Reprint from

aequationes mathematicae

Vol. 7, fasc. 1, 1972 pages 36–58

The Effect of Quadrature Errors in the Numerical Solution of Two-Dimensional Boundary Value Problems by Variational Techniques

R. J. HERBOLD and R. S. VARGA
(Cincinnati, Ohio, U.S.A. and Kent, Ohio, U.S.A.)

BIRKHÄUSER VERLAG BASEL

The Effect of Quadrature Errors in the Numerical Solution of Two-Dimensional Boundary Value Problems by Variational Techniques

R. J. Herbold and R. S. Varga (Cincinnati, Ohio, U.S.A. and Kent, Ohio, U.S.A.)

Reprint from
aequationes mathematicae
Vol. 7, fasc. 1, 1972 BIRKHÄUSER VERLAG BASEL pages 36–58

The Effect of Quadrature Errors in the Numerical Solution of Two-Dimensional Boundary Value Problems by Variational Techniques[1])

R. J. HERBOLD and R. S. VARGA (Cincinnati, Ohio, U.S.A. and Kent, Ohio, U.S.A.)

1. Introduction

In [8], quadrature schemes were prescribed for the numerical solution of a class of one-dimensional boundary value problems by variational technique described in [4]. This paper is the analogue of [8] for the real two-dimensional nonlinear boundary value problem

$$\left.\begin{aligned} \frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} &= f(x,y,u),\ (x,y)\in G \\ u(x,y) &= 0, \quad (x,y)\in \partial G, \end{aligned}\right\} \tag{1.1}$$

where G is a rectangle in the (x, y)-plane. We point out that the results we present here extend easily to higher dimensions and to a larger class of nonlinear differential operators ([5, § 3]), but, for ease of presentation, we will restrict ourselves to the problem (1.1). With certain assumptions on $f(x, y, u)$, the unique solution of (1.1) can be approximated by applying the classical Ritz-Galerkin procedure to the variational formulation of the problem by minimizing over finite-dimensional subspaces ([5], [13, p. 188]). For sequences of piecewise-polynomial Hermite and spline subspaces, upper bounds for the rates of convergence of these approximations can be theoretically deduced from [1], [2], and [5]. These approximate solutions are, however, not precisely obtained in practical computation on a digital computer since certain integrals arising in the Ritz-Galerkin formulation are replaced by quadrature formulas.

The object of this paper is to investigate the errors introduced in the approximate solutions by such quadrature formulas. In particular, we shall obtain bounds for the errors introduced by such quadrature schemes, as they apply to finite-dimensional piecewise-polynomial Hermite and spline subspaces, and we shall determine when these quadrature errors are *consistent* with (i.e., the same order as) the approximation errors of the Ritz-Galerkin method. Numerical results based on such consistent quadrature schemes are also presented.

---

1) This research was supported in part by AEC Grant AT(11-1)-2075.

Received June 8, 1970

2. Formulation of the Problem

Let us assume that G is the open rectangle $(a, b)\times(c, d)$ in the (x, y)-plane; $a<x<b$, $c<y<d$, and let ∂G denote its boundary. We define S to be the set of all continuous functions $w(x, y)$ defined on $\bar{G}$, the closure of G, such that $w(x, y)$ is piecewise continuously differentiable over $\bar{G}$, and $w(x, y)=0$ when $(x, y)\in\partial G$. The first eigenvalue, Λ_1, associated with the Helmholtz equation $\Delta u+\lambda u=0$, over $\bar{G}$ is given by

$$\Lambda_1 = \inf_{\substack{w \in S \\ w \not\equiv 0}} \frac{\displaystyle\iint_G \left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} dx\, dy}{\displaystyle\iint_G (w)^2\, dx\, dy}. \tag{2.1}$$

The problem which we wish to consider is approximating the function $\phi(x, y)\in C(\bar{G})\cap C^2(G)$, where $\phi(x, y)$ satisfies (1.1). We assume that $f(x, y, u)\in C^0(\bar{G}\times R)$, and moreover that there exists a constant γ such that

$$\frac{f(x, y, u_1) - f(x, y, u_2)}{u_1 - u_2} \geqslant \gamma > -\Lambda_1, \quad (x, y)\in\bar{G}, \quad -\infty < u_1, \quad u_2 < +\infty, \quad u_1 \neq u_1. \tag{2.2}$$

If $\phi(x, y)$ is a solution of (1.1) when condition (2.2) holds, then it is shown in [3, p. 86] that $\phi(x, y)$ (which is certainly in S) strictly minimizes the functional

$$F[w] = \iint_G \left\{\frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \int_0^{w(x,y)} f(x, y, \eta)\, d\eta\right\} dx\, dy \tag{2.3}$$

over S, i.e.,

$$F[\phi] = \inf_{w \in S} F[w].$$

A consequence of this fact is that if a solution exists, it is unique. We will make the hypothesis that a solution of (1.1) exists; see [5] and [10] for further details.

As in the one-dimensional problem, we now consider a finite-dimensional subspace S_M of S, of dimension M, and minimize the functional (2.3) over S_M. If the functions $\{w_i(x, y)\}_{i=1}^{M}$ form a basis for S_M, then any element in S_M can be written as

$$w(x, y) = \sum_{j=1}^{M} u_j w_j(x, y).$$

We know from [5, Theorem 3.3] that, given any finite-dimensional subspace S_M of

S, there exists in S_M, one and only one $\hat{w}(x, y)$ such that

$$F[\hat{w}] = \inf_{w \in S_M} F[w].$$

In order to minimize the functional $F[w]$ over S_M spanned by $\{w_i(x, y)\}_{i=1}^M$, following system of equations must be solved:

$$A\mathbf{u} + \mathbf{k}(\mathbf{u}) = \mathbf{0}, \qquad (2$$

where $\mathbf{u} \equiv (u_1, u_2, \ldots, u_M)^T$, and where $A = (a_{i,j})$ is an $M \times M$ real, symmetric mat and $\mathbf{k}(\mathbf{u}) = (k_1(\mathbf{u}), \ldots, k_M(\mathbf{u}))^T$ is a column vector, which we now define: letting

$$\langle w, v\rangle = \iint_G \left\{\left(\frac{\partial w}{\partial x} \cdot \frac{\partial v}{\partial x}\right) + \left(\frac{\partial w}{\partial y} \cdot \frac{\partial v}{\partial y}\right)\right\} dx\, dy, \quad w, v \in S, \qquad (2$$

then

$$a_{i,j} = \langle w_i, w_j\rangle, \quad 1 \leqslant i, j \leqslant M, \qquad (2$$

and

$$k_i(\mathbf{u}) = \iint_G f\left(x, y, \sum_{j=1}^M u_j w_j\right) w_i\, dx\, dy, \quad 1 \leqslant i \leqslant M. \qquad (2$$

Denoting the solution of (2.4) by $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_M)^T$, the approximation to t solution $\phi(x, y)$ of (1.1) is then

$$\hat{w}(x, y) = \sum_{j=1}^M \hat{u}_j w_j(x, y).$$

The inner product (2.5) defined on set S induces the norm

$$\|w\|_D = (\langle w, w\rangle)^{1/2} = \left(\iint_G \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dx\, dy\right)^{1/2} \qquad (2.$$

on S. This norm is easily seen to be equivalent to the Sobolev norm [15, p. 55]

$$\|w\|_{1,2} \equiv \left\{\iint_G \left((w)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right) dx\, dy\right\}^{1/2}.$$

The quantity

$$\|w\|_\gamma = \left\{\iint_G \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \gamma (w)^2\right] dx\, dy\right\}^{1/2},$$

for $\gamma > -\Lambda_1$, is also a norm on S ([3, p. 97]) and is associated with the inner produc

$$\langle w, v\rangle_\gamma \equiv \iint_G \left\{\frac{\partial w}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial y} + \gamma w v\right\} dx\, dy, w, v \in S.$$

This norm is equivalent to $\|\cdot\|_D$ in (2.8) and is the norm basically used in the sections to follow. Note that $\|\cdot\|_D \equiv \|\cdot\|_0$.

Let $\{S_{M_i}\}_{i=1}^{\infty}$ be a sequence of finite-dimensional subspace of S and denote by $\hat{w}_{M_i}$ the unique element which minimizes $F[w]$ over S_{M_i}. From [5, Corollary 3.1] we know that if $\lim_{i\to\infty}\{\inf_{w\in S_{M_i}}\|w-g\|_0\}=0$ for all $g\in S$, $\partial f(x,y,u)/\partial u \in C^0(\bar{G}\times R)$ and if there exist two constants γ and K such that

$$-\Lambda_1 < \gamma \leqslant \frac{\partial f(x,y,u)}{\partial u} \leqslant K, \quad (x,y)\in\bar{G}, \quad -\infty < u < +\infty, \tag{2.9}$$

then

$$\lim_{i\to\infty} \|\phi - \hat{w}_{M_i}\|_0 = 0.$$

The hypothesis that $\partial f(x,y,u)/\partial u$ be bounded is rather restrictive. We can make other requirements on the function f which enable us to omit the hypothesis that $\partial f/\partial u$ be bounded above and to obtain the convergence of the sequence $\{\hat{w}_{M_i}\}_{i=1}^{\infty}$. In [5, Theorems 4.1 and 4.2] it is shown that one can derive an a priori pointwise bound for the solution $\phi(x,y)$ of (1.1) when one of the following conditions hold:

$$\lim_{|u|\to\infty} \inf \frac{f(x,y,u)}{u} \geqq 0, \quad (x,y)\in\bar{G}, \tag{2.10}$$

$$\frac{\partial f(x,y,u)}{\partial u} \geqq \gamma > -\frac{1}{\varrho}, \quad (x,y)\in\bar{G}, \quad -\infty < u < +\infty, \tag{2.11}$$

where ϱ is a positive quantity determined as follows. Let $\Psi(x,y)$ be the (unique) solution of $\Delta u(x,y) = -1$, $(x,y)\in G$; $u(x,y)=0$, $(x,y)\in\partial G$; then $\varrho = \sup_{(x,y)\in G}|\Psi(x,y)| > 0$.

