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Extended L_p-Error Bounds for Spline and L-Spline Interpolation*

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1. INTRODUCTION

Our basic aim here is to extend and improve the error bounds for spline and L-spline interpolation recently given by Swartz and Varga [11]. In so doing, we also extend some recent results of Scherer [9]. To illustrate one such improvement, consider the interpolation of a given function $f \in C^k[a, b]$, with $0 \leqslant k < 2m$, by a smooth polynomial spline $s \in C^{2m-2}[a, b]$, of local degree $2m - 1$ on each segment of a uniform partition Δ of $[a, b]$, where s is uniquely determined from f by means of

$$\begin{aligned} (f - s)(x_i) &= 0, \qquad 1 \leqslant i \leqslant N - 1, \\ D^j(f - s)(a) = D^j(f - s)(b) &= 0 \qquad \text{for} \quad 0 \leqslant j \leqslant \min(k, m - 1), \\ D^j s(a) = D^j s(b) &= 0 \qquad \text{if} \quad k < j \leqslant m - 1, \end{aligned} \tag{1.1}$$

with $x_i \equiv a + ih$, $h = (b - a)/N$, $0 \leqslant i \leqslant N$. It is known from [11, Theorem 7.4] that there exists a constant K, independent of f and h, such that

$$Kh^{k-j}\omega_\infty(D^k f, h) \geqslant \begin{cases} \| D^j(f - s)\|_{L_\infty[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_\infty[a,b]}, & \text{if} \quad k < j \leqslant 2m - 1, \end{cases} \tag{1.2}$$

where ω_∞ denotes the usual L_∞-modulus of continuity. If $f \in W_p{}^k[a, b]$ with $1 \leqslant k \leqslant 2m$, and $2 \leqslant p \leqslant \infty$, one can deduce from (1.2) (cf. [11, Corollary 7.5]) that

$$\begin{aligned} &Kh^{k-j+(1/q)-(1/p)} \| D^k f \|_{L_p[a,b]} \\ &\quad \geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k - 1, \quad p \leqslant q \leqslant \infty, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k - 1 < j \leqslant 2m - 1, \quad p \leqslant q \leqslant \infty. \end{cases} \end{aligned} \tag{1.3}$$

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For $0 \leqslant k < 2m$ and $2 \leqslant p \leqslant \infty$, the above results can be improved and extended (cf., Theorem 6.2) to

$$Kh^{k-j+(1/q)-(1/p)}\omega_p(D^k f, h)$$
$$\geqslant \begin{cases} \| D^j(f-\tilde{s})\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k-1 \quad \text{if} \quad k > 0, \quad p \leqslant q \leqslant \infty, \\ \| D^k(f-\tilde{s})\|_{L_p[a,b]}, & j = k, \quad p = q, \\ \| D^j\tilde{s}\|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m-1, \quad p \leqslant q \leqslant \infty, \end{cases} \tag{1.4}$$

where ω_p denotes the L_p-modulus of continuity (cf. (2.2)), and $\tilde{s} \in C^{2m-2}[a, b]$ is again a smooth polynomial spline, of local degree $2m - 1$ on each segment of Δ, which interpolates f in a manner similar to (1.1) (cf. (6.9)).

We shall also obtain here improved interpolation error bounds for L-spline interpolation (Section 4), Hermite L-spline interpolation (Section 4), and an improved stability analysis for L-spline and Hermite L-spline interpolation (Section 5), as originally considered in [11]. In Section 6, we extend the results of [11] concerning polynomial spline interpolation on uniform partitions of $[a, b]$ for general boundary interpolation of the second integral relation type (cf. (6.4)). Finally, in Section 7, we give some improved interpolation error bounds for smooth spline interpolation, where the spline is locally of *even* degree on each segment of the partition, which extend certain recent results of Scherer [9].

2. Notation

For $-\infty < a < b < +\infty$, and for any extended real number p satisfying $1 \leqslant p \leqslant \infty$, let $L_p[a, b]$ denote as usual the Banach space of real-valued Lebesgue-measurable functions f defined on $[a, b]$ such that $\int_a^b |f(t)|^p \, dt < \infty$ if $1 \leqslant p < \infty$, and such that f is essentially bounded on $[a, b]$ if $p = +\infty$, endowed with the norm

$$\|f\|_{L_p[a,b]} = \begin{cases} \left(\int_a^b |f(t)|^p \, dt\right)^{1/p}, & 1 \leqslant p < \infty, \\ \text{ess. sup}\{|f(t)| : t \in [a, b]\}, & p = \infty. \end{cases}$$

