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A NOTE ON LACUNARY INTERPOLATION BY SPLINES*

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Dedicated to Professor I. J. Schoenberg on the occasion of his seventieth birthday, April 21, 1973 (old) and May 4, 1973 (new).

Abstract. In the previous paper by A. Meir and A. Sharma, error bounds for lacunary interpolation of certain functions by deficient quintic splines are developed. In this note, we extend their results to a wider class of functions and indicate that the extended results are best possible. In addition, a stability result for such interpolation is also presented.

1. Introduction. In the preceding paper [1] by A. Meir and A. Sharma, error bounds have been developed for lacunary interpolation of certain functions by deficient quintic splines. More precisely, let $S_{n,5}^{(3)}$ be the class of quintic splines $s(x)$ such that

(i) $s \in C^3[0, 1]$,

(ii) $s \in \pi_5$ on each $[v/n, (v + 1)/n]$, $0 \leqq v \leqq n - 1$,

and given $f \in C^3[0, 1]$, let s_n be the unique element (cf. [1, Theorem 1]) (for n odd) in $S_{n,5}^{(3)}$ which interpolates f in the sense that

(i) $(f - s_n)(v/n) = 0$, $0 \leqq v \leqq n$,

(ii) $D^2(f - s_n)(v/n) = 0$, $0 \leqq v \leqq n$,

(iii) $D^3(f - s_n)(0) = D^3(f - s_n)(1) = 0$.

We call this interpolant s_n the *Meir–Sharma interpolant* of f. If $\omega(f; \delta)$ denotes the usual modulus of continuity of f, and if $\| \cdot \|_\infty \equiv \| \cdot \|_{L_\infty[0,1]}$, then Meir and Sharma [1] have established the following.

THEOREM A. *Let $f \in C^4[0, 1]$, let n be an odd integer, and let s_n be its unique Meir–Sharma interpolant in $S_{n,5}^{(3)}$. Then*

$$\|D^j(f - s_n)\|_\infty \leqq 75n^{j-3}\omega(D^4f; 1/n) + 8n^{j-4}\|D^4f\|_\infty, \qquad 0 \leqq j \leqq 4. \tag{1}$$

The purpose of this note is to extend their results to a wider class of functions, to exhibit a stability result for their interpolation process, and to describe evidence indicating that the results to be established are best possible.

2. Main result. We begin with the preliminary result of the following lemma.

LEMMA 1. *Let $f \in C^6[0, 1]$, let n be an odd integer, and let s_n be its unique Meir–Sharma interpolant in $S_{n,5}^{(3)}$. Then*

$$\|D^j(f - s_n)\|_\infty \leqq 20n^{j-5}\|D^6f\|_\infty, \qquad 0 \leqq j \leqq 4. \tag{2}$$

Proof. Let $\tilde{s}$ be any piecewise quintic function in $C^4[0, 1]$ such that $D^4\tilde{s}(x)$ is the continuous piecewise linear interpolation of $D^4f(x)$ in the points v/n, $0 \leqq v \leqq n$. Since the Meir–Sharma interpolant of $\tilde{s}$ in $S_{n,5}^{(3)}$ is evidently $\tilde{s}$ (cf. [1, Theorem 1]), then the Meir–Sharma interpolant of $f - \tilde{s}$ is $s_n - \tilde{s}$. Hence, from

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Theorem A for any integer j with $0 \leqq j \leqq 4$,

$$\begin{aligned} \|D^j(f - s_n)\|_\infty &= \|D^j\{(f - \tilde{s}) - (s_n - \tilde{s})\}\|_\infty \\ &\leqq 75n^{j-3}\omega(D^4(f - \tilde{s}); 1/n) + 8n^{j-4}\|D^4(f - \tilde{s})\|_\infty . \end{aligned} \tag{3}$$

But as $D^4\tilde{s}$ is the continuous piecewise linear interpolant of $D^4 f$, then it is well known that

$$\|D^4(f - \tilde{s})\|_\infty \leqq \frac{1}{8n^2}\|D^6 f\|_\infty .$$

Similarly,

$$\omega\left(D^4(f - \tilde{s}); \frac{1}{n}\right) \leqq \frac{1}{4n^2}\|D^6 f\|_\infty .$$

Then, substituting in (3) gives the desired result. Q.E.D.