Consider the problem (1.1) when either condition (2.10) or (2.11) holds. Assume that M is the constant such that the unique solution $\phi(x,y)$ of (1.1) satisfies $|\phi(x,y)| \leqslant M$ for all $(x,y)\in\bar{G}$. Then, as noted in [3, p. 96], solving (1.1) is equivalent to solving

$$\left.\begin{aligned} &\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x,y,u), \quad (x,y)\in G, \\ &u(x,y) = 0, \quad (x,y)\in\partial G, \end{aligned}\right\} \tag{2.12}$$

where

$$g(x,y,u) = \begin{cases} f(x,y,M), & M < u \\ f(x,y,u), & |u| \leqq M, \\ f(x,y,-M), & u < -M. \end{cases}$$

Therefore, suppose $\partial f(x,y,u)/\partial u \in C^0(\bar{G}\times R)$. If $f(x,y,u)$ does not satisfy the condition that $\partial f/\partial u$ is bounded above but it does satisfy either (2.10) or (2.11), then we can solve the corresponding problem (2.12) which satisfies (2.9) and the sequence

$\{w_{M_i}(x, y)\}_{i=1}^{\infty}$ converges in the norm $\|\cdot\|_D$ to $\phi(x, y)$, the unique solution of (1. as well as (2.12). Hence, we will assume that $\partial f/\partial u(x, y, u) \in C^0(\bar{G} \times R)$, and th either $\partial f/\partial u$ is bounded above or either (2.10) or (2.11) holds.

For any subspace S_M of S, it can be shown (cf. [3, p. 100], [5, Theorem 4.4]) th there exists a positive constant C, independent of the choice of S_M, such that t following error bound is valid:

$$\|\hat{w} - \phi\|_D \leqslant C \inf_{w \in S_M} \|w - \phi\|_D, \qquad (2.1$$

where $\hat{w}$ is such that $F[\hat{w}] = \inf_{w \in S_M} F[w]$. This implies that rigorous bounds can deduced from the two-dimensional interpolation error estimates of [1] and [2] f the quantities $\|\hat{w} - \phi\|_D$ and $\|\hat{w} - \phi\|_\gamma$. Unfortunately, Sobolev's Imbedding Theore [15, p. 174] in two dimensions for this case does not allow us to deduce error boun in the uniform norm, as was the case in one dimension.

Let us consider the system of equations (2.4) which must be solved. For piecewis polynomial Hermite and spline subspaces, the matrix entries $a_{i,j}$ of the matrix A a easily computed since this involves only the integration of polynomials, which easily automated on a digital computer. The quantities $\mathbf{k}(\mathbf{u})$ in (2.7) are more difficu because f is not in general a piecewise-polynomial. This prompts us to use a quadratu scheme to evaluate the quantities in (2.7), which then generates a new system nonlinear equations

$$\mathbf{0} = A\mathbf{u} + \tilde{\mathbf{k}}(\mathbf{u}), \qquad (2.1$$

where $\tilde{k}_i(\mathbf{u})$ is obtained by applying quadrature scheme to $k_i(\mathbf{u})$. The solution $\tilde{\mathbf{u}}$ (2.14) in turn generates a new function

$$\tilde{w} = \sum_{i=1}^{M} \tilde{u}_i w_i \quad \text{in} \quad S_M.$$

In the next two sections, we shall discuss the choice of quadrature schemes for given sequence of piecewise-polynomial subspaces $\{S_{M_i}\}_{i=1}^{\infty}$ of S so that the approx imations $\{\tilde{w}_{M_i}\}_{i=1}^{\infty}$, determined successively from (2.14), have the same general ord of accuracy as the theoretical approximations $\{\hat{w}_{M_i}\}_{i=1}^{\infty}$, determined successivel from (2.4).

Without loss of generality, from here on in our discussion we will assume that G i the open unit square $(0, 1) \times (0, 1)$.

3. Linear Case

Let us suppose that the function f in (1.1) is not a function of u. In this cas $f(x, y)$ satisfies (2.2) with $\gamma = 0$, and $\partial f/\partial u$ is bounded above by zero. The integrals o

(2.7) are also independent of u, and in this case, we have

$$k_i = \iint_G f(x, y)\, w_i(x, y)\, dx\, dy = L[f(x, y)\, w_i(x, y)], \quad 1 \leqslant i \leqslant M, \tag{3.1}$$

where the integral in (3.1) is regarded as a bounded linear functional, L, on $C^1(\bar{G})$. With the subspace S_M, we associate a linear functional L_M which is to approximate L, and we define

$$\tilde{\mathbf{k}}_i = L_M[f(x, \bar{y})\, w_i(x, y)], \quad 1 \leqslant i \leqslant M \tag{3.2}$$

as the approximations of k_i in (3.1). The matrix problem of (2.4) now reduces to

$$A\mathbf{u} + \mathbf{k} = \mathbf{0} \tag{3.3}$$

and the use of the approximate linear functional L_M gives the associated matrix problem

$$A\mathbf{u} + \tilde{\mathbf{k}} = \mathbf{0}. \tag{3.4}$$

As previously noted, A is a real symmetric matrix, and since we can easily verify that

$$\mathbf{u}^T A \mathbf{u} = \left\| \sum_{j=1}^{M} u_j w_j \right\|_D^2, \tag{3.5}$$

A is then also positive definite. Hence, each of the matrix problems (3.3) and (3.4) admits a unique solution, denoted by $\hat{\mathbf{u}}$ and $\tilde{\mathbf{u}}$ respectively, and the associated functions in S_M are $\hat{w}_M = \sum_{j=1}^{M} \hat{u}_j w_j$ and $\tilde{w}_M = \sum_{j=1}^{M} \tilde{u}_j w_j$. From (3.3) and (3.4), it follows that $A(\hat{\mathbf{u}} - \tilde{\mathbf{u}}) = (\tilde{\mathbf{k}} - \mathbf{k})$, and premultiplying by $(\hat{\mathbf{u}} - \tilde{\mathbf{u}})^T$ and using (3.5) gives

$$(\hat{\mathbf{u}} - \tilde{\mathbf{u}})^T A(\hat{\mathbf{u}} - \tilde{\mathbf{u}}) = \|\hat{w}_M - \tilde{w}_M\|_D^2 = (\hat{\mathbf{u}} - \tilde{\mathbf{u}})^T (\tilde{\mathbf{k}} - \mathbf{k}).$$

Using the definitions of the functionals L and L_M, the last quantity above can be expressed as

$$(L_M - L)[f(x, y)(\hat{w}_M(x, y) - \tilde{w}_M(x, y))],$$

and thus

$$\|\hat{w}_M - \tilde{w}_M\|_D^2 = (L_M - L)[f(x, y)(\hat{w}_M(x, y) - \tilde{w}_M(x, y))]. \tag{3.6}$$

This equation will be used repeatedly in this section.

Our object now is to bound $\|\hat{w}_M - \tilde{w}_M\|_D$ for certain quadrature schemes L_M, after making further assumptions on f and the subspace S_M. Let

$$\Delta_x: 0 = x_0' < x_1' < \cdots < x_{N_x+1}' = 1, \quad \Delta_y: 0 = y_0' < y_1' < \cdots < y_{N_y+1}' = 1$$

be a partition of the unit square $\bar{G}$ and denote by Δ the set of points (x_i', y_j'), $0 \leqslant i \leqslant N_x + 1$, $0 \leqslant j \leqslant N_y + 1$. We will always assume that $\bar{\Delta} \leqslant \delta_0 \Delta$, where δ_0 is a fixed positive number, and where

$$\bar{\Delta} \equiv \max_{\substack{0 \leqslant i \leqslant N_x \\ 0 \leqslant j \leqslant N_y}} \{|x_{i+1}' - x_i'|, |y_{j+1}' - y_j'|\}$$

and $\underline{\Delta}$ is the corresponding minimum. We now restrict our attention to subspaces $S_M(\Delta)$ of S of piecewise-polynomial functions. More precisely, for any $w(x, y) \in S_M(\Delta)$, $w(x, y)$ is a piecewise-polynomial of degree n_0 in each variable and a polynomial in each cell

$$[x_i', x_{i+1}'] \times [y_j', y_{j+1}'], \quad 0 \leqslant i \leqslant N_x, \quad 0 \leqslant j \leqslant N_y, \quad \text{of} \quad \Delta .$$

Such subspaces include the Hermite and spline subspaces as special cases (cf. [1] and [2]). We also assume that the function f of (1.1) is such that $f(x, y) \in C^{m_0}([x_i', x_{i+1}'] \times [y_j', y_{j+1}'])$, $0 \leqslant i \leqslant N_x$, $0 \leqslant j \leqslant N_y$. This latter hypothesis is of course valid if $f(x, y) \in C^{m_0}(\bar{G})$, but it also holds for functions $f(x, y)$ whose m_0-th derivative is piecewise continuous on G, with points of discontinuity on the boundaries of the cells of Δ. The important point is that since f is given, the quantity m_0 and the possible points of discontinuity of the partial derivatives of f can be determined directly.