More generally, $W_p^k[a, b]$, with k a positive integer and $1 \leqslant p \leqslant \infty$, denotes the Sobolev space of all real-valued functions f defined on $[a, b]$ whose $(k-1)$st derivative is absolutely continuous, and for which $D^k f \in L_p[a, b]$. (Here, $D^k \equiv (d/dx)^k$.) We also set $W_p^0[a, b] \equiv L_p[a, b]$. The norm on $W_p^k[a, b]$ is given, as usual, by

$$\|f\|_{W_p^k[a,b]} = \sum_{j=0}^{k} \| D^j f\|_{L_p[a,b]}.$$

For any $f \in W_p^k[a, b]$, it is well known (cf. Hestenes [6] and Whitney [14]) that f can be extended to a function $\tilde{f}$ in $W_p^k[2a - b, 2b - a]$. One way of concretely achieving this is to set (cf., Johnen [7])

$$\begin{aligned}\tilde{f}(x) &= \sum_{i=0}^{k} c_i f(b + 2^{-i}(b - x)), && x \in (b, 2b - a],\\ &= f(x), && x \in [a, b],\\ &= \sum_{i=0}^{k} c_i f(a + 2^{-i}(a - x)), && x \in [2a - b, a),\end{aligned}$$

where the real numbers c_i, $0 \leqslant i \leqslant k$, uniquely solve the following Vandermond system of linear equations:

$$\sum_{i=0}^{k} c_i(-2)^{-ij} = 1, \qquad 0 \leqslant i \leqslant k.$$

We remark that the mapping $f \to \tilde{f}$, as defined in (2.1), is a bounded linear transformation from $W_p^k[a, b]$ to $W_p^k[2a - b, 2b - a]$. We shall assume throughout that $f \in W_p^k[a, b]$ ($k \geqslant 0$ and $1 \leqslant p \leqslant \infty$) is extended by means of (2.1), if values of $f(x)$ are required for $x \in [2a - b, 2b - a] - [a, b]$.

With the above convention, for any $f \in L_p[a, b]$ and for any $0 < t \leqslant b - a$, we set

$$\omega_p(f, t) = \sup_{|h| \leqslant t} \left\{ \left(\int_a^b |f(x + h) - f(x)|^p \, dx \right)^{1/p} \right\}, \tag{2.2}$$

where $\omega_p(f, t)$ is called the *L_p-modulus of continuity* of f on $[a, b]$. As is well known, (cf. Achieser [1]) $\omega_p(f, t)$ is a nondecreasing function of t, for which

$$\lim_{t \downarrow 0} \omega_p(f, t) = 0 \qquad \text{if} \quad \begin{cases} f \in L_p[a, b], & 1 \leqslant p < \infty, \\ f \in C^0[a, b], & p = \infty, \end{cases} \tag{2.3}$$

where $C^k[a, b]$, $k \geqslant 0$, denotes the set of all real-valued functions $g(t)$. defined on $[a, b]$, such that $D^i g$ is continuous on $[a, b]$ for all $0 \leqslant i \leqslant k$. In addition, if $f \in W_\tau^1[a, b]$ with $1 \leqslant \tau \leqslant p$, it can be verified, upon representing the integral of (2.2) as an appropriate sum, each term of which is an integral over an interval of length at most t, that

$$\omega_p(f, t) \leqslant 4t^{1+(1/p)-(1/\tau)} \| Df \|_{L_\tau[a,b]} \, . \tag{2.4}$$

For any $0 < h \leqslant 2(b - a)$ and for any $f \in L_p[a, b]$, we define

$$f_h(x) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(t) \, dt, \qquad x \in [a, b], \tag{2.5}$$

as the *Stekloff function of f* (cf. Achieser [1]). With Fubini's Theorem and the Hölder inequality, it directly follows from (2.2) and (2.5) that

$$\|f - f_h\|_{L_p[a,b]} \leqslant \omega_p(f, h/2). \tag{2.6}$$

More generally, if $f \in W_p{}^k[a, b]$, then evidently $f_h \in W_{k+1}^p[a, b]$ for any $0 < h \leqslant 2(b - a)$, and, moreover, from (2.5) we verify that

$$D^j f_h(x) = (D^j f)_h\,(x), \qquad x \in [a, b], \quad 0 \leqslant j \leqslant k. \tag{2.7}$$

Thus, from (2.6) and (2.7), if $f \in W_p{}^k[a, b]$, then

$$\| D^j(f - f_h)\|_{L_p[a,b]} \leqslant \omega_p(D^j f, h/2), \qquad 0 \leqslant j \leqslant k. \tag{2.8}$$

