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EXTENDED ERROR BOUNDS FOR SPLINE AND L-SPLINE INTERPOLATION

Stephen Demko and Richard S. Varga

1. Introduction

The basic object of this paper is to extend many of the error bounds for spline and L-spline interpolation, as given in Swartz and Varga [1] and Scherer [2]. To describe these extensions, the following standard notation is used. If $-\infty < a < b < +\infty$, and if N is a positive integer, then

(1) $\Delta\colon\ a = x_0 < x_1 < \dots < x_N = b$

is a partition of the interval $[a,b]$ with knots x_i, and we set $\overline{\Delta} \equiv \max_{0\le i\le N-1} \{x_{i+1}-x_i\}$, and $\underline{\Delta} \equiv \min_{0\le i\le N-1} \{x_{i+1}-x_i\}$. For $\sigma \geq 1$, $P_\sigma(a,b)$ then denotes all partitions Δ of $[a,b]$ for which $\overline{\Delta}/\underline{\Delta} \leq \sigma$. Next, if π_n is the set of all real polynomials of degree at most n and $W_p^k[a,b]$ is the usual Sobolev space with norm $\|\cdot\|_{W_p^k[a,b]}$, then for nonnegative integers n and m with $n \geq m \geq 0$,

(2) $Sp(n,m,\Delta) \equiv \{s \in W_\infty^m[a,b]\colon s(x) \in \pi_n \text{ for } x \in [x_i,x_{i+1}],\ 0\le i\le N-1\}$

is the space of polynomial splines on $[a,b]$ relative to the partition Δ. Similarly, if $z = (z_1, z_2, \dots, z_{N-1})$ is an $(N-1)$-tuple of positive integers z_i with $1 \le z_i \le m$, then as in [1], with $\mu \equiv \max_{1\le i\le N-1} \{z_i\}$,

(3) $Sp(L,\Delta,z) = \{s \in W_\infty^{2m-\mu}[a,b]\colon L^*L\, s(x)=0 \text{ on } (a,b) - \{x_i\}_{i=1}^{N-1},$

$\text{and } D^k s(x_i-) = D^k s(x_i+) \text{ for all } 0\le k\le 2m-1-z_i,\ 0<i<N\}$

is the space of L-splines on $[a,b]$, with incidence vector z.

A typical result from [1, Theorem 7.4] is the following.

<u>Theorem 1.</u> Given $f \in C^k[a,b]$ with $0 \le k < 2m$ and given $\Delta \in \mathcal{P}_1(a,b)$, let s be the unique interpolant of f in $Sp(2m-1,\ 2m-1,\Delta)$ such that

$$(4)\quad \begin{cases} (f-s)(x_i) = 0, & 1 \le i \le N-1, \\ D^j(f-s)(a) = D^j(f-s)(b) = 0 & \text{for } 0 \le j \le \min(k,m-1), \\ D^j s(a) = D^j s(b) = 0 & \text{if } \min(k,m-1) < j \le m-1. \end{cases}$$

Then,

$$(5)\quad K(\bar{\Delta})^{k-j}\omega_\infty(D^k f,\bar{\Delta}) \ge \begin{cases} \|D^j(f-s)\|_{L_\infty[a,b]}, & 0 \le j \le k, \\ \|D^j s\|_{L_\infty[a,b]}, & \text{if } k < j \le 2m-1, \end{cases}$$

where ω_∞ denotes the usual modulus of continuity.

From the above result, one deduces (cf. [1, Corollary 7.5])

<u>Corollary 2.</u> With the hypotheses of Theorem 1, if $f \in W_p^{k+1}[a,b]$ with $0 \le k < 2m$ and $2 \le p \le \infty$, then for $p \le q \le \infty$,

$$(6)\quad K(\bar{\Delta})^{k+1-j+\frac{1}{q}-\frac{1}{p}}\|D^{k+1}f\|_{L_p[a,b]} \ge \begin{cases} \|D^j(f-s)\|_{L_q[a,b]}, & 0 \le j \le k, \\ \|D^j s\|_{L_q[a,b]}, & \text{if } k < j \le 2m-1. \end{cases}$$

The basic results of this paper can be described then as extensions of Theorem 1, and improvements of Corollary 2.

