Numer. Math. 20, 252—256 (1973)
© by Springer-Verlag 1973

On Minimal Gerschgorin Sets for Families of Norms*

Richard S. Varga and Bernard W. Levinger

Received June 26, 1972

* Research supported in part by the Atomic Energy Commission under Grant AT(11-1)-2075.

Summary. In this note, the minimal Gerschgorin set G is defined for a matrix A, relative to a matrix D and a family $\mathscr{F}$ of norms. This minimal Gerschgorin set is shown to be an inclusion region for the eigenvalues of a related collection $\hat{\Omega}$ of matrices, i.e.,

$$\sigma(\hat{\Omega}) \subseteq G.$$

The main result is a necessary and sufficient condition for equality to hold in the above inclusion. In addition, examples are given, one for which equality does not hold in the above inclusion.

1. Introduction

Let $\mathscr{F}$ be a finite or infinite non-empty family of norms on $\mathbb{C}^n$. If $[\mathbb{C}^n]$ denotes the set of all $n \times n$ complex matrices, let A and D be two arbitrary but fixed matrices in $[\mathbb{C}^n]$. If λ is an eigenvalue of A, written $\lambda \in \sigma(A)$, there exists an $\boldsymbol{x} \neq \boldsymbol{0}$ in $\mathbb{C}^n$ with $A\boldsymbol{x} = \lambda \boldsymbol{x}$, or equivalently, $(A - D)\boldsymbol{x} = (\lambda - D)\boldsymbol{x}$. For $\lambda \notin \sigma(D)$, $(\lambda - D)$ is invertible, and hence

$$(\lambda - D)^{-1}(A - D)\boldsymbol{x} = \boldsymbol{x}. \tag{1}$$

Using the standard notation $\varrho(S) \equiv \max\{|\lambda| : \det(\lambda I - S) = 0\}$ for the spectral radius of any $S \in [\mathbb{C}^n]$ and $\|S\|_\phi \equiv \sup\{\phi(Sy) : \phi(y) = 1\}$ for the induced operator norm of S for a norm ϕ on $\mathbb{C}^n$, then (1) implies, as is well-known, that

$$1 \leqq \varrho\{(\lambda - D)^{-1}(A - D)\} \leqq \|(\lambda - D)^{-1}(A - D)\|_\phi. \tag{2}$$

As the last inequality holds for any $\phi \in \mathscr{F}$, then

$$1 \leqq \varrho\{(\lambda - D)^{-1}(A - D)\} \leqq \inf\{\|(\lambda - D)^{-1}(A - D)\|_\phi : \phi \in \mathscr{F}\}. \tag{3}$$

Defining the set $G = G_{\mathscr{F}}(A; D)$ in the complex plane $\mathbb{C}$ by

$$G \equiv \sigma(D) \cup \{z \in \mathbb{C} : z \notin \sigma(D) \text{ and } \|(z - D)^{-1}(A - D)\|_\phi \geqq 1 \ \forall \phi \in \mathscr{F}\}, \tag{4}$$

the inequalities of (3) give

$$\sigma(A) \subseteq G. \tag{5}$$

The set G is called the *minimal Gerschgorin set* for A, relative to $\mathscr{F}$ and D, and (5) establishes that G is an inclusion region for the eigenvalues of A. Note that since the set

$$\sigma(D) \cup \{z \in \mathbb{C} : z \notin \sigma(D) \text{ and } \|(z - D)^{-1}(A - D)\|_\phi \geqq 1\}$$

is a closed and bounded set in $\mathbb{C}$ for each $\phi \in \mathscr{F}$, then so is their intersection, G.

Of course, many matrices $B \in [\mathbb{C}^n]$ have their spectra in G. In particular, let $\hat{\Omega} = \hat{\Omega}_{\mathscr{F}}(A; D)$ be the subset of $[\mathbb{C}^n]$ of all matrices B for which

$$\|(z-D)^{-1}(B-D)\|_\phi \leqq \|(z-D)^{-1}(A-D)\|_\phi \ \forall \phi \in \mathscr{F}, \quad \forall z \notin \sigma(D). \tag{6}$$

Clearly, $\hat{\Omega}$ is nonempty since it trivially contains A and D. For any $B \in \hat{\Omega}$, consider any $\lambda \in \sigma(B)$ with $\lambda \notin \sigma(D)$. The argument leading to (3) similarly applied to B gives

$$1 \leqq \varrho\{(\lambda - D)^{-1}(B-D)\} \leqq \|(\lambda-D)^{-1}(B-D)\|_\phi \leqq \|(\lambda-D)^{-1}(A-D)\|_\phi \ \forall \phi \in \mathscr{F}, \tag{7}$$

the last inequality following from (6). Hence, from (4), $\lambda \in G$. If $\lambda \in \sigma(B) \cap \sigma(D)$, then λ is again, by definition, in G. Thus, if $\sigma(\hat{\Omega})$ denotes the collection of all eigenvalues of all $B \in \hat{\Omega}$, the above discussion has established