We remark that this useful trick in the above lemma, of using $\tilde{s}$ to deduce interpolation error bounds for smoother functions from interpolation results for less smooth functions, can be found in Swartz [2, p. 19], where it is attributed to C. deBoor.

We next state a special case of a result from Swartz and Varga [3, Lemma 3.2].

LEMMA 2. *Let $f \in C^k[0, 1]$, where $0 \leqq k < 6$, and let n be any positive integer. Then, there exists a unique Hermite spline interpolant g of f, with $g \in C^6[0, 1]$ and with $g \in \pi_{13}$ on each $[\nu/n, (\nu + 1)/n]$, such that*

(i) $D^j(f - g)(\nu/n) = 0, \quad 0 \leqq j \leqq k, \quad 0 \leqq \nu \leqq n,$

(ii) $D^j g(\nu/n) = 0, \quad k < j \leqq 6, \quad 0 \leqq \nu \leqq n.$

Moreover, there exists a constant K, independent of f and n, such that

$$Kn^{j-k}\omega\left(D^k f; \frac{1}{n}\right) \geqq \begin{cases} \|D^j(f - g)\|_\infty, & 0 \leqq j \leqq k, \\ \|D^j g\|_\infty, & k < j \leqq 6. \end{cases} \tag{4}$$

Given $f \in C^k[0, 1]$ where $0 \leqq k < 6$, and given n an odd positive integer, let g be its unique Hermite spline interpolant, in the sense of Lemma 2. Since g admits a unique Meir–Sharma interpolant, say $\tilde{s}_n$, we call $\tilde{s}_n$ the *generalized* Meir–Sharma interpolate of f in $S^{(3)}_{n,5}$. Note, of course, that if $f \in C^k[0, 1]$ with $k \geqq 3$, then the generalized Meir–Sharma interpolant of f in $S^{(3)}_{n,5}$ reduces identically to the Meir–Sharma interpolant of f in $S^{(3)}_{n,5}$.

This brings us to our main result.

THEOREM 1. *Let $f \in C^k[0, 1]$, where $0 \leqq k < 6$, let n be an odd positive integer, and let $\tilde{s}_n$ be the unique generalized Meir–Sharma interpolant of f in $S^{(3)}_{n,5}$. Then, their exists a constant K, independent of f and n, such that*

$$Kn^{1+j-k}\omega\left(D^k f; \frac{1}{n}\right) \geqq \|D^j(f - \tilde{s}_n)\|_\infty, \qquad 0 \leqq j \leqq \min(k, 4). \tag{5}$$

Proof. If g is the unique interpolant of f of Lemma 2, and $\tilde{s}_n$ is the unique generalized Meir–Sharma interpolant of f in $S^{(3)}_{n,5}$, then, by the triangle inequality,

$$\|D^j(f - \tilde{s}_n)\|_\infty \leqq \|D^j(f - g)\|_\infty + \|D^j(g - \tilde{s}_n)\|_\infty .$$

The first term on the right side of the above inequality can be bounded above from Lemma 2 by $Kn^{j-k}\omega(D^k f; 1/n)$ for $0 \leqq j \leqq k$. The second term on the right can be bounded above from Lemma 1 by

$$\|D^j(g - \tilde{s}_n)\|_\infty \leqq 20n^{j-5}\|D^6 g\|_\infty, \qquad 0 \leqq j \leqq 4.$$

But, again by Lemma 2, $\|D^6 g\|_\infty \leqq Kn^{6-k}\omega(D^k f; 1/n)$, so that

$$\|D^j(g - \tilde{s}_n)\|_\infty \leqq K' n^{1+j-k}\omega\left(D^k f; \frac{1}{n}\right), \qquad 0 \leqq j \leqq 4.$$

Combining these inequalities establishes (5) for $0 \leqq j \leqq \min(4, k)$. Q.E.D.