2. <u>Basic Comparison Theorem</u>

Given any $f \in L_p[a,b]$ with $1 \le p \le \infty$, let

$$(7)\quad \omega_p(f,t) \equiv \sup_{|h| \le t}\left\{\int_a^b |f(x+h) - f(x)|^p dx\right\}^{1/p}, \quad 0 < t \le (b-a),$$

denote the <u>p^{th} modulus of continuity of f</u>, where we assume that f has been suitably extended to an L_p-function on $[2a-b, 2b-a]$. As is well known,

$$(8)\quad \lim_{t\to 0} \omega_p(f,t) = 0 \text{ if } \begin{cases} f\in L_p[a,b], & 1\le p<\infty, \\ f\in C^0[a,b], & p=\infty. \end{cases}$$

Moreover, if $f\in W^1_\tau[a,b]$ with $1\le\tau\le p$, then

$$(9)\quad \omega_p(f,t) \le 4t^{1+\frac{1}{p}-\frac{1}{\tau}}\|Df\|_{L_\tau[a,b]} \cdot$$

Next, for any $0<h\le 2(b-a)$, we define from f the function

$$(10)\quad f_h(x) = \frac{1}{h}\int_{x-h/2}^{x+h/2} f(t)dt, \qquad x\in[a,b],$$

the so-called Stekloff function of f. If $f\in W^k_p[a,b]$, then $f_h\in W^{k+1}_p[a,b]$, and moreover, it can be verified that

$$(11)\quad \begin{cases} D^j f_h(x) = (D^j f)_h(x), \quad x\in[a,b], & 0\le j\le k, \\ \|D^j(f-f_h)\|_{L_p[a,b]} \le \omega_p(D^j f, h/2), & 0\le j\le k, \\ \|D^{k+1} f_h\|_{L_p[a,b]} \le \frac{1}{h}\,\omega_p(D^k f,h). \end{cases}$$

With the function f_h, the following comparison theorem, the analogue of Swartz and Varga [1, Lemma 3.2], can be proved by means of a Peano Kernel Theorem argument.

<u>Theorem 3.</u> Given $f\in W^k_p[a,b]$ with $0\le k<2m$ and given $\Delta\in P_\sigma(a,b)$, let g be the unique interpolant of f in $Sp(4m+1,2m+1,\Delta)$ such that

$$(12)\quad \begin{cases} D^j(f-g)(x_i) = 0, \quad 0\le j\le k-1 \text{ if } k>0, & 0\le i\le N, \\ D^k(f_h-g)(x_i) = 0, & 0\le i\le N, \\ D^j g(x_i) = 0, \quad k<j\le 2m, & 0\le i\le N. \end{cases}$$

Then, with $h = \overline{\Delta}$,

(13)

$$K(\overline{\Delta})^{k-j+\frac{1}{q}-\frac{1}{p}}\omega_p(D^k f,\overline{\Delta}) \ge \begin{cases} \|D^j(f-g)\|_{L_q[a,b]}, & 0\le j\le k-1 \text{ if } k>0, p\le q\le\infty, \\ \|D^k(f-g)\|_{L_p[a,b]}, & j=k,\ p=q, \\ \|D^j g\|_{L_q[a,b]}, & k<j\le 2m,\ p\le q\le\infty. \end{cases}$$

With the above comparison theorem, we give the following result and sketch its proof.

<u>Theorem 4.</u> Given $f \in W_p^k[a,b]$ with $0 \le k < 2m$ and with $2 \le p \le \infty$, and given $\Delta \in \mathcal{P}_1(a,b)$, let s be the unique interpolant of f in $Sp(2m-1,\ 2m-1,\Delta)$ such that

$$(14)\quad \begin{cases} (f-s)(x_i) = 0, & 1 \le i \le N-1, \quad \text{if } k>0, \\ (f_h-s)(x_i) = 0, & 1 \le i \le N-1, \quad \text{if } k=0, \\ D^j(f-s)(a) = D^j(f-s)(b) = 0 & \text{for } 0 \le j \le \min(k-1,m-1) \text{ if } k>0, \\ D^k(f_h-s)(a) = D^k(f_h-s)(b) = 0 & \text{if } k \le m-1, \\ D^j s(a) = D^j s(b) = 0 & \text{if } k<j \le m-1. \end{cases}$$