As our first choice for the bounded linear functional L_M, consider a quadrature scheme of the form

$$\int_{x_0}^{x_m} \int_{y_0}^{y_m} \sigma(x, y)\, dy\, dx \doteqdot \sum_{i, j=0}^{m} \alpha_{i, j} \sigma(\tau_i, \eta_j), \tag{3.7}$$

where $x_0 \leqslant \tau_0 < \tau_1 < \cdots < \tau_m \leqslant x_m$, $y_0 \leqslant \eta_0 < \eta_1 < \cdots < \eta_m \leqslant y_m$ are selected points of $[x_0, x_m]$ and $[y_0, y_m]$. Given any $\sigma(x, y) \in C^{m_0}([x_0, x_m] \times [y_0, y_m])$ and m_0 determined from f, it is always possible to select a quadrature scheme of the form (3.7) such that the quadrature error of (3.7) satisfies

$$\left.\begin{aligned} &\left| \sum_{i, j=0}^{m} \alpha_{i, j} \sigma(\tau_i, \eta_j) - \int_{x_0}^{x_m} \int_{y_0}^{y_m} \sigma(x, y)\, dy\, dx \right| \\ &\qquad \leqslant K \delta^{m_0+2} \max_{i=0, m_0} \left\{ \left\| \frac{\partial^{m_0} \sigma}{\partial x^i\, \partial y^{m_0-i}} \right\|_{L^\infty([x_0, x_m] \times [y_0, y_m])} \right\} \end{aligned}\right\} \tag{3.8}$$

where $\delta \equiv \max(x_m - x_0, y_m - y_0)$, and where K is independent of $(y_m - y_0)$ and $(x_m - x_0)$. Quadrature schemes such as (3.7)–(3.8) are very easily generated by taking in essence the Cartesian product of one-dimensional quadrature schemes of the form

$$\int_{y_0}^{y_m} \sigma(t)\, dt \doteqdot \sum_{i=0}^{m} \alpha_i \sigma(\tau_i)$$

with

$$\left| \sum_{i=0}^{m} \alpha_i \sigma(\tau_i) - \int_{y_0}^{y_m} \sigma(t)\, dt \right| \leqslant K (y_m - y_0)^{m_0+1} \left\| \frac{d^{m_0} \sigma}{dx^{m_0}} \right\|_{L^\infty[y_0, y_m]},$$

and applying it twice to

$$\int_{x_0}^{x_m} \int_{y_0}^{y_m} \sigma(x, y)\, dy\, dx .$$

See also [11, ch. 4]. Writing k_i in (3.1) as

$$k_i = \sum_{l=0}^{N_x} \sum_{k=0}^{N_y} \int_{x'_l}^{x'_{l+1}} \int_{y'_k}^{y'_{k+1}} f(x, y)\, w_i(x, y)\, dy\, dx,$$

we apply (3.7) on each cell $[x'_l, x'_{l+1}] \times [y'_k, y'_{k+1}]$, and this in turn defines the linear functional L_M of (3.2) which we will denote as

$$L_M[f(x, y)\, w_i(x, y)] = \tilde{k}_i = \sum_{l=0}^{M_0} \sum_{k=0}^{M_1} \beta_{l,k} f(x_l, y_k)\, w_i(x_l, y_k) \tag{3.9}$$

for simplicity of notation. This brings us to

THEOREM 1. *Assuming that f in (1.1) is independent of u, let $\partial^{m_0} f(x, y)/\partial x^{m_0}$ and $\partial^{m_0} f(x, y)/\partial y^{m_0}$ be continuous on each cell of Δ. If Q is any collection of quasi-uniform partitions of $\bar{G} = [0, 1] \times [0, 1]$, i.e., there exists a fixed positive constant δ_0 such that $\delta_0 \underline{\Delta} \geqslant \bar{\Delta}$ for all $\Delta \in Q$, and if for each $\Delta \in Q$, $S_M(\Delta)$ is any finite-dimensional subspace of S such that for any $v(x, y) \in S_M(\Delta)$, $v(x, y)$ is a polynomial of degree n_0 in each variable on each cell of Δ, then for $m_0 \geqslant n_0 + 2$, the linear functional L_M defined in (3.9) is such that*

$$\|\hat{w}_M - \tilde{w}_M\|_D \leqslant B(\bar{\Delta})^{m_0 - n_0 - 2}, \tag{3.10}$$

where B is a constant independent of Δ.

Proof. Expressing $(L_M - L)[f(\hat{w}_M - \tilde{w}_M)]$ in (3.6) as a sum of terms and applying (3.8) to each of these terms gives

$$\left.\begin{aligned} \|\hat{w}_M - \tilde{w}_M\|_D^2 &= (L_M - L)[f(\hat{w}_M - \tilde{w}_M)] \\ &\leqslant \sum_{l=0}^{N_x} \sum_{k=0}^{N_y} K(\bar{\Delta})^{m_0+2} \max_{i=0,\, m_0} \left\| \frac{\partial^{m_0}[f(\hat{w}_M - \tilde{w}_M)]}{\partial x^i\, \partial y^{m_0 - i}} \right\|_{L^\infty_{l,k}}, \end{aligned}\right\} \tag{3.11}$$

where

$$\|\cdot\|_{L^\infty_{l,k}} \equiv \|\cdot\|_{L^\infty([x'_l, x'_{l+1}] \times [y'_k, y'_{k+1}])}$$

and K is independent of Δ. By hypothesis, there exists a constant C_1, independent of Δ, such that

$$\max_{0 \leqslant k \leqslant N_y} \max_{0 \leqslant l \leqslant N_x} \max \left\{ \left\| \frac{\partial^{j_1} f(x, y)}{\partial x^{j_1}} \right\|_{L^\infty_{l,k}}, \left\| \frac{\partial^{j_2} f(x, y)}{\partial y^{j_2}} \right\|_{L^\infty_{l,k}} \right\} \leqslant C_1,$$

for $0 \leqslant j_1, j_2 \leqslant m_0$. Consequently, using the Leibnitz formula for differentiation of a product, the sum of (3.11) is bounded above by

$$C_1 K(\bar{\Delta})^{m_0+2} \sum_{l=0}^{N_x} \sum_{k=0}^{N_y} \max_{\substack{i=0 \\ i=m_0}} \left\{ \sum_{r=0}^{i} \sum_{s=0}^{m_0-1} \binom{i}{r} \binom{m_0 - i}{s} \left\| \frac{\partial^{r+s}(\hat{w}_M - \tilde{w}_M)}{\partial x^r\, \partial y^s} \right\|_{L^\infty_{l,k}} \right\}. \tag{3.12}$$

By the assumption that the elements of $S_M(\Delta)$ are piecewise-polynomials of degree n_0 in each variable, the upper limits on the sums on r and s in (3.12) are at most n_0. Since $\hat{w}_M - \tilde{w}_M$ is a polynomial in each cell of Δ then, by a theorem of Markov [12, p. 138], there exists a constant C_2, independent of Δ, such that

$$\left\| \frac{\partial^{i+j}(\hat{w}_M - \tilde{w}_M)}{\partial x^i \, \partial y^j} \right\|_{L^\infty_{l,k}} \leqslant C_2 \left\| \frac{\hat{w}_M - \tilde{w}_M}{(\bar{\Delta})^{i+j}} \right\|_{L^\infty_{l,k}}, \tag{3.13}$$

for all $0 \leqslant i, j \leqslant n_0$, $0 \leqslant l \leqslant N_x$, and $0 \leqslant k \leqslant N_y$. Using (3.13), we can deduce from (3.11) and (3.12) that there is a constant C_3, independent of Δ, such that

$$\|\hat{w}_M - \tilde{w}_M\|_D^2 \leqslant C_3 (N_x + 1)(N_y + 1)(\bar{\Delta})^{m_0 + 2 - n_0} \|\hat{w}_M - \tilde{w}_M\|_{L^\infty(G)}. \tag{3.14}$$

Realizing that $\max(N_x + 1, N_y + 1) \leqslant \underline{\Delta}^{-1}$ and since we are assuming $\bar{\Delta} \leqslant \delta_0 \underline{\Delta}$, we are assured from (3.14) that there exists a constant C_4 such that

$$\|\hat{w}_M - \tilde{w}_M\|_D^2 \leqslant C_4 (\bar{\Delta})^{m_0 - n_0} \|\hat{w}_M - \tilde{w}_M\|_{L^\infty(G)}. \tag{3.15}$$