Finally, for $f \in W_p{}^k[a, b]$, it follows from (2.5) that

$$D^{k+1} f_h(x) = (1/h)\{D^k f(x + h/2) - D^k f(x - h/2)\}$$

for almost all x in $[a, b]$, from which we deduce that

$$\| D^{k+1} f_h \|_{L_p[a,b]} \leqslant (1/h)\ \omega_p(D^k f, h). \tag{2.9}$$

For a positive integer N,

$$\Delta\colon a = x_0 < x_1 < \cdots < x_N = b \tag{2.10}$$

denotes a partition of $[a, b]$ with knots x_i. The collection of all such partitions of $[a, b]$ is called $\mathscr{P}(a, b)$. We define $\bar{\Delta} = \max\{(x_{i+1} - x_i)\colon 0 \leqslant i \leqslant N - 1\}$ and $\underline{\Delta} = \min\{(x_{i+1} - x_i)\colon 0 \leqslant i \leqslant N - 1\}$ for each partition Δ of the form (2.10). For any real number σ with $\sigma \geqslant 1$, $\mathscr{P}_\sigma(a, b)$ then denotes the subset of all partitions Δ in $\mathscr{P}(a, b)$ for which $\bar{\Delta} \leqslant \sigma\underline{\Delta}$. In particular, $\mathscr{P}_1(a, b)$ is the collection of all *uniform* partitions of $[a, b]$.

If π_n denotes the collection of all real algebraic polynomials of degree at most n, then for any nonnegative integers n and m with $n \geqslant m \geqslant 0$, the *polynomial spline space* $\mathrm{Sp}(n, m, \Delta)$ is defined (cf. Scherer [9]) by

$$\mathrm{Sp}(n, m, \Delta) = \{s(x)\colon s \in W_\infty{}^m[a, b],\ s(x) \in \pi_n \text{ for } x \in (x_i, x_{i+1}),\\ i = 0, 1, \ldots, N - 1\}. \tag{2.11}$$

We remark that $\mathrm{Sp}(n, m, \Delta)$ is a finite-dimensional subspace of $W_\infty{}^m[a, b]$. If $m \geqslant 1$, then as $W_\infty{}^m[a, b] \subset C^{m-1}[a, b]$, each element of $\mathrm{Sp}(n, m, \Delta)$ is in $C^{m-1}[a, b]$.

Since we shall make use of the related concept of L-splines, we describe them briefly. Given the differential operator L or order m,

$$Lu(x) \equiv \sum_{j=0}^{m} c_j(x)\, D^j u(x), \qquad m \geqslant 1,$$

where $c_j \in C^j[a, b]$, $0 \leqslant j \leqslant m$, with $c_m(x) \geqslant \delta > 0$ for all $x \in [a, b]$, and given a partition Δ of the form (2.10), for $N > 1$ let $z = (z_1, z_2, \ldots, z_{N-1})$, the incidence vector, be an $(N-1)$-tuple of positive integers with $1 \leqslant z_i \leqslant m$, $1 \leqslant i \leqslant N-1$. Then, the *L-spline space* $\mathrm{Sp}(L, \Delta, z)$ is (cf. Ahlberg, Nilson, and Walsh [2] and Schultz and Varga [8]) the collection of all real-valued functions w defined on $[a, b]$ such that

$$L^*Lw(x) = 0, \qquad x \in (a, b) - \{x_i\}_{i=1}^{N-1},$$
$$D^k w(x_i-) = D^k w(x_i+) \qquad \text{for} \quad 0 \leqslant k \leqslant 2m - 1 - z_i, \quad 1 \leqslant i \leqslant N-1, \tag{2.12}$$

where L^* is the formal adjoint of L. From (2.12), we see that

$$\mathrm{Sp}(L, \Delta, z) \subset W_\infty^{2m-\mu}[a, b] \qquad \text{where} \quad \mu \equiv \max\{z_i: 1 \leqslant i \leqslant N-1\}. \tag{2.13}$$

Moreover, on comparing the definitions of (2.11) and (2.12), we see that $\mathrm{Sp}(D^m, \Delta, \tilde{z}) = \mathrm{Sp}(2m-1, 2m-l, \Delta)$ if $\tilde{z} = (l, l, \ldots, l)$, where $1 \leqslant l \leqslant m$.

In what is to follow, we shall denote throughout any generic constant which is independent of the functions considered and is independent of the maximum mesh spacing $\bar{\Delta}$, by the symbol K. These constants, however, in general do depend upon n, m, a, b, the various norms and orders of derivatives used, as well as upon σ if $\Delta \in \mathscr{P}_\sigma(a, b)$.

3. Basic Comparison Functions

As in Swartz and Varga [11], the key idea here is an elementary one, based on the triangle inequality. From known interpolation errors for smooth functions g, error bounds for less smooth functions, f, are determined as follows. A smooth piecewise polynomial interpolant g of f is constructed, and bounds for $f - g$ are determined (Theorem 3.5). A spline interpolant, s, of f is then defined, which is also the spline interpolant of this smooth g. Then, bounds for $f - s$ will follow from known bounds for $f - g$ and $g - s$.