Proposition 1. Let A and D be any matrices in $[\mathbb{C}^n]$, and let $\mathscr{F}$ be any family of norms on $\mathbb{C}^n$. Then,

$$\sigma(\hat{\Omega}) \subseteq G. \tag{8}$$

For a related but somewhat more general result, see also Kovarik [1, Prop. 1].

The purpose of this note is to give a necessary and sufficient condition that equality hold in (8), i.e., $\sigma(\hat{\Omega}) = G$, and to give examples both where equality in (8) holds and where equality in (8) fails.

2. Preliminary Results

We now derive some properties of the set $\hat{\Omega}$.

Lemma 1. The set $\hat{\Omega}$ is star-shaped with respect to D, i.e., for any $B \in \hat{\Omega}$ and any $t \in \mathbb{C}$ with $0 \leqq |t| \leqq 1$, then $\{D + t(B-D)\} \in \hat{\Omega}$.

Proof. If $B \in \hat{\Omega}$ and $\tilde{B} \equiv D + t(B-D)$ where $0 \leqq |t| \leqq 1$, then for any $\phi \in \mathscr{F}$ and any $z \notin \sigma(D)$,

$$\begin{aligned}\|(z-D)^{-1}(\tilde{B}-D)\|_\phi &= |t|\,\|(z-D)^{-1}(B-D)\|_\phi \leqq \|(z-D)^{-1}(B-D)\|_\phi \\ &\leqq \|(z-D)^{-1}(A-D)\|_\phi,\end{aligned}$$

the last inequality following from (6), since B is by hypothesis an element of $\hat{\Omega}$. Thus, by definition, $\tilde{B} \in \hat{\Omega}$. Q.E.D.

Lemma 2. $\hat{\Omega}$ is a compact subset of $\mathbb{C}^{n^2}$.

Proof. Here, it is convenient to regard each matrix $B \in [\mathbb{C}^n]$ as a point in $\mathbb{C}^{n^2}$. We first show that $\hat{\Omega}$ is **bounded.** Given any two matrices S and T in $[\mathbb{C}^n]$ with T nonsingular, it is well known (cf. Ostrowski [2, II.4 and II.16]) that, for any norm ϕ on $\mathbb{C}^n$,

$$\|TS\|_\phi \geqq \|S\|_\phi \cdot (\|T^{-1}\|_\phi)^{-1}.$$

Now, fix any $\phi \in \mathscr{F}$ and any $z \notin \sigma(D)$. For any $B \in \hat{\Omega}$, it follows from the above inequality that

$$\|(z-D)^{-1}(B-D)\|_\phi \geqq \|B-D\|_\phi \cdot (\|z-D\|_\phi)^{-1},$$

and as $\|B-D\|_\phi \geqq \|B\|_\phi - \|D\|_\phi$, we see from (6) that

$$\|B\|_\phi \leqq \|D\|_\phi + \|z-D\|_\phi \cdot \{\|(z-D)^{-1}(A-D)\|_\phi\}$$

for any $B\in\hat{\Omega}$. Thus, $\hat{\Omega}$ is a bounded subset of $\mathbb{C}^{n^2}$. That is a closed subset of $\hat{\Omega}$ $\mathbb{C}^{n^2}$ follows easily from (6). Thus, $\hat{\Omega}$ is a compact subset of $\mathbb{C}^{n^2}$. Q.E.D.

3. Main Result

For any $z\notin\sigma(D)$, it is convenient now to define

$$\nu(z) \equiv \sup\{\varrho((z-D)^{-1}(B-D)) : B\in\hat{\Omega}\}, \tag{9}$$

and

$$\eta(z) \equiv \inf\{\|(z-D)^{-1}(A-D)\|_\phi : \phi\in\mathscr{F}\}. \tag{10}$$

Evidently, $G=\sigma(D)\cup\{z\in\mathbb{C}: z\notin\sigma(D) \text{ and } \eta(z)\geqq 1\}$, and it follows from the last two inequalities of (7) that

$$\nu(z)\leqq\eta(z), \quad \forall z\notin\sigma(D).$$

In addition, the first inequality of (7) gives that each $z\in\sigma(\hat{\Omega})$ not in $\sigma(D)$ must satisfy $\nu(z)\geqq 1$. This brings us to

Theorem 1. Let A and D be any matrices in $[\mathbb{C}^n]$, and let $\mathscr{F}$ be any non-empty family of norms on $\mathbb{C}^n$. Then

$$\sigma(\hat{\Omega})=G \tag{11}$$

if and only if $\nu(z)\geqq 1$ for all $z\in G$ not in $\sigma(D)$.