We now wish to relate $\|\hat{w} - \tilde{w}\|_{L^\infty(G)}$ and $\|\hat{w} - \tilde{w}\|_D$, using the fact that $\hat{w}_M - \tilde{w}_M$ is a piecewise-polynomial and $\hat{w}_M(x, y) - \tilde{w}(x, y) = 0$ if $x = 0$ or $y = 0$. In [9], it is proved if $p_n(x)$ is a polynomial of degree n over the interval $[-1, +1]$ then there is a constant A, independent of n, such that

$$\int_{-1}^{+1} \left| \frac{dp_n(x)}{dx} \right| dx \leqslant An^2 \int_{-1}^{+1} |p_n(x)| \, dx$$

and hence by a change of variable

$$\int_a^b \left| \frac{dq_n(x)}{dx} \right| dx \leqslant \frac{2An^2}{(b-a)} \int_a^b |q_n(x)| \, dx$$

for any polynomial $q_n(x)$ of degree n defined on $[a, b]$. Therefore, for $z \in [a, b]$, we have the string of inequalities

$$\left.\begin{aligned} |q_n(z)| - |q_n(a)| &\leqslant |q_n(z) - q_n(a)| \\ &= \left| \int_a^z \frac{dq_n(x)}{dx} \, dx \right| \leqslant \int_a^z \left| \frac{dq_n(x)}{dx} \right| dx \\ &\leqslant \int_a^b \left| \frac{dq_n(x)}{dx} \right| dx \leqslant \frac{2An^2}{(b-a)} \int_a^b |q_n(x)| \, dx. \end{aligned}\right\} \tag{3.16}$$

Suppose we have a piecewise-polynomial $\sigma(x)$ defined on $[0,1]$, where $\Delta_0 : 0 < x_0 < x_1 < \cdots < x_{N+1} = 1$ is a partition of $[0,1]$, with $\bar{\Delta}_0 \leqslant \delta_1 \Delta_0$, $\delta_1 > 0$, and $\sigma(x)$ is a polynomial of degree n on $[x_i, x_{i+1}]$, $0 \leqslant i \leqslant N$. Also, we assume $\sigma(0) = 0$ and denote by $\sigma_i(x)$ the function $\sigma(x)$ restricted to $[x_i, x_{i+1}]$. Using (3.16) on $\sigma_0(x)$ with $a = x_0 = 0$ and $b = x_1$, we see that

$$\|\sigma_0(x)\|_{L^\infty[x_0, x_1]} \leqslant \frac{2An^2}{(x_1 - x_0)} \int_{x_0}^{x_1} |\sigma_0(x)| \, dx .$$

From this inequality, we see that

$$|\sigma_0(x_1)| \leqslant \frac{2An^2}{(x_1 - x_0)} \int_{x_0}^{x_1} |\sigma_0(x)| \, dx ,$$

so using (3.16) on $\sigma_1(x)$ with $a = x_1$ and $b = x_2$ we obtain

$$\|\sigma_1(x)\|_{L^\infty[x_1, x_2]} \leqslant \frac{2An^2}{(x_1 - x_0)} \int_{x_0}^{x_1} |\sigma_0(x)| \, dx + \frac{2An^2}{(x_2 - x_1)} \int_{x_1}^{x_2} |\sigma_1(x)| \, dx .$$

Continuing, we see that

$$\|\sigma(x)\|_{L^\infty[0, 1]} \leqslant \sum_{j=0}^{N} \frac{2An^2}{(x_{j+1} - x_j)} \int_{x_j}^{x_{j+1}} |\sigma_j(x)| \, dx$$

$$\leqslant \frac{2An^2}{\Delta_0} \int_0^1 |\sigma(x)| \, dx .$$

Hence, using the Cauchy-Schwarz inequality,

$$\|\sigma(x)\|_{L^\infty[0, 1]} \leqslant \frac{2An^2}{\Delta_0} \|\sigma(x)\|_{L^2[0, 1]} .$$

Since $\bar{\Delta}_0 \leqslant \delta_1 \Delta_0$, we can let $\beta_0 = 2An^2 \delta_1$ and we have

$$\|\sigma(x)\|_{L^\infty[0, 1]} \leqslant \frac{\beta_0}{\bar{\Delta}_0} \|\sigma(x)\|_{L^2[0, 1]} .$$

Applying this to $\hat{w}(x, y) - \tilde{w}(x, y)$, we see that

$$\begin{aligned}\|\hat{w} - \tilde{w}\|_{L^\infty(G)} &= \sup_{0 \leqslant x \leqslant 1} \sup_{0 \leqslant y \leqslant 1} |\hat{w}(x, y) - \tilde{w}(x, y)| \\ &\leqslant \sup_{0 \leqslant x \leqslant 1} \left| \frac{\beta}{\underline{\Delta}} \sqrt{\int_0^1 |\hat{w} - \tilde{w}|^2 \, dy} \right| \\ &\leqslant \frac{\beta_0}{\underline{\Delta}} \sqrt{\int_0^1 \left[\sup_{0 \leqslant x \leqslant 1} |\hat{w} - \tilde{w}| \right]^2 dy} \\ &\leqslant \frac{\beta_0}{\underline{\Delta}} \sqrt{\int_0^1 \left[\frac{\beta_0}{\underline{\Delta}} \sqrt{\int_0^1 |\hat{w} - \tilde{w}|^2 \, dx} \right]^2 dy},\end{aligned}$$

so that

$$\|\hat{w} - \tilde{w}\|_{L^\infty(G)} \leqslant \frac{\beta_0^2}{\underline{\Delta}^2} \|\hat{w} - \tilde{w}\|_{L^2(G)}.$$

Hence, (3.15) becomes

$$\|\hat{w} - \tilde{w}\|_D^2 \leqslant C_4 (\bar{\Delta})^{m_0 - n_0 - 2} \beta_0^2 \|\hat{w} - \tilde{w}\|_{L^2(G)},$$

and since $\|\cdot\|_{L^2(G)} \leqslant \|\cdot\|_{1,2} \leqslant C_0 \|\cdot\|_D$ for some C_0, let $B = C_4 \beta_0^2 C_0$ and we have

$$\|\hat{w} - \tilde{w}\|_D \leqslant B (\bar{\Delta})^{m_0 - n_0 - 2},$$

where B is independent of Δ. Q.E.D.

If we have a sequence $\{S_{M_i}(\Delta_i)\}_{i=1}^\infty$ of finite-dimensional subspaces of S with $\bar{\Delta}_i \leqslant \delta_1 \underline{\Delta}_i$ such that the elements of any $S_{M_i}(\Delta_i)$ are piecewise-polynomials of *fixed* degree n_0, and if $\lim_{i \to \infty} \bar{\Delta}_i = 0$, then from (3.10) if m_0, dependent only on f, satisfies $m_0 > n_0 + 2$, we evidently have

$$\lim_{i \to \infty} \| \hat{w}_{M_i} - \tilde{w}_{M_i} \|_D = 0.$$

Hence, the quadrature error, introduced by computing $\tilde{w}_{M_i}$ rather than $\hat{w}_{M_i}$, tends to zero with i. This error, however, may or may not be small, relative to $\|\hat{w}_{M_i} - \phi\|_D$. This brings us to

DEFINITION 1. Let Q be a collection of partitions Δ of $\bar{G}$, and for each $\Delta \in Q$, let $S_M(\Delta)$ be a finite-dimensional subspace of S consisting of elements which are polynomials of fixed degree n_0 in each cell defined by Δ, and let $\hat{w}_M$, the function which minimizes $F[w]$ of (2.3) over $S_M(x)$, satisfy

$$\|\hat{w}_M - \varphi\|_N \leqslant K (\bar{\Delta})^l \quad \text{for all} \quad \Delta \in Q, \tag{3.17}$$

where K and l are positive constants independent of Δ, $\varphi(x, y)$ is the solution of (1.1), and $\|\cdot\|_N$ is some norm on the space S. Then, the choice of linear functionals in (3.2) is *consistent* in the norm $\|\cdot\|_N$ with the bounds of (3.17) if there exists a constant K_0, independent of Δ, such that

$$\|\hat{w}_M - \tilde{w}_M\|_N \leqslant K_0 (\bar{\Delta})^l \quad \text{for all} \quad \Delta \in Q.$$

From the triangle inequality, the bounds of (3.17) for the norm $\|\cdot\|_D$, and the results of Theorem 1, it follows that

$$\|\tilde{w} - \varphi\|_D \leqslant \|\hat{w} - \tilde{w}\|_D + \|\hat{w} - \varphi\|_D \leqslant B(\bar{\Delta})^{m_0 - n_0 - 2} + K(\bar{\Delta})^l, \quad \Delta \in Q.$$

Hence, $m_0 - n_0 - 2 \geqslant l$ gives a consistent choice of functionals in (3.9) in the norm $\|\cdot\|_D$ which preserves the asymptotic accuracy of (3.17) in this norm. Note that even if this choice is not consistent in the norm $\|\cdot\|_D$, i.e., if $1 \leqslant m_0 - n_0 - 2 < l$, it follows that when the collection Q is a sequence of partitions $\{\Delta_i\}_{i=1}^{\infty}$ with $\bar{\Delta}_i \leqslant \delta_1 \underline{\Delta}_i$ for some fixed positive δ_1, then the associated sequence $\{\tilde{w}_{M_i}\}_{i=1}^{\infty}$ converges in the norm $\|\cdot\|_D$ to $\varphi(x, y)$ as $i \to \infty$ when $\lim_{i \to \infty} \bar{\Delta}_i = 0$.

Another way to approximate the quantities k_i, $1 \leqslant i \leqslant M$, in (3.1) is by replacing $f(x, y)$ by an interpolate, calling it $\tilde{f}(x, y)$, and evaluating $\iint_{Cr} \tilde{f}(x, y)\, w_i(x, y)\, dx\, dy$ exactly. We assume that we have a partition Δ determined by the partitions Δ_x: $0 = x_0 < x_1 < \cdots < x_{N_x+1} = 1$ and Δ_y: $0 = y_0 < y_1 < \cdots < y_{N_y+1} = 1$ of the x and y axis. Note that if $\{w_i(x, y)\}_{i=1}^{M}$ and $\tilde{f}(x, y)$ are polynomials on each cell $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ of Δ, then the integral $\iint_G \tilde{f}(x, y)\, w_i(x, y)\, dx\, dy$ is simply the sum of integrals of polynomials over the cells of Δ, and is therefore easy to calculate on a digital computer. We now determine how accurate an approximation $\tilde{f}(x, y)$ must be to $f(x, y)$ in order for this type of quadrature scheme to be useful in our variational technique.