To begin, we state an interpolation result of Swartz and Varga [11, Corollary 3.3].

Lemma 3.1. *Given $f \in W_p^{k+1}[a, b]$ with $0 \leqslant k < 2m$, and given $\Delta \in \mathscr{P}_\sigma(a, b)$, let $\hat{g}$ be the unique interpolant of f in* $\mathrm{Sp}(4m+1, 2m+1, \Delta)$ *such that*

$$D^j(f - \hat{g})(x_i) = 0, \qquad 0 \leqslant j \leqslant k, \quad 0 \leqslant i \leqslant N,$$
$$D^j \hat{g}(x_i) = 0, \qquad k < j \leqslant 2m, \quad 0 \leqslant i \leqslant N. \tag{3.1}$$

Then,

$$K(\bar{\Delta})^{k+1-j+(1/q)-(1/p)} \| D^{k+1}f \|_{L_p[a,b]} \geqslant \begin{cases} \| D^j(f-\hat{g})\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \quad p \leqslant q \leqslant \infty, \\ \| D^j \hat{g} \|_{L_q[a,b]}, & k < j \leqslant 2m, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{3.2}$$

We remark that since $\| \psi \|_{L_q[a,b]} \leqslant (b-a)^{(1/q)-(1/p)} \| \psi \|_{L_p[a,b]}$ for any $\psi \in L_p[a,b]$ and for any q with $1 \leqslant q \leqslant p$, the upper bounds of (3.2) can be trivially extended to the full range of q, i.e., $1 \leqslant q \leqslant \infty$, simply by replacing the exponent of $\bar{\Delta}$ in (3.2) by $k+1-j+\min(0,(1/q)-(1/p))$. This same extension of course applies to all subsequent bounds developed.

With Lemma 3.1, we prove the following:

LEMMA 3.2. *Given $f \in W_p^k[a,b]$ with $0 \leqslant k < 2m$, and given $\Delta \in \mathscr{P}_\sigma(a,b)$, let $\tilde{g}$ be the unique interpolant of f_h (defined in (2.5)) in* $\mathrm{Sp}(4m+1, 2m+1, \Delta)$, *in the sense of* (3.1). *With $h = \bar{\Delta}$, then*

$$K(\bar{\Delta})^{k-j+(1/q)-(1/p)} \omega_p(D^k f, \bar{\Delta}) \geqslant \begin{cases} \| D^k(f-\tilde{g})\|_{L_p[a,b]}, & k = j, \quad p = q, \\ \| D^j \tilde{g} \|_{L_q[a,b]}, & k < j \leqslant 2m, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{3.3}$$

Proof. To establish the first inequality of (3.3), the triangle inequality gives

$$\| D^k(f-\tilde{g})\|_{L_p[a,b]} \leqslant \| D^k(f-f_h)\|_{L_p[a,b]} + \| D^k(f_h-\tilde{g})\|_{L_p[a,b]}. \tag{3.4}$$

Since $\tilde{g}$ is the unique interpolant of $f_h \in W_p^{k+1}[a,b]$ in $\mathrm{Sp}(4m+1, 2m+1, \Delta)$, in the sense of (3.1), the last term of (3.4) can be bounded above from (3.2) of Lemma 3.1 by $K\bar{\Delta} \| D^{k+1}f_h \|_{L_p[a,b]}$. But, it follows from (2.9) with $h = \bar{\Delta}$ that $\| D^{k+1} f_h \|_{L_p[a,b]} \leqslant (\bar{\Delta})^{-1} \omega_p(D^k f, \bar{\Delta})$, whence

$$\| D^k(f_h - \tilde{g})\|_{L_p[a,b]} \leqslant K\omega_p(D^k f, \bar{\Delta}).$$

Similarly, from (2.8) and the nondecreasing property of $\omega_p(D^k f, t)$, we have that

$$\| D^k(f-f_h)\|_{L_p[a,b]} \leqslant \omega_p(D^k f, \bar{\Delta}),$$

which then gives the desired first inequality of (3.3). The second inequality of (3.3) similarly follows from (2.9) and the second inequality of (3.2) Q.E.D.

The next lemma is well known, but for completeness, a short proof is given.