Proof. First, suppose that $\nu(z)\geqq 1$ for all $z\in G$ not in $\sigma(D)$. To show that (11) holds, let z be an arbitrary point in G. If $z\in\sigma(D)$, then, as $D\in\hat{\Omega}$, it necessarily follows that $z\in\sigma(\hat{\Omega})$. If $z\notin\sigma(D)$, then, by hypothesis, $1\leqq\nu(z)<+\infty$. It is well-known that the spectral radius of a matrix is a continuous function of the entries of the matrix. Since $\hat{\Omega}$ is a compact subset of $\mathbb{C}^{n^2}$ from Lemma 2, there is evidenthy a $B\in\hat{\Omega}$ such that

$$\varrho\{(z-D)^{-1}(B-D)\}=\nu(z).$$

Moreover, if we write $B=D+S$, it is obvious from (6) that $D+e^{i\theta}S$ is also an element of $\hat{\Omega}$ for each real θ. Thus, without loss of generality, we may assume that $(z-D)^{-1}(B-D)$ has $\nu(z)$ as an eigenvalue, i.e., there is an $\boldsymbol{x}\neq\boldsymbol{0}$ in $\mathbb{C}^n$ with $(z-D)^{-1}(B-D)\boldsymbol{x}=\nu(z)\boldsymbol{x}$, or equivalently,

$$\left\{D+\left(\frac{1}{\nu(z)}\right)(B-D)\right\}\boldsymbol{x}=z\boldsymbol{x}. \tag{12}$$

But, using the fact from Lemma 1 that $\hat{\Omega}$ is star-shaped with respect to D, the matrix $\tilde{B}\equiv D+\left(\frac{1}{\nu(z)}\right)(B-D)$ is an element of $\hat{\Omega}$, having z as an eigenvalue, i.e., $z\in\sigma(\hat{\Omega})$. Thus, $G\subseteq\sigma(\hat{\Omega})$. But, with the reverse inclusion from (8), then $\sigma(\hat{\Omega})=G$.

Conversely, assume that there is a z in G not in $\sigma(D)$ for which $\nu(z)<1$. But then, z cannot be in $\sigma(\hat{\Omega})$ since if it were, $\nu(z)$ would be at least unity from (7). Thus, $\sigma(\hat{\Omega})\subsetneqq G$. Q.E.D.

4. Examples

It is natural to ask when equality holds in (11). Particular known results do in fact establish (11) in special cases, and it is worthwhile to recall one such example. If $\{e_j\}_{j=1}^n$ denotes the canonical basis in $\mathbb{R}^n$, let the family $\mathscr{F}_1$ of norms on $\mathbb{C}^n$ be defined as

$$\mathscr{F}_1 = \{\phi\colon \text{ there exist positive numbers } \phi_1, \phi_2, \ldots, \phi_n \text{ such that} \quad (13)$$
$$\phi\left(\sum_{j=1}^n c_j e_j\right) = \max_{1\leq i\leq n} [|c_j|/\phi_j]\}.$$

For any $A=(a_{i,j})\in[\mathbb{C}^n]$, choose $D=\operatorname{diag}(A)=\operatorname{diag}(a_{1,1}, a_{2,2}, \ldots, a_{n,n})$. For this choice of D and $\mathscr{F}_1$, the set $G_{\mathscr{F}_1}(A;\operatorname{diag}(A))$ reduces exactly to the original minimal Gerschgorin set considered in [4]. In particular, from [4, Theorem 6], it is known that

$$\sigma(\hat{\Omega}_{\mathscr{F}_1}(A;\operatorname{diag}(A))) = G_{\mathscr{F}_1}(A, \operatorname{diag}(A)),$$

i.e., equality holds in (11) for this example. Similarly, if $\mathscr{F}_2$ is the collection of **all** norms on $\mathbb{C}^n$, then for arbitrary A and D in $[\mathbb{C}^n]$, equality holds in (11). For infinite dimensional analogues which similarly establish equality in (11), see Kovarik [1].