The interpolation of $f(x, y)$ in (3.1) by $\tilde{f}(x, y)$ would generate a system of equations (3.4) where

$$\tilde{k}_i = L_M[f w_i] = L(\tilde{f} w_i] = \iint_G \tilde{f}(x, y)\, w_i(x, y)\, dx\, dy, \quad 1 \leqslant i \leqslant M. \quad (3.18)$$

This in turn again serves to define $\tilde{w}_M(x, y) = \sum_{j=1}^{M} \tilde{u}_j w_j(x, y)$ from the solution of (3.4).

THEOREM 2. *Assuming that f in* (1.1) *is independent of u, let Δ be any partition of $\bar{G} = [0,1] \times [0,1]$ such that f is continuous on each cell of Δ and let $S_M(\Delta)$ be any finite-dimensional subspace of S such that for every $v(x, y) \in S_M(\Delta)$, $v(x, y)$ is a polynomial of degree n_0 in each variable on each cell of Δ. If $\tilde{f}_M(x, y; \Delta)$ is a continuous piecewise-polynomial interpolate of $f(x, y)$ such that $\tilde{f}(x, y; \Delta)$ is a polynomial on each cell defined by Δ, let L_M be the associated linear functional of* (3.18). *Then, there* *exists a constant K, independent of Δ, such that*

$$\|\hat{w}_M - \tilde{w}_M\|_D \leqslant K \|\tilde{f}_M - f\|_{L^2(G)}. \tag{3.19}$$

Proof. From (3.6), we have

$$\|\hat{w}_M - \tilde{w}_M\|_D^2 = \iint_G [\tilde{f}_M(x, y; \Delta) - f(x, y)] [\hat{w}_M(x, y) - \tilde{w}_M(x, y)] \, dx \, dy,$$

and using the Cauchy-Schwarz inequality gives

$$\|\hat{w}_M - \tilde{w}_M\|_D^2 \leqslant \|\tilde{f}_M - f\|_{L^2(G)} \|\hat{w} - \tilde{w}\|_{L^2(G)}. \tag{3.20}$$

Now $\|\hat{w}_M - \tilde{w}_M\|_{L^2(G)} \leqslant \|\hat{w} - \tilde{w}\|_{1,2}$, and $\|\cdot\|_D$ is equivalent to $\|\cdot\|_{1,2}$, so there exists a constant K such that

$$\|\hat{w}_M - \tilde{w}_M\|_{L^2(G)} \leqslant K \|\hat{w}_M - \tilde{w}_M\|_D.$$

Hence from (3.20),

$$\|\hat{w}_M - \tilde{w}_M\|_D \leqslant K \|\tilde{f}_M - f\|_{L^2(G)}. \qquad \text{Q.E.D.}$$

From inequality (3.19), it is now clear how the piecewise-polynomial interpolate is to be chosen so as to have a consistent quadrature scheme in some norm. For example, if $f(x, y) \in C^4(\bar{G})$, then the piecewise cubic Hermite interpolate $\tilde{f}(x, y)$, relative to a partition Δ on G, satisfies (cf. [1])

$$\|f - \tilde{f}\|_{L^2(G)} \leqslant K_1 (\bar{\Delta})^4.$$

4. Nonlinear Case

We now define what we mean by a consistent quadrature scheme for the general problem (1.1), where f is a function of u, as well as x and y.

If we approximate the integrals $k_i(\mathbf{u})$, $i \leqslant i \leqslant M$, in (2.7) by a quadrature scheme and denote these approximations by $\tilde{k}_i(\mathbf{u})$, we have the following new system to solve:

$$A\mathbf{u} + \tilde{\mathbf{k}}(\mathbf{u}) = \mathbf{0}. \tag{4.1}$$

Naturally, since f is dependent on u, we are not assured that the system (4.1) has a unique solution. As in the previous section, we will denote by $\hat{w}_M(x, y)$ the approximation generated by the system (2.4) using subspace S_M.

Let $\Delta_x: 0 = x'_0 < x'_1 < \cdots < x'_{N_x+1} = 1$, $\Delta_y: 0 = y'_0 < y'_0 < \cdots < y'_{N_y+1} = 1$ be a partition of the unit square $\bar{G}$ and, as in §3, denote by Δ the set of points (x'_i, y'_j), $0 \leqslant i \leqslant N_x + 1$, $0 \leqslant j \leqslant N_y + 1$. Again we assume $\bar{\Delta} \leqslant \delta_0 \underline{\Delta}$, where δ_0 is a fixed positive constant, and we restrict our attention to subspaces $S_M(\Delta)$ of S of piecewise-polynomial functions. Writing $k_i(\mathbf{u})$ in (2.7) as the sum

$$k_i(\mathbf{u}) = \sum_{l=0}^{N_x} \sum_{k=0}^{N_y} \int_{x'_l}^{x'_{l+1}} \int_{y'_k}^{y'_{k+1}} f\left(x, y, \sum_{j=1}^{M} u_j w_j\right) w_i \, dy \, dx$$

and applying the quadrature scheme (3.7) to the $(N_x+1)(N_y+1)$ integrals in this sum, we obtain an approximation which we will denote, in simplified notation, as

$$\tilde{k}_i(\mathbf{u}) = \sum_{l=0}^{M_0} \sum_{k=0}^{M_1} \beta_{l,k} f\left(x_l, y_k, \sum_{j=1}^{M} u_j w_j\right) w_i(x_l, y_k). \tag{4.2}$$

Substituting these $\tilde{k}_i(\mathbf{u})$ for $k_i(\mathbf{u})$ in (2.4) generates system (4.1). The following theorem gives sufficient conditions for the system of equations (4.1) to have a unique solution.

THEOREM 3. *Suppose that S_M is a finite-dimensional subspace of S spanned by the linearly independent set $\{w_i(x, y)\}_{i=1}^{M}$ and let the quadrature scheme* (3.7), *used as described above to obtain the approximations $\tilde{k}_i(\mathbf{u})$ in* (4.1), *satisfy the following conditions:*

$$\alpha_{i,j} \geqslant 0, \quad 0 \leqslant i, j \leqslant m; \quad \sum_{i=0}^{m} \sum_{j=0}^{m} \alpha_{i,j} = (x_m - x_0)(y_m - y_0);$$

and, in the notation of (4.2)

$$\sum_{l=0}^{M_0} \sum_{k=0}^{M_1} \beta_{l,k} w_i(x_l, y_k)\, w_j(x_l, y_k) = \int_0^1 \int_0^1 w_i w_j \, dy \, dx,$$

for $1 \leqslant i, j \leqslant m$. Then the system (4.1) *has a unique solution $\tilde{\mathbf{u}}$.*

Proof. We define the following functional on S_M:

$$H[w] = \int_0^1 \int_0^1 \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dy\, dx + \sum_{l=0}^{M_0} \sum_{k=0}^{M_1} \beta_{l,k} \int_0^{w(x_l, y_k)} f(x_l, y_k, \eta)\, d\eta$$

where $w(x, y) = \sum_{j=1}^{M} u_j w_j(x, y)$. Note that the system $0 = \partial H[w]/\partial u_i$, $1 \leqslant i \leqslant M$, is exactly system (4.1) where the $\tilde{k}_i(\mathbf{u})$, $1 \leqslant i \leqslant M$, are described in (4.2). Hence, in order to prove that (4.1) has a unique solution, it suffices to show that $H[w]$ has a unique stationary value over S_M. The remainder of the proof is simply the two-dimensional analogue of the proof of Theorem 4 in [8]. See [7, p. 157] for complete details.

THEOREM 4. *Let Q be any collection of quasi-uniform partitions Δ of $\bar{G}$, i.e., there exists a constant $\delta_0 > 0$ such that $\delta_0 \underline{\Delta} \geqslant \bar{\Delta}$ for all $\Delta \in Q$, and for each $\Delta \in Q$, let $S_M(\Delta)$ be a finite-dimensional subspace of S consisting of piecewise-polynomial Hermite or spline functions of degree n_0. Assume that $\partial^{m_0} f(x, y, w(x, y))/\partial x^{m_0}$ and $\partial^{m_0} f(x, y, w(x, y))/\partial y^{m_0}$ are continuous for all $w(x, y) \in S_M(\Delta)$ for all $\Delta \in Q$ and that $\phi(x, y) \in C^{n_0+1}(\bar{G})$. If the quadrature scheme* (3.7), *used to approximate the $k_i(\mathbf{u})$ in* (2.7), *satisfies the hypotheses of Theorem* 3 *for each subspace $S_M(\Delta)$, and $m_0 \geqslant n_0$, then* *there exists a positive constant K such that*

$$\|\hat{w}-\tilde{w}\|_\gamma \leqslant K(\bar{\Delta})^s \quad \text{for all} \quad \Delta \in Q,$$

where $s=\min(m_0-n_0-2, n_0)$.