LEMMA 3.3. *For $u \in W_p^1[c,d]$ where $-\infty < c < d < \infty$, assume that there is some $x_0 \in [c,d]$ for which $u(x_0) = 0$. Then, for any q with $1 \leqslant q \leqslant \infty$,*

$$\| u \|_{L_q[c,d]} \leqslant (d-c)^{1+(1/q)-(1/p)} \| Du \|_{L_p[c,d]}. \tag{3.5}$$

Proof. Clearly, $\| u \|_{L_q[c,d]} \leqslant (d-c)^{1/q} \| u \|_{L_\infty[c,d]} = (d-c)^{1/q} \, | u(\xi)|$ for some $\xi \in [c, d]$. Since $u(\xi) = \int_{x_0}^{\xi} Du(t)\, dt$, then from Hölder's inequality, $| u(\xi)| \leqslant | \xi - x_0 |^{1-1/p} (\int_{x_0}^{\xi} | Du(t)|^p \, dt)^{1/p} \leqslant (d-c)^{1-1/p} \| Du \|_{L_p[c,d]}$, whence $\| u \|_{L_q[c,d]} \leqslant (d-c)^{1+(1/q)-(1/p)} \| Du \|_{L_p[c,d]}$. Q.E.D.

The result of Lemma 3.3 can be immediately applied as follows:

COROLLARY 3.4. *Given* $f \in W_p^k[a, b]$ *with* $0 \leqslant k < 2m$, *and given* $\Delta \in \mathscr{P}_\sigma(a, b)$, *let* g *be the unique interpolant of* f *in* $\mathrm{Sp}(4m+1, 2m+1, \Delta)$, *such that*

$$\begin{aligned} D^j(f-g)(x_i) &= 0, \qquad 0 \leqslant j \leqslant k-1 \quad \textit{if} \quad k > 0, \quad 0 \leqslant i \leqslant N, \\ D^k(f_h - g)(x_i) &= 0, \qquad 0 \leqslant i \leqslant N, \\ D^j g(x_i) &= 0, \qquad k < j \leqslant 2m, \quad 0 \leqslant i \leqslant N. \end{aligned} \tag{3.6}$$

If $h = \bar{\Delta}$ *and if* $k > 0$, *then*

$$K(\bar{\Delta})^{k-j+(1/q)-(1/p)} \| D^k(f-g)\|_{L_p[a,b]} \geqslant \| D^j(f-g)\|_{L_q[a,b]}, \qquad 0 \leqslant j \leqslant k-1, \quad p \leqslant q \leqslant \infty. \tag{3.7}$$

Proof. If $k > 0$ and if $0 \leqslant j \leqslant k-1$, then from (3.6), $D^j(f-g)(x_i) = 0$, $0 \leqslant i \leqslant N$. Since $D^j(f-g) \in W_p^{k-j}[a, b]$, the inequality of (3.5) can be applied on each interval $[x_i, x_{i+1}]$, $0 \leqslant i \leqslant N-1$, of $[a, b]$, which gives, for $p \leqslant q \leqslant \infty$,

$$\begin{aligned} \| f-g \|_{L_q[a,b]} &\leqslant \bar{\Delta} \, \| D(f-g)\|_{L_q[a,b]} \leqslant \cdots \leqslant (\bar{\Delta})^{k-1} \| D^{k-1}(f-g)\|_{L_q[a,b]} \\ &\leqslant K(\bar{\Delta})^{k+(1/q)-(1/p)} \| D^k(f-g)\|_{L_p[a,b]}, \end{aligned}$$

from which (3.7) follows. Q.E.D.

It is not difficult to show that the unique interpolant g in $\mathrm{Sp}(4m+1, 2m+1, \Delta)$ of f, in the sense of (3.6), has the following representation: for $x \in [x_i, x_{i+1}]$, and $h_i \equiv x_{i+1} - x_i$, then for $k > 0$,

$$\begin{aligned} g(x) = \sum_{j=0}^{k-1} \frac{D^j f(x_i)}{j!} (x - x_i)^j &+ \frac{h_i^k}{(k-1)!} \int_0^1 Q\left(\frac{x - x_i}{h_i}; t\right) D^k f(x_i + h_i t)\, dt \\ &+ h_i^k \left[D^k f_h(x_i) \cdot \phi_{0,k}\left(\frac{x - x_i}{h_i}\right) + D^k f_h(x_{i+1}) \cdot \phi_{1,k}\left(\frac{x - x_i}{h_i}\right) \right], \end{aligned} \tag{3.8}$$

where (cf. Swartz and Varga [11]) $\phi_{0,k}(x)$ and $\phi_{1,k}(x)$ are the unique polynomials of degree $4m+1$ such that

$$D^j \phi_{i,k}(0) = \delta_{j,k} \cdot \delta_{i,0}, \qquad D^j \phi_{i,k}(1) = \delta_{j,k} \cdot \delta_{i,1}, \qquad 0 \leqslant j \leqslant 2m, \quad i = 0, 1,$$

and where $Q(y; t)$, for $t \in [0, 1]$, is the unique polynomial interpolation, as a function of y, of $(y - t)_+^{k-1}$ such that $Q(y; t)$ is a polynomial of degree $4m + 1$ in y with