Several comments are in order. First, if $f \in C^4[0, 1]$, then we see that the special case $k = 4$ of Theorem 1 effectively reduces to the result of Theorem A above due to Meir and Sharma [1]. Second, as it is easy to see for $f \in C^6[0, 1]$ that the error bounds of (2) follow directly from the error bounds of (5), it is natural to ask if the exponent of n in (2), namely $j - 5$, is best possible, particularly since other quintic piecewise polynomials, such as quintic splines, have a known interpolation error bound like that of (2), but with the exponent of n in (2) replaced by $j - 6$. Based on a computer test of the cases $f(x) = x^6$ and $f(x) = x^7$ using $n = 11, 21, 41$, and 81, we have observed numerically that

$$\|f - \tilde{s}_n\|_\infty = O(n^{-5}).$$

But, as the corresponding inequality of (2) in these cases also gives the same bound, we thus believe the exponents of n in both (2) and (5) are *best possible*. Finally, Meir and Sharma also consider a somewhat different type of lacunary interpolation (cf. [1, Theorem 3]) in $S_{n,5}^{(3)}$ for which the interpolation error bound of (1) of Theorem A is also valid. We remark that the result of Theorem 1 applies *without change* to this interpolation as well.

We conclude this note with a stability result, along the lines of those in [3].

THEOREM 2. *Let $f \in C^k[0, 1]$, where $0 \leqq k < 6$, let n be an odd positive integer, and let $\hat{s}_n$ be the unique Meir–Sharma interpolant of the following data*:

(i) $\hat{s}_n(\nu/n) = \alpha_{\nu,0}, \quad 0 \leqq \nu \leqq n,$

(ii) $D^2\hat{s}_n(\nu/n) = \alpha_{\nu,2}, \quad 0 \leqq \nu \leqq n,$

(iii) $D^3\hat{s}_n(0) = \alpha_{0,3}, \quad D^3\hat{s}_n(1) = \alpha_{n,3},$

where we suppose that there exists a function $F(f, n)$ such that

(a) $$n^{-k}F(f, n) \geqq \max_{0 \leqq \nu \leqq n} |f(\nu/n) - \alpha_{\nu,0}|,$$

(b) $$n^{2-k}F(f, n) \geqq \begin{cases} \max\limits_{0 \leqq \nu \leqq n} |D^2 f(\nu/n) - \alpha_{\nu,2}| & \text{if } k \geqq 2, \\ \max\limits_{0 \leqq \nu \leqq n} |\alpha_{\nu,2}| & \text{if } k < 2, \end{cases}$$

(c) $$n^{3-k}F(f, n) \geqq \begin{cases} \max\,[|D^3 f(0) - \alpha_{0,3}|, |D^3 f(1) - \alpha_{n,3}|] & \text{if } k \geqq 3, \\ \max\,[|\alpha_{0,3}|, |\alpha_{n,3}|] & \text{if } k < 3. \end{cases}$$

Then there is a constant K, independent of f, F, and n, such that

$$(6) \qquad Kn^{1+j-k}[\omega(D^k f; 1/n) + F(f, n)] \geqq \|D^j(f - \hat{s}_n)\|_\infty, \qquad 0 \leqq j \leqq \min(k, 4).$$

Remarks. Should these bounds of (6) be sharp, they would indicate that the Meir–Sharma interpolant is *less* stable than many local piecewise quintic approximants, such as quintic spline interpolants of continuity class $C^4[0, 1]$. In this latter case, if for example $f \in C^6[0, 1]$ and if s is its unique quintic spline interpolant in $C^4[0, 1]$, defined by $(f - s)(\nu/n) = 0$, $0 \leqq \nu \leqq n$, $D^j(f - s)(0) = D^j(f - s)(1) = 0, j = 1, 2$, then

$$\|D^j(f - s)\|_\infty \leqq Kn^{j-6}\|D^6 f\|_\infty, \qquad 0 \leqq j \leqq 5.$$

But if perturbed data are similarly interpolated, i.e., $\tilde{s}$ is the unique quintic spline interpolant such that $\tilde{s}(\nu/n) = \alpha_{\nu,0}, 0 \leqq \nu \leqq n, D^j\tilde{s}(0) = \alpha_{0,j}, D^j\tilde{s}(1) = \alpha_{n,j}, j = 1, 2$, where $|\alpha_{\nu,j} - D^j f(\nu/n)| \leqq Kn^{j-6}\|D^6 f\|_\infty$, then the above interpolation error bounds hold *also* for $\tilde{s}$ (cf. [3]). Loosely speaking, in this latter case one may perturb the data by the order of associated interpolation error, without affecting the order of magnitudes of the resulting global error bounds. This seems no longer to be true for the Meir–Sharma interpolant, for, according to Theorem 2, one similarly needs $O(n^{n-6})$ accuracy for the data for $D^j f(\nu/n)$, to obtain global approximation accuracies of order $O(n^{j-5})$.

