple manipulation of the expression in (2.3) shows that there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\{1, 2,...\}$ such that for $0 < \epsilon < 1$, there is a positive integer $k_1(\epsilon)$ for which

$$a_{n_k+1} \geqslant \left\{\frac{\rho e B(1-\epsilon)}{n_k}\right\}^{(n_k+1)/\rho} \qquad \forall\, k \geqslant k_1(\epsilon).$$

For this subsequence, the left side of (4.11) is bounded below by

$$2\left\{\frac{\alpha[\rho e B(1-\epsilon)]^{1/\rho}}{4}\right\}^{(n_k+1)} = 2\left\{\frac{[e(1-\epsilon)]^{1/\rho}}{2^{2+1/\rho}}\right\}^{(n_k+1)}, \qquad \forall\, k \geqslant k_1(\epsilon).$$

Hence, from (4.11), we have that

$$\Gamma\left(\frac{(1-\epsilon)^{1/\rho}}{e^{\epsilon/\rho}\cdot 2^{2+1/\rho}}\right)^{n_k} \leqslant \frac{1}{q^{n_k} - e^{n_k(1+\epsilon)/2\rho}} \qquad \forall\, k \geqslant k_2(\epsilon), \tag{4.12}$$

where $\Gamma \equiv 2[e(1-\epsilon)]^{1/\rho}/2^{2+1/\rho}$. Clearly, the above inequality can hold for all n_k sufficiently large only if

$$q \leqslant \frac{2^{2+1/\rho} \cdot e^{\epsilon/\rho}}{(1-\epsilon)^{1/\rho}},$$

and as ϵ is arbitrary,

$$q \leqslant 2^{2+1/\rho}. \tag{4.13}$$

But then, as $1/q$ in (4.4) can be chosen arbitrarily close to $\overline{\lim}_{n\to\infty}(\lambda_{0,n})^{1/n}$, we have the desired result (4.1). Q.E.D.

We remark that for entire function of perfectly regular growth $(1, B)$ with nonnegative coefficients, the lower bound of (4.1) is $1/8$. For the special case $f(z) = e^z$, it has been shown in [2] by using better lower bounds for K_n that $1/6$ is a lower bound for $\overline{\lim}_{n\to\infty}(\lambda_{0,n})^{1/n}$.

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