$$D_y Q(y_l ; t) = \begin{cases} D_y^j (y_l - t)_+^{k-1}, & 0 \leqslant j < k \\ 0, & k \leqslant j \leqslant 2m \end{cases}, \qquad l = 0 \text{ or } 1,$$

where $y_0 = 0$ and $y_1 = 1$, and D_y denotes differentiation with respect to the first variable y. Similarly, the unique interpolant $\tilde{g}$ in $\mathrm{Sp}(4m + 1, 2m + 1, \Delta)$ of f_h, in the sense of (3.1), has the following representation: for $x \in [x_i, x_{i+1}]$ and for $k > 0$,

$$\tilde{g}(x) = \sum_{j=0}^{k-1} \frac{D^j f_h(x_i)}{j!} (x - x_i)^j + \frac{h_i^k}{(k-1)!} \int_0^1 Q\left(\frac{x - x_i}{h_i}; t\right) D^k f_h(x_i + h_i t)\, dt$$
$$+ h_i^k \left[D^k f_h(x_i) \cdot \phi_{0,k}\left(\frac{x - x_i}{h_i}\right) + D^k f_h(x_i) \cdot \phi_{1,k}\left(\frac{x - x_i}{h_i}\right)\right]. \tag{3.9}$$

For the case $k = 0$, the representations of (3.8) and (3.9) remain valid with the sum and integral terms deleted.

With these representations for g and $\tilde{g}$, we now prove the main result of this section, which will be repeatedly used in subsequent developments.

THEOREM 3.5. *Given $f \in W_p^k[a, b]$ with $0 \leqslant k < 2m$, and given $\Delta \in \mathscr{P}_\sigma(a, b)$, let g be the unique interpolant of f in* $\mathrm{Sp}(4m + 1, 2m + 1, \Delta)$, *in the sense of* (3.6). *Then, with $h = \bar{\Delta}$,*

$$K(\bar{\Delta})^{k-j+(1/q)-(1/p)}\, \omega_p(D^k f, \bar{\Delta})$$
$$\geqslant \begin{cases} \| D^j(f - g)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k - 1 \quad \text{if} \quad k > 0, \quad p \leqslant q \leqslant \infty, \\ \| D^k(f - g)\|_{L_p[a,b]}, & j = k, \quad p = q, \\ \| D^j g \|_{L_q[a,b]}, & k < j \leqslant 2m, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{3.10}$$

Proof. Assume first that $k = 0$. If $\tilde{g}$ is the unique interpolant of f_h in $\mathrm{Sp}(4m + 1, 2m + 1, \Delta)$, in the sense of (3.1), it follows from (3.6) that $\tilde{g} = g$. Hence, the inequalities of (3.3) for the case $j = k = 0$ directly establish the second and third inequalities of (3.10).

Next, assume that $0 < k < 2m$. To establish the first and second inequalities of (3.10), it is sufficient, because of (3.7) of Corollary 3.4, to show that

$$\| D^k(f - g)\|_{L_p[a,b]} \leqslant K\omega_p(D^k f, \bar{\Delta}). \tag{3.11}$$

Again, if $\tilde{g}$ is the unique interpolant of f_h in $\mathrm{Sp}(4m+1, 2m+1, \Delta)$ in the sense of (3.1), then by the triangle inequality,

$$\| D^k(f-g)\|_{L_p[a,b]} \leqslant \| D^k(f-\tilde{g})\|_{L_p[a,b]} + \| D^k(\tilde{g}-g)\|_{L_p[a,b]}. \quad (3.12)$$

The first term on the right of (3.12) is, from the first inequality of (3.3) of Lemma 3.2, bounded above by $K\omega_p(D^kf, \bar{\Delta})$. To bound the last term of (3.12), we make use of the representations of (3.8) and (3.9) for g and $\tilde{g}$. For any $x \in [x_i, x_{i+1}]$, and for any $k \leqslant j \leqslant 2m$, it follows from (3.8) and (3.9) that

$$D^j(\tilde{g}-g)(x) = \frac{h_i^{k-j}}{(k-1)!} \int_0^1 D_y{}^j Q\left(\frac{x-x_i}{h_i}; t\right) \{D^kf_h(x_i+h_it) - D^kf(x_i+h_it)\}\, dt.$$

Because $D_y{}^l Q(y; t)$ is uniformly bounded on $[0,1] \times [0,1]$ for any $0 \leqslant l \leqslant 2m$, then by Hölder's inequality,

$$| D^j(\tilde{g}-g)(x)| \leqslant K h_i^{k-j-(1/p)} \left(\int_{x_i}^{x_{i+1}} | D^kf_h(u) - D^kf(u)|^p\, du\right)^{1/p}, \quad x \in [x_i, x_{i+1}].$$