We also note that Theorem 2 implies (cf. [3]) that, independent of the continuity class for f, we may supply Meir–Sharma data (for example, $\alpha_{\nu,2}$) via appropriate derivatives (for example, $D^2Q(\nu/n)$) of a quintic Lagrange polynomial interpolant Q of f in any six contiguous points within $O(1/n)$ of the location at which the data is required, without affecting global approximation error bounds.

Proof of Theorem 2. We follow here basically the proof of Theorem 5.1 of [3]. Let $\tilde{s}_n$ be the generalized Meir–Sharma interpolant of f in $S_{n,5}^{(3)}$ (Theorem 1), and let $\hat{s}_n$ be the unique interpolant in $S_{n,5}^{(3)}$ of the given approximate data of Theorem 2.

Define a Hermite spline function $g(x)$, with $g \in C^6[0, 1]$ and with $g \in \pi_{13}$ on each $[\nu/n, \nu/(n+1)]$, such that

$$g(\nu/n) = (f - \hat{s}_n)(\nu/n), \qquad 0 \leqq \nu \leqq n,$$

$$D^2 g(\nu/n) = \begin{Bmatrix} D^2(f - \hat{s}_n)(\nu/n) & \text{if} \quad k \geqq 2 \\ -D^2\hat{s}_n(\nu/n) & \text{if} \quad k < 2 \end{Bmatrix}, \qquad 0 \leqq \nu \leqq n,$$

$$D^3 g(0) = \begin{Bmatrix} D^3(f - \hat{s}_n)(0) & \text{if} \quad k \geqq 3 \\ -D^3\hat{s}_n(0) & \text{if} \quad k < 3 \end{Bmatrix}, \quad D^3 g(1) = \begin{Bmatrix} D^3(f - \hat{s}_n)(1) & \text{if} \quad k \geqq 3 \\ -D^3\hat{s}_n(1) & \text{if} \quad k < 3 \end{Bmatrix},$$

$$D^j g(\nu/n) = 0 \quad \text{otherwise}, \quad 0 \leqq \nu \leqq n, \quad j = 1, 3, 4, 5, 6.$$

It is readily verified that the Meir–Sharma interpolant of g in $S_{n,5}^{(3)}$ is $\tilde{s}_n - \hat{s}_n$.

Now, for $0 \leqq j \leqq \min(k, 4)$, we have from the triangle inequality

$$\|D^j(f - \hat{s}_n)\|_\infty \leqq \|D^j(f - \tilde{s}_n)\|_\infty + \|D^j g\|_\infty + \|D^j\{g - (\tilde{s}_n - \hat{s}_n)\}\|_\infty. \tag{7}$$

We note that, by Theorem 1, the first term on the right side of (7) is bounded above by $Kn^{1+j-k}\omega(D^k f; 1/n)$ for $0 \leqq j \leqq \min(k, 4)$. Next, from the hypotheses of Theorem 2 and from Lemma 4.3 of [3], the second term on the right of (7) can be bounded above by

$$\|D^j g\|_\infty \leqq Kn^{j-k}F(f, n), \qquad 0 \leqq j \leqq 6. \tag{8}$$

Finally, since $\tilde{s}_n - \hat{s}_n$ is the Meir–Sharma interpolant of g in $S_{n,5}^{(3)}$, then the third term on the right of (7) can be bounded above from the case $k = 5$ of Theorem 1 by

$$\|D^j\{g - (\tilde{s}_n - \hat{s}_n)\}\|_\infty \leqq Kn^{j-4}\omega(D^5g; 1/n), \qquad 0 \leqq j \leqq 4. \tag{9}$$

But, using the case $j = 5$ of (8), then $\omega(D^5g; 1/n) \leqq 2\|D^2g\|_\infty \leqq Kn^{5-k}F(f, n)$. Combining this inequality with that of (9) then yields

$$\|D^j\{g - (\tilde{s}_n - \hat{s}_n)\}\|_\infty \leqq Kn^{1+j-k}F(f, n). \qquad \text{Q.E.D.}$$

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