We now establish that equality **cannot** hold in general in (11), and we make use of Theorem 1 to show this. Specifically, if $\mathscr{F}_3$ is the family of norms of (13) for the special case $n=2$, choose

$$A=\begin{bmatrix}1 & 3\\ 5 & 4\end{bmatrix};\quad D=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}.$$

Starting with the observation (cf. [5]) that, for any $S=(s_{i,j})\in[\mathbb{C}^2]$,

$$\inf\{\|S\|_\phi\colon \phi\in\mathscr{F}_3\} = \varrho(|S|) \tag{14}$$

where $|S|\equiv(|s_{i,j}|)$, it follows from (14) that

$$\begin{aligned}\eta(z) &\equiv \inf\{\|(z-D)^{-1}(A-D)\|_\phi\colon \phi\in\mathscr{F}_3\} = \varrho\{|(z-D)^{-1}(A-D)|\}\\ &= (4+\sqrt{4+3|z-1|\cdot|z-4|})/(|z|\cdot|z-5|), \quad \forall z\notin\sigma(D),\end{aligned} \tag{15}$$

the last expression resulting from direct computation. Thus, as

$$G\equiv\sigma(D)\cup\{z\in\mathbb{C}\colon z\notin\sigma(D) \text{ and } \eta(z)\geqq 1\},$$

it follows that

$$G=\{z\in\mathbb{C}\colon 4+\sqrt{4+3|z-1|\cdot|z-4|}\geqq|z|\cdot|z-5|\}. \tag{16}$$

Next, a short calculation based on (6), which uses the variability of **both** z and $\phi\in\mathscr{F}_3$, shows that $B\in\hat{\Omega}$ if and only

$$B-D=\begin{bmatrix}0 & c_{1,2}\\ c_{2,1} & 0\end{bmatrix}$$

and $|c_{1,2}|\leqq 1$ and $|c_{2,1}|\leqq 3$. From this, it follows that

$$|(z-D)^{-1}(B-D)|\leqq|(z-D)^{-1}(A-D)| \quad \forall z\in\sigma(D), \quad \forall B\in\hat{\Omega}, \tag{17}$$

in the sense of nonnegative matrices. Choose now any z_0 for which $\eta(z_0)=1$, i.e., z_0 is a boundary point of G. If we for convenience set $S=(z_0-D)^{-1}(B-D)$ for any $B\in\hat{\Omega}$ and $T=|(z_0-D)^{-1}(A-D)|$, then $|S|\leqq T$ from (17), and hence from (15) and the Perron-Frobenius theorem on nonnegative matrices (cf. [3, p. 28]),

$$\varrho(S)\leqq\varrho(T)=\eta(z_0)=1,$$

with equality holding throughout only if S has the representation (cf. [3, p. 29])

$$S=e^{i\mu}NTN^{-1} \tag{18}$$

where N is a diagonal matrix with diagonal entries all having modulus unity. By direct verification, it turns out, however, that S has the representation of (18) if and only if z_0 is real. Thus, for any non-real z_0 with $\eta(z_0)=1$, (which clearly exist from (15)), it necessarily follows that $\nu(z_0)<1=\eta(z_0)$ and hence, from Theorem 1,

$$\sigma(\hat{\Omega}_{\mathscr{F}_3}(A;D))\subset G_{\mathscr{F}_3}(A;D).$$

It is interesting to note in the last example that while $\sigma(\hat{\Omega})\subsetneqq G$, it is however true that the real points of the boundary, ∂G, of G are in $\sigma(\hat{\Omega})$, i.e.,

$$\partial G\cap\sigma(\hat{\Omega})\neq\phi. \tag{19}$$

It is an open question if (19) is valid for any choice of matrices A and D in $[\mathbb{C}^n]$, and for any non-empty family $\mathscr{F}$ of norms on $\mathbb{C}^n$.

References

1. Kovarik, Z. V.: Spectrum localization for certain classes of operators acting on C-spaces. To appear.
2. Ostrowski, A. M.: On some metrical properties of operator matrices and matrices partitioned into blocks. J. Math. Anal. Appl. **2**, 161–209 (1962).
3. Varga, R. S.: Matrix iterative analysis. New Jersey, Englewood Cliffs: Prentice-Hall, Inc. 1962, 322 pp.
4. Varga, R. S.: Minimal Gerschgorin sets. Pacific J. Math. **15**, 719–729 (1965).
5. Varga, R. S.: On a connection between infima of norms and eigenvalues of associated operators. Lin. Alg. Appl. (to appear).

Richard S. Varga
Department of Mathematics
Kent State University
Kent, Ohio 44242 U.S.A.

Bernard W. Levinger
Department of Mathematics
Colorado State University
Fort Collins, Colorado 80521 U.S.A.