Proof. For any partition $\Delta \in C$, let $\{w_i(x,y)\}_{i=1}^M$ be a basis for $S_M(\Delta)$. Denote by $\bar{w}(x,y)=\sum_{j=1}^M \bar{u}_j w_j$ the interpolate in S_M of $\phi(x,y)$, the unique solution of (1.1). We define

$$\varepsilon_i = (A\bar{\mathbf{u}})_i + k_i(\bar{\mathbf{u}}), \quad 1 \leqslant i \leqslant M,$$

where A and $k_i(\cdot)$ are defined in (2.6) and (2.7) respectively. Thus,

$$A\bar{\mathbf{u}} = -\tilde{\mathbf{k}}(\bar{\mathbf{u}}) + \boldsymbol{\varepsilon} + \tilde{\mathbf{k}}(\bar{\mathbf{u}}) - \mathbf{k}(\bar{\mathbf{u}}). \tag{4.3}$$

Recalling that $\tilde{\mathbf{u}}$ satisfies

$$A\tilde{\mathbf{u}} = -\tilde{\mathbf{k}}(\tilde{\mathbf{u}}), \tag{4.4}$$

subtracting (4.4) from (4.3) and premultiplying by $(\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T$, we obtain

$$(\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T A(\bar{\mathbf{u}}-\tilde{\mathbf{u}}) = (\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T(\tilde{\mathbf{k}}(\tilde{\mathbf{u}})-\tilde{\mathbf{k}}(\bar{\mathbf{u}})) + (\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T(\boldsymbol{\varepsilon}+\tilde{\mathbf{k}}(\bar{\mathbf{u}})-\mathbf{k}(\bar{\mathbf{u}})). \tag{4.5}$$

Letting $\tilde{w}=\sum_{j=1}^M \tilde{u}_j w_j$, it is easy to show from the hypotheses of Theorem 3 and (2.2), that

$$\gamma \int_0^1 \int_0^1 (\bar{w}-\tilde{w})^2 \, dy \, dx \leqslant (\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T(\tilde{\mathbf{k}}(\bar{\mathbf{u}})-\tilde{\mathbf{k}}(\tilde{\mathbf{u}})).$$

Therefore, from (4.5) and (3.5), it follows from the definition of $\|\cdot\|_\gamma$, that

$$\|\bar{w}-\tilde{w}\|_\gamma^2 \leqslant (\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T \boldsymbol{\varepsilon} + (\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T(\tilde{\mathbf{k}}(\bar{\mathbf{u}})-\mathbf{k}(\bar{\mathbf{u}})). \tag{4.6}$$

The term $(\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T(\tilde{\mathbf{k}}(\bar{\mathbf{u}})-\mathbf{k}(\bar{\mathbf{u}}))$ in (4.6) is just the error in applying the quadrature scheme (3.7) on the cells of partition Δ, to the function $f(x,y,\bar{w})(\bar{w}-\tilde{w})$. From (3.8) we see that this error is bounded above by

$$\sum_{l=0}^{N_x}\sum_{k=0}^{N_y} K_1(\bar{\Delta})^{m_0+2} \max_{i=0,m_0} \left\{ \left\| \frac{\partial^{m_0} f(x,y,\bar{w})(\bar{w}-\tilde{w})}{\partial x^i \, \partial y^{m_0-i}} \right\|_{L^\infty l,k} \right\},$$

where

$$\|\cdot\|_{L^\infty l,k} = \|\cdot\|_{L^\infty([x'_l, x'_{l+1}]\times[y'_k, y'_{k+1}])}.$$

Then, using using the Markov theorem as in the proof of Theorem 1, the assumed continuity properties of the function f and the fact that derivatives of the Hermite or spline interplate $\bar{w}$ are bounded ([2] and [1]) since we are assuming that $\phi(x,y)\in C^{n_0+1}(\bar{G})$, we see that there exist constants $K_{l,k}$, such that

$$(\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T(\mathbf{k}(\bar{\mathbf{u}})-\tilde{\mathbf{k}}(\bar{\mathbf{u}})) \leqslant \sum_{l=0}^{N_x}\sum_{k=0}^{N_y} K_{l,k}(\bar{\Delta})^{m_0+2-n_0}\|\bar{w}-\tilde{w}\|_{L^\infty(G)}.$$

Therefore, by the same argument as in §3, there is a constant K_1 such that

$$|(\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T(\mathbf{k}(\bar{\mathbf{u}})-\tilde{\mathbf{k}}(\bar{\mathbf{u}}))| \leqslant K_1 (N_x+1)(N_y+1)(\bar{\Delta})^{m_0-n_0}\|\bar{w}-\tilde{w}\|_{L^2(G)}.$$

Now, since $\underline{\Delta}\max(N_x+1, N_y+1)\leqslant 1$ and $\bar{\Delta}\leqslant\delta_0\underline{\Delta}$, then

$$K_1(N_x+1)(N_y+1)(\bar{\Delta})^{m_0-n_0}\|\bar{w}-\tilde{w}\|_{L^2}\leqslant K_1\,\delta_0^2\,(\underline{\Delta})^{m_0-n_0-2}\|\bar{w}-\tilde{w}\|_{L^2(G)}.$$

As noted in the previous section, $\|\cdot\|_{L^2}\leqslant\|\cdot\|_{1,2}$ and $\|\cdot\|_{1,2}$ is equivalent to $\|\cdot\|_\gamma$, so there is a constant K_2 such that

$$|(\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T(\mathbf{k}(\bar{\mathbf{u}})-\tilde{\mathbf{k}}(\bar{\mathbf{u}}))| \leqslant K_2(\bar{\Delta})^{m_0-n_0-2}\|\bar{w}-\tilde{w}\|_\gamma. \tag{4.7}$$

Combining (4.6) and (4.7), we have

$$\|\bar{w}-\tilde{w}\|_\gamma^2\leqslant|(\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T\,\boldsymbol{\varepsilon}|+K_2(\bar{\Delta})^{m_0-n_0-2}\|\bar{w}-\tilde{w}\|_\gamma. \tag{4.8}$$

As in one dimension ([8, §4] and [7, pp. 62–66]), it can be verified that there exists a constant K_3 such that

$$|(\bar{\mathbf{u}}-\tilde{\mathbf{u}})^T\,\boldsymbol{\varepsilon}|\leqslant K_3\|\bar{w}-\hat{w}\|_\gamma\|\bar{w}-\tilde{w}\|_\gamma.$$

Therefore, (4.8) becomes

$$\|\bar{w}-\tilde{w}\|_\gamma\leqslant K_3\|\bar{w}-\hat{w}\|_\gamma+K_2(\bar{\Delta})^{m_0-n_0-2},$$

and hence

$$\begin{aligned}\|\hat{w}-\tilde{w}\|_\gamma &\leqslant \|\hat{w}-\bar{w}\|_\gamma+\|\bar{w}-\tilde{w}\|_\gamma\\ &\leqslant(1+K_3)\|\hat{w}-\bar{w}\|_\gamma+K_2(\bar{\Delta})^{m_0-n_0-2}\\ &\leqslant(1+K_3)\{\|\hat{w}-\phi\|_\gamma+\|\phi-\bar{w}\|_\gamma\}+K_2(\bar{\Delta})^{m_0-n_0-2}.\end{aligned}$$

However, $\|\hat{w}-\phi\|_\gamma\leqslant C\|\phi-\bar{w}\|_\gamma$ from (2.13), so that

$$\|\hat{w}-\tilde{w}\|_\gamma\leqslant(1+K_3)(1+C)\|\phi-\bar{w}\|_\gamma+K_2(\bar{\Delta})^{m_0-n_0-2}.$$

Then, as $\bar{w}$ is the interpolation of ϕ in S_M, the error bounds of [2] give us that $\|\phi-\bar{w}\|_\gamma\leqslant K(\bar{\Delta})^{n_0}$, from which the desired conclusion of Theorem 4 follows. Q.E.D.

This brings us to the analogue of Definition 1 for the nonlinear problem.

DEFINITION 2. Let Q be any collection of quasi-uniform partitions of G, and for each $\Delta\in Q$, let $S_M(\Delta)$ be a finite-dimensional Hermite or spline subspace of S. Let $\hat{w}_M$, the function which minimizes $F[w]$ of (2.3) over $S_M(\Delta)$, satisfy

$$\|\hat{w}_M-\phi\|_N\leqslant K(\bar{\Delta})^l\quad\text{for all}\quad\Delta\in Q, \tag{4.9}$$

where K and l are positive constants independent of Δ, ϕ is the solution of (1.1), and $\|\cdot\|_N$ is some norm on S. Then, the choice of quadrature schemes in (3.7) is *consistent* in the norm $\|\cdot\|_N$ with the bounds of (4.9) if there exists a positive constant K_0, independent of Δ, such that

$$\|\hat{w}_M-\tilde{w}_M\|_N\leqslant K_0(\bar{\Delta})^l\quad\text{for all}\quad\Delta\in Q.$$

COROLLARY. *If the hypotheses of Theorem* 4 *hold, then the quadrature scheme* (3.7) *is consistent in the norm* $\|\cdot\|_\gamma$ *with the bounds* $\|\hat{w}_M-\phi\|_\gamma \leqslant K_4(\bar{\Delta})^{n_0}$ *deduced from* [2] *and* [1], *if* $m_0 \geqslant 2n_0+2$.