Upon integrating the above expression with respect to x, summing on i, $0 \leqslant i \leqslant N-1$, and upon applying Jensen's inequality, it follows that

$$\| D^j(\tilde{g}-g)\|_{L_q[a,b]} \leqslant K(\bar{\Delta})^{k-j+(1/q)-(1/p)} \| D^k(f_h - f)\|_{L_p[a,b]}, \quad k \leqslant j \leqslant 2m, \quad p \leqslant q \leqslant \infty.$$

Using (2.8), this implies that

$$\| D^j(\tilde{g}-g)\|_{L_q[a,b]} \leqslant K(\bar{\Delta})^{k-j+(1/q)-(1/p)}\, \omega_p(D^kf, \bar{\Delta}), \quad k \leqslant j \leqslant 2m, \quad p \leqslant q \leqslant \infty. \quad (3.13)$$

Thus, with $j = k$ and $p = q$, then $\| D^k(\tilde{g}-g)\|_{L_p[a,b]} \leqslant K\omega_p(D^kf, \bar{\Delta})$, which establishes the first and second inequalities of (3.10).

Finally, to establish the third inequality of (3.10) when $0 < k < 2m$ and $p \leqslant q \leqslant \infty$, we have by the triangle inequality that

$$\| D^j g\|_{L_q[a,b]} \leqslant \| D^j(g-\tilde{g})\|_{L_q[a,b]} + \| D^j\tilde{g}\|_{L_q[a,b]}, \qquad k < j \leqslant 2m. \quad (3.14)$$

The inequality of (3.13) then suitably bounds the first term on the right of (3.14), and the second inequality of (3.3) then suitably bounds the last term of (3.14). Q.E.D.

4. L-Spline Interpolation

Here, as well as in subsequent developments, we make use of the following modified convention. If $\Delta \in \mathscr{P}(a, b)$ and if $D^j g \in L_q[x_i, x_{i+1}]$ for each subinterval $[x_i, x_{i+1}]$ of $[a, b]$ defined by Δ, then $\| D^j g \|_{L_q[a,b]}$ is defined by

$$\begin{aligned} \| D^j g \|_{L_q[a,b]} &= \left(\sum_{i=0}^{N-1} \| D^j g \|^q_{L_q[x_i, x_{i+1}]} \right)^{1/q}, \qquad 1 \leqslant q < \infty, \\ &= \max(\| D^j g \|_{L_\infty[x_i, x_{i+1}]} : 0 \leqslant i \leqslant N-1), \qquad q = \infty. \end{aligned} \tag{4.1}$$

For the L-spline spaces $\mathrm{Sp}(L, \Delta, z)$ as defined in Section 2, we state an interpolation result of Swartz and Varga [11, Corollary 3.6].

Lemma 4.1. *Given $f \in W_p^{k+1}[a, b]$ with $0 \leqslant k < 2m$ and $2 \leqslant p \leqslant \infty$, and given $\Delta \in \mathscr{P}_\sigma(a, b)$, let s be the unique interpolant of f in $\mathrm{Sp}(L, \Delta, z)$ such that for $z_0 = m = z_N$,*

$$\begin{aligned} D^j(f - s)(x_i) &= 0, \qquad 0 \leqslant j \leqslant \min(k, z_i - 1), \quad 0 \leqslant i \leqslant N, \\ D^j s(x_i) &= 0, \qquad \text{if} \quad k < j \leqslant z_i - 1, \quad 0 \leqslant i \leqslant N, \end{aligned} \tag{4.2}$$

then,

$$K(\bar{\Delta})^{k+1-j+(1/q)-(1/2)} \| f \|_{W_p^{k+1}[a,b]} \geqslant \begin{cases} \| D^j(f - s) \|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \quad p \leqslant q \leqslant \infty, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m - 1, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{4.3}$$

For polynomial splines, i.e., $L = D^m$, $\| f \|_{W_p^{k+1}[a,b]}$ can be replaced in (4.3) by $\| D^{k+1} f \|_{L_p[a,b]}$.

The following application of Theorem 3.5 is then an improvement of the above result.