Then, with $h = \overline{\Delta}$,

$$(15)\quad K(\overline{\Delta})^{k-j+\frac{1}{q}-\frac{1}{p}}\omega_p(D^k f,\overline{\Delta}) \ge \begin{cases} \|D^j(f-s)\|_{L_q[a,b]}, & 0 \le j \le k-1 \text{ if } k>0, p \le q \le \infty, \\ \|D^k(f-s)\|_{L_p[a,b]}, & j=k,\ p=q, \\ \|D^j s\|_{L_q[a,b]}, & \text{if } k<j \le 2m-1,\ p \le q \le \infty. \end{cases}$$

<u>Proof:</u> To sketch the proof of this result, write $f-s = (f-g)+(g-s)$, where g is the interpolant of f in $Sp(4m+1,2m+1,\Delta)$, in the sense of (12). Applying (13) of Theorem 3 then suitably bounds the derivatives of $(f-g)$. Next, because of the definition of g in (13), s, as defined in (14), is also the unique interpolant of g in $Sp(2m-1,2m-1,\Delta)$, in the sense of (4). Applying the first inequality of (6) of Corollary 2 then yields

$$\|D^j(g-s)\|_{L_q[a,b]} \le K(\overline{\Delta})^{k+1-j+\frac{1}{q}-\frac{1}{p}}\|D^{k+1}g\|_{L_p[a,b]} \quad \text{for } 0 \le j \le k,\ p \le q \le \infty.$$

But, from the last inequality of (13) for $j = k+1$, $\|D^{k+1}g\|_{L_p[a,b]} \le K(\overline{\Delta})^{-1}\omega_p(D^k f,\overline{\Delta})$, whence $\|D^j(g-s)\|_{L_q[a,b]} \le K(\overline{\Delta})^{k-j+\frac{1}{q}-\frac{1}{p}}\omega_p(D^k f,\overline{\Delta})$, from which the first two inequalities of (15) follow. Similarly, upon writing $s=g-(g-s)$, the above technique, i.e., using the bounds of (13) and (6), yields the third inequality of (15). Q.E.D.

Corollary 5. With the hypotheses of Theorem 4, let $\{\Delta_i\}_{i=1}^{\infty} \in P_1(a,b)$ with $\lim_{i\to\infty} \overline{\Delta}_i = 0$, and let s_i be the unique interpolant of f in $Sp(2m-1, 2m-1, \Delta_i)$, in the sense of (14) with $h = \overline{\Delta}_i$. Then,

$$(16) \quad \lim_{i\to\infty} \|D^k(f-s_i)\|_{L_p[a,b]} = 0.$$

It is interesting to remark that, since the error bounds of (15) for the special case $p=q=\infty$ essentially reduce for smooth functions to the bounds of (5), then Theorem 4 can be viewed as an extension of Theorem 1. Similarly, the error bounds of (15) represent a sharpening of the error bounds of (6) (with k replaced by k-1) in two ways. First, the added factor $\omega_p(D^k f, \overline{\Delta})$ in (15) tends to zero as $\overline{\Delta} \to 0$ if $2 \le p < \infty$ (cf. (8)). Second, one obtains the additional estimate for $\|D^k(f-s)\|_{L_p[a,b]}$ in (15) which does not appear in (6) (with k replaced by k-1).

3. Extensions

The basic comparison function g, as defined in Theorem 3, can similarly be systematically used to obtain improved error bounds for L-spline, Hermite L-spline, and polynomial spline interpolation under general boundary conditions, thereby extending the general error bounds of Swartz and Varga [1], both for uniform and nonuniform partitions of [a,b]. The same can also be done to extend the stability-type error bounds of [1]. Finally, new interpolation error bounds for even-ordered polynomial splines, determined from integral-type interpolation conditions, as considered, for example, in Scherer [2], also can be deduced from the basic comparison Theorem 3.

References

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Department of Mathematics
Kent State University
Kent, Ohio 44240