When f in (1.1) is a function of u as well as x and y, approximating the integrals $k_i(\mathbf{u})$ in (2.7) by interpolating $f(x, y, \sum_{j=1}^M u_j w_j)$ by some $\tilde{f}(x, y, \sum_{j=1}^M u_j w_j)$ and then evaluating

$$\iint_G \tilde{f}\left(x, y, \sum_{j=1}^M u_j w_j\right) w_i \, dy \, dx$$

exactly, is not advantageous. The reason, as shown in [8, §4], is that the system of equations which is generated cannot be shown to be the gradient of some functional set to zero and hence, we cannot be sure that the attractive techniques available for minimizing functionals are applicable.

We should mention here that if we restrict ourselves to piecewise Hermite interpolation subspaces, then these results may be generalized from rectangles to rectangular polygons, as in [2].

5. Numerical Examples

We now discuss some examples of problems of the form (1.1) whose solutions have been approximated by the techniques discussed in the previous sections.

Consider the problem

$$\left.\begin{aligned} \Delta u(x, y) &= 6xye^x e^y (xy + x + y - 3), \quad (x, y) \in G, \\ u(x, y) &= 0, \quad (x, y) \in \partial G, \end{aligned}\right\} \tag{5.1}$$

where G is the open unit square $(0, 1) \times (0, 1)$. The quantity Λ_1 of (2.1) in this case is $2\pi^2$, as noted in [6, p. 249]. For this problem, γ in (2.2) can be chosen to be zero. Since $6xye^xe^y(xy+x+y-3)$ is independent of u, then (2.9) holds with $K=0$. The unique solution to (5.1) is

$$u(x, y) = 3e^x e^y (x - x^2)(y - y^2).$$

The first way that the solution of (5.1) was approximated was by minimizing the functional

$$F[w] = \iint_G \left\{ \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + 6xye^x e^y (xy + x + y - 3)\, w(x, y) \right\} dy\, dy \tag{5.2}$$

over the piecewise linear Hermite subspace $H_0^1(\pi_N)$ of S, described in [2, §4], where π_N is the uniform mesh on G with mesh size $h_N = 1/(N+1)$. The dimension of $H_0^1(\pi_N)$ is N^2. Denoting the basis functions of $H^1(\pi_N)$ as $\{w_i(x, y)\}_{i=1}^{N^2}$, let $\sum_{j=1}^{N^2} \hat{u}_j w_j(x, y) = \hat{w}_N(x, y)$ be the unique minimum of $F[w]$ in (5.2) over $H_0^1(\pi_N)$.

By [2, Theorem 16] and the fact that $\|\cdot\|_{1,2}$ is equivalent to $\|\cdot\|_D$ we know that

$$\|\phi - \hat{w}_N\|_D \leqslant K h_N, \quad N \geqslant 1, \tag{5.3}$$

where K is a constant independent of N. The two-dimensional four-point Gaussian quadrature scheme generated as in (3.7) from the one-dimensional two-point Gaussian quadrature scheme with weight function unity, was used to approximate the integrals of the form (3.1). If we denote by $\tilde{w}_N(x, y)$ the approximation resulting from the minimization of $F[w]$ using the above mentioned quadrature scheme, then by Theorem 1, with $m_0=4$ and $n_0=1$, we can easily verify that there is a constant K', independent of N, such that

$$\|\tilde{w}_N - \hat{w}_N\|_D \leqslant K' h_N.$$

Therefore, by Definition 1, the two-dimensional four-point Gaussian quadrature scheme is *consistent* in the norm $\|\cdot\|_D$ with the bounds of (5.3). Consequently, we have

$$\|\phi - \tilde{w}_N\|_D \leqslant K h_N, \quad N \geqslant 1,$$

where K is a constant independent of N. The numerical results are given in Table 1. The quantity α in this table is

$$\alpha \equiv \log\left(\frac{\|\phi - \tilde{w}_{n_1}\|_{L^\infty(G)}}{\|\phi - \tilde{w}_{n_2}\|_{L^\infty(G)}}\right) \Big/ \log\left(\frac{h_{n_1}}{h_{n_2}}\right), \tag{5.4}$$

which is an estimate of the rate of convergence of our approximations, defined in terms of successive values of the mesh spacing h. Equation (5.4) is derived from the fact that as $h_N \to 0$, we have

$$\|\phi - \tilde{w}_N\|_{L^\infty(G)} \sim K(h_N)^\alpha$$

for some constants α and K which are independent of h_N. Then, for two successive values of h, $h_{n_1} > h_{n_2}$, we have asymptotically

$$\frac{\|\phi - \tilde{w}_{n_1}\|_{L^\infty(G)}}{\|\phi - \tilde{w}_{n_2}\|_{L^\infty(G)}} \sim \left(\frac{h_{n_1}}{h_{n_2}}\right)^\alpha, \tag{5.5}$$

and (5.4) follows from (5.5). We see from Table 1 that the accuracy seems to be $O(h_N^2)$ in the norm $\|\cdot\|_{L^\infty(G)}$.

Table 1

Subspace $H_0^1(\pi_N)$

h_N	dimension of $H_0^1(\pi_N)$	$\|\phi - \tilde{w}_N\|_{L^\infty(G)}$	α
1/7	36	$3.10 \cdot 10^{-2}$	–
1/8	49	$2.43 \cdot 10^{-2}$	1.84
1/9	64	$1.96 \cdot 10^{-2}$	1.85
1/10	81	$1.60 \cdot 10^{-2}$	1.94
1/11	100	$1.33 \cdot 10^{-2}$	1.97

The solution of problem (5.1) was also approximated by minimizing $F[w]$ in (5.2) over the piecewise cubic Hermite subspace $H_0^2(\pi_N)$ of S ([2, 4]), where π_N is the same as above. The dimension of $H_0^2(\pi_N)$ is $4N^2+8N+4$, and if we denote by $\hat{w}_N(x, y)$ the element of $H_0^2(\pi_N)$ which uniquely minimizes $F[w]$ over $H_0^2(\pi_N)$, then, again using [2, Theorem 16], we can deduce that

$$\|\phi - \hat{w}_N\|_D \leqslant K h_N^3, \quad N \geqslant 1, \tag{5.6}$$

for some constant K, independent of N. For the piecewise cubic polynomial subspace $H_0^2(\pi_N)$, the two-dimensional sixteen-point Gaussian quadrature scheme derived from the one-dimensional fourpoint Gaussian quadrature scheme with weight function unity, was used to approximate the integrals in the system generated by minimizing $F[w]$. For this quadrature scheme, $m_0=8$ and by Theorem 1, with $n_0=3$, we can verify that

$$\|\hat{w}_N - \tilde{w}_N\|_D \leqslant K h_N^3, \quad N \geqslant 1,$$

where $\tilde{w}_N$ is the minimum of $F[w]$ over $H_0^2(\pi_N)$ using the quadrature scheme. Hence, by Definition 1, this Gaussian quadrature scheme is *consistent* in the norm $\|\cdot\|_D$ with (5.6) and we have

$$\|\phi - \tilde{w}_N\|_D \leqslant K h_N^3, \quad N \geqslant 1,$$

where K is a constant independent of N. The numerical results are given in Table 2.

Table 2

Subspace $H_0^2(\pi_N)$

h_N	dimension of $H_0^2(\pi_N)$	$\|\phi - \tilde{w}_N\|_{L^\infty(G)}$	α
1/3	36	$9.11 \cdot 10^{-4}$	–
1/4	64	$3.15 \cdot 10^{-4}$	3.70
1/5	100	$1.32 \cdot 10^{-4}$	3.92
1/6	144	$7.06 \cdot 10^{-5}$	3.93

The α values in this table indicate that for the subspace $H_0^2(\pi_N)$ the accuracy in the norm $\|\cdot\|_{L^\infty(G)}$ is probably $O(h_N^4)$.

The final subspace over which we minimized $F[w]$ in (5.2) was the piecewise-cubic-polynomial spline ([1]) subspace $Sp_0^2(\pi_N)$ of S, where π_N is once again the uniform mesh on the unit square. From [6], we can easily deduce that if $\hat{w}_N(x, y)$ is the minimum of $F[w]$ over $Sp_0^2(\pi_N)$, then

$$\|\phi - \hat{w}_N\|_D \leqslant K h_N^3, \quad N \geqslant 1. \tag{5.7}$$

The dimension of $Sp_0^2(\pi_N)$ is N^2+4N+4 and the two-dimensional, sixteen-point Gaussian quadrature scheme was used to approximate the integrals involved in minimizing $F[w]$. Letting $\tilde{w}_N$ denote the approximations resulting from minimizing $F[w]$ over $Sp_0^2(\pi_N)$ using the quadrature scheme, by Theorem 1 we see that

$$\|\hat{w}_N - \tilde{w}_N\|_D \leqslant K h_N^3, \quad N \geqslant 1,$$

and hence by Definition 1, the quadrature scheme is *consistent* in $\|\cdot\|_D$ with (5.7). Consequently

$$\|\phi - \tilde{w}_N\|_D \leqslant K h_N^3, \quad N \geqslant 1,$$

where K is a constant independent of N. The numerical results are given in Table 3. They indicate that the accuracy in the norm $\|\cdot\|_{L^\infty(G)}$ is $O(h_N^4)$.

Table 3

Subspace $Sp_0^2(\pi_N)$

h_N	dimension of $Sp_0^2(\pi_N)$	$\|\phi - \tilde{w}_N\|_{L^\infty(G)}$	α
1/3	16	$1.08\cdot 10^{-3}$	–
1/4	25	$3.57\cdot 10^{-4}$	3.84
1/5	36	$1.53\cdot 10^{-4}$	3.84
1/6	49	$7.66\cdot 10^{-5}$	3.82
1/7	64	$4.19\cdot 10^{-5}$	3.94

As our second and final example, we consider

$$\left.\begin{aligned} \Delta u(x, y) = (u)^3 &+ (-2 + (1-2x)^2)\,(e^{y(1-y)} - 1 + u) \\ &+ (-2 + (1-2y)^2)\,(e^{x(1-x)} - 1 + u) \\ &- (e^{x(1-x)} - 1)^3\,(e^{y(1-y)} - 1)^3, \quad (x, y) \in G, \end{aligned}\right\} \tag{5.8}$$

$$u(x, y) = 0, \quad (x, y) \in \partial G, \tag{5.9}$$

where G is the open unit square. As in problem (5.1), Λ_1 in (2.1) is $2\pi^2$ and γ in (2.2) can be set equal to -4. Denoting the right side of (5.8) by $f(x, y, u)$, $\partial f/\partial u$ is in $C^0(\bar{G} \times R)$, but it is unbounded as $|u| \to +\infty$. We now show that (2.11) is satisfied. We first estimate the quantity $-\varrho^{-1}$, where ϱ is defined as follows. Let $\Psi(x, y)$ be the (unique) solution of

$$\left.\begin{aligned} \Delta u(x, y) &= 1, \quad (x, y) \in G, \\ u(x, y) &= 0, \quad (x, y) \in \partial G, \end{aligned}\right\} \tag{5.10}$$

where G is the open unit unit square. Then

$$\varrho = \sup_{(x, y) \in G} |\Psi(x, y)| > 0.$$

To estimate ϱ, we use the following maximum principle [14, p. 56]. If G' is a domain in the (x, y) plane with boundary $\partial G'$ and $\phi(x, y)$ is a solution to the problem

$$\Delta u(x, y) = -F(x, y), \quad (x, y) \in G'$$
$$u(x, y) = f(x, y), \quad (x, y) \in \partial G',$$

where $F(x, y) < 0$ for $(x, y) \in G'$, then

$$\max_{(x,y)\in G'} |\phi(x, y)| \leqslant \max_{\partial G'} |f| + \tfrac{1}{4}R^2 \max_{G'} |F|$$

where R is the radius of a circle containing G'. For our problem (5.10), $f=0$, $F \equiv -1$, and we may let $R = \sqrt{2}/2$. Therefore,

$$\varrho = \sup_{(x,y)\in G} |\Psi(x, y)| \leqslant \frac{1}{4}\left(\frac{2}{2}\right)^2 = \frac{1}{8},$$

so that $-\varrho^{-1} \leqslant -8$. Since $\gamma = -4$ and $-\varrho^{-1} \leqslant -8$, then (2.11) is satisfied for the problem (5.8)–(5.9). The unique solution to (5.8)–(5.9) is

$$u(x, y) = (e^{x(1-x)} - 1)(e^{y(1-y)} - 1).$$

In approximating the solution of (5.8)–(5.9) by the variational method, the function which we minimize is

$$F[w] = \iint_G \left\{ \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \int_0^w f(x, y, \eta)\, d\eta \right\} dy\, dx, \tag{5.11}$$

where $f(x, y, u)$ represents the right side of (5.8).

The first subspace over which we minimized $F[w]$ in (5.11) was the piecewise linear Hermite subspace $H_0^1(\pi_N)$, where π_N is again the uniform mesh on the unit square with mesh size $h_N = 1/(N+1)$. If we denote by $\hat{w}_N(x, y)$ the minimum of $F[w]$ over $H_0^1(\pi_N)$, then we can deduce from [2, Theorem 16] that there is a constant K such that

$$\|u - \hat{w}_N\|_\gamma \leqslant K h_N, \quad N \geqslant 1. \tag{5.12}$$

By Theorem 3 and Theorem 4 we see that when the two-dimensional, four-point Gaussian quadrature scheme is used to approximate the integrals arising in the minimization of $F[w]$, the resulting approximation $\tilde{w}_N(x, y)$ satisfies

$$\|\hat{w}_N - \tilde{w}_N\|_\gamma \leqslant K h_N, \quad N \geqslant 1.$$

Hence, by Definition 2, this quadrature scheme is *consistent* in the norm $\|\cdot\|_\gamma$ with bounds (5.12). The numerical results are given in Table 4.

The subspaces $H_0^2(\pi_N)$ of piecewise cubic Hermite polynomials and $Sp_0^2(\pi_N)$ of piecewise cubic spline polynomials were also used to approximate the solution of (5.8)–(5.9). By [2, Theorem 16] and [1], these subspaces yield, through the minimization of $F[w]$, approximations which are $O(h_N)$ in the norm $\|\cdot\|_\gamma$. Next, by Theorems

Table 4

Subspace $H_0^1(\pi_N)$

h_N	dimension of $H_0^1(\pi_N)$	$\|u-\tilde{w}_N\|_{L^\infty(G)}$
1/4	9	$7.22\cdot 10^{-3}$
1/5	16	$5.22\cdot 10^{-3}$
1/6	25	$3.46\cdot 10^{-3}$
1/7	36	$2.69\cdot 10^{-3}$
1/8	49	$2.01\cdot 10^{-3}$

3 and 4, we can verify that the two-dimensional, sixteen-point Gaussian quadrature scheme is consistent by Definition 2 in the norm $\|\cdot\|_\gamma$ with these $O(h_N^3)$ bounds. Letting $\tilde{w}_N(x, y)$ denote the approximations generated by minimizing $F[w]$ over the Hermite and spline subspaces and using the above-mentioned quadrature scheme, the numerical results are given in Tables 5 and 6.

Table 5

Subspace $H_0^2(\pi_N)$

h_N	dimension of $H_0^2(\pi_N)$	$\|u-\tilde{w}_N\|_{L^\infty(G)}$
1/2	16	$4.55\cdot 10^{-4}$
1/3	36	$1.05\cdot 10^{-4}$
1/4	64	$4.06\cdot 10^{-5}$
1/5	100	$8.67\cdot 10^{-6}$
1/6	144	$5.28\cdot 10^{-6}$

Table 6

Subspace $Sp_0^2(\pi_N)$

h_N	dimension of $Sp_0^2(\pi_N)$	$\|u-\tilde{w}_N\|_{L^\infty(G)}$
1/2	9	$4.44\cdot 10^{-2}$
1/3	16	$2.13\cdot 10^{-4}$
1/4	25	$4.92\cdot 10^{-5}$
1/5	36	$1.96\cdot 10^{-5}$
1/6	49	$9.76\cdot 10^{-6}$

REFERENCES

[1] BIRKHOFF, G. and DE BOOR, C. R., *Piecewise Polynomial Interpolation and Approximation*, in *Approximation of Functions* (Editor: H. L. Garabedian), *Proc. Sympos. General Motors Res. Lab., 1964* (Elsevier Publ., New York, N.Y. 1965), pp. 164–190.

[2] BIRKHOFF, G., SCHULTZ, M. H. and VARGA, R. S., *Piecewise Hermite Interpolation in One and Two Variables with Applications to Partial Differential Equations*, Numer. Math. *11*, 232–256 (1968).

[3] Ciarlet, P. G., *Variational Methods for Non-Linear Boundary-Value Problems* (Doctoral Thesis, Case Institute of Technology, 1966 (103 pp.)).
[4] Ciarlet, P. G., Schultz, M. H. and Varga, R. S., *Numerical Methods of High-Order Accuracy for Nonlinear Boundary Value Problems. I. One Dimensional Problem*, Numer. Math. *9*, 394–430 (1967).
[5] Ciarlet, P. G., Schultz, M. H. and Varga, R. S., *Numerical Methods of High-Order Accuracy for Nonlinear Boundary Value Problems. V. Monotone Operator Theory*, Numer. Math. *13*, 51–77 (1969).
[6] Courant, R. and Hilbert, D., *Methods of Mathematical Physics, Vol. 1* Interscience Publ., New York, N.Y. 1953).
[7] Herbold, R. J., *Consistent Quadrature Schemes for the Numerical Solution of Boundary Value Problems by Variational Techniques* (Doctoral Thesis, Case Western Reserve University, 1968 (189 pp.)).
[8] Herbold, R. J., Schultz, M. H. and Varga, R. S., *The Effect of Quadrature Errors in the Numerical Solution of Boundary Value Problems by Variational Techniques*, Aequationes Math. *3*, 247–270 (1969).
[9] Hille, E., Szegö, G. and Tamarkin, J., *On Some Generalizations of a Theorem of A. Markoff*, Duke Math. J. *3*, 729–739 (1937).
[10] Levinson, N., *Dirichlet Problem for* $\Delta u = f(P, u)$, J. Math. Mech. *12*, 567–575 (1963).
[11] Sard, A., *Linear Approximation* (Amer. Math. Soc., Providence, R.I. 1963 [Mathematical Survey, Vol. 9]).
[12] Todd, J. (Editor), *A Survey of Numerical Analysis* (McGraw Hill, New York-Toronto-London 1962).
[13] Varga, R. S., *Matrix Iterative Analysis* (Prentice Hall, Englewood Cliffs, N.J. 1962).
[14] Weinberger, H. F., *A First Course in Partial Differential Equations with Complex Variables and Transform Methods* (Blaisdell Publ., New York, N.Y. 1965).
[15] Yosida, K., *Functional Analysis* (Springer Verlag, Berlin 1965).

Procter and Gamble and
Kent State University