Theorem 4.2. *Given $f \in W_p^{k}[a, b]$ with $0 \leqslant k < 2m$ and $2 \leqslant p \leqslant \infty$, and given $\Delta \in \mathscr{P}_\sigma(a, b)$, let s be the unique interpolant of f in $\mathrm{Sp}(L, \Delta, z)$ such that for $z_0 = m = z_N$,*

$$\begin{aligned} D^j(f - s)(x_i) &= 0, \qquad 0 \leqslant j \leqslant \min(k - 1, z_i - 1) \;\; \text{if} \;\; k > 0, \;\; 0 \leqslant i \leqslant N, \\ D^k(f_h - s)(x_i) &= 0, \qquad \text{if} \quad k \leqslant z_i - 1, \quad 0 \leqslant i \leqslant N, \\ D^j s(x_i) &= 0, \qquad \text{if} \quad k < j \leqslant z_i - 1, \quad 0 \leqslant i \leqslant N. \end{aligned} \tag{4.4}$$

Then, with $h = \bar{\Delta}$,

$$K(\bar{\Delta})^{k-j+(1/q)-(1/2)} \{ \omega_p(D^k f, \bar{\Delta}) + \bar{\Delta} \cdot \| f \|_{W_p^k[a,b]} \} \geqslant \begin{cases} \| D^j(f - s) \|_{L_q[a,b]}, & 0 \leqslant j \leqslant k - 1 \quad \text{if} \quad k > 0, \quad p \leqslant q \leqslant \infty, \\ \| D^k(f - s) \|_{L_p[a,b]}, & j = k, \quad p = q, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m - 1, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{4.5}$$

For polynomial splines, i.e., $L = D^m$, the term $\bar{\Delta} \cdot \|f\|_{W_p^k[a,b]}$ can be deleted in (4.5).

Proof. Let g be the unique interpolant of f in Sp$(4m+1, 2m+1, \Delta)$ in the sense of (3.6). Consider first the special case $k = 0$. By the triangle inequality,

$$\|f - s\|_{L_p[a,b]} \leqslant \|f - f_h\|_{L_p[a,b]} + \|f_h - g\|_{L_p[a,b]} + \|g - s\|_{L_p[a,b]}. \tag{4.6}$$

From (2.6), the first term on the right of (4.6) is bounded above by $\omega_p(f, \bar{\Delta})$. Next, since $f_h \in W_p^1[a, b]$ and since the unique interpolant $\hat{g}$ of f_h in Sp$(4m+1, 2m+1, \Delta)$ in the sense of (3.1) is such that $\hat{g} = g$, then upon applying (2.9) and (3.2) of Lemma 3.1 for the case $q = p$ and $j = k = 0$, we similarly have that $\|f_h - g\|_{L_p[a,b]} \leqslant K\omega_p(f, \bar{\Delta})$. Next, since s is also by definition the unique interpolant of g in Sp(L, Δ, z) in the sense of (4.2) with $k = 0$, the bounds of (4.3) of Lemma 4.1 imply that

$$\|g - s\|_{L_p[a,b]} \leqslant K(\bar{\Delta})^{1+(1/p)-(1/2)} \|g\|_{W_p^1[a,b]}. \tag{4.7}$$

Now, $\|Dg\|_{L_p[a,b]} \leqslant K\|Df_h\|_{L_p[a,b]}$ from (3.2), so that $\|Dg\|_{L_p[a,b]} \leqslant K(\bar{\Delta})^{-1}\,\omega_p(f, \bar{\Delta})$, using (2.9). Similarly from (3.10),

$$\|g\|_{L_p[a,b]} \leqslant \|g - f\|_{L_p[a,b]} + \|f\|_{L_p[a,b]} \leqslant K\omega_p(f, \bar{\Delta}) + \|f\|_{L_p[a,b]}.$$

Thus,

$$\|g\|_{W_p^1[a,b]} \equiv \|g\|_{L_p[a,b]} + \|Dg\|_{L_p[a,b]} \leqslant K(\bar{\Delta})^{-1} \cdot \omega_p(f, \bar{\Delta}) + \|f\|_{L_p[a,b]}, \tag{4.8}$$

and combining the above inequality with (4.7) yields

$$\|g - s\|_{L_p[a,b]} \leqslant K(\bar{\Delta})^{(1/p)-(1/2)} (\omega_p(f, \bar{\Delta}) + \bar{\Delta} \cdot \|f\|_{L_p[a,b]}).$$

The above bound, in conjunction with the other bounds for (4.6), then gives that $\|f - s\|_{L_p[a,b]} \leqslant K(\bar{\Delta})^{(1/p)-(1/2)} \{\omega_p(f, \bar{\Delta}) + \bar{\Delta} \cdot \|f\|_{L_p[a,b]}\}$, the desired second inequality of (4.5) for the case $k = 0$. Finally, to obtain the desired third inequality of (4.5) for the case $k = 0$, one simply combines the second inequality of (4.3), i.e., $\|D^j s\|_{L_q[a,b]} \leqslant K(\bar{\Delta})^{1-j+(1/q)-(1/2)} \|g\|_{W_p^1[a,b]}$, with (4.8).

Assume now that $0 < k < 2m$. Write $D^j(f - s) = D^j(f - g) + D^j(g - s)$ or $D^j s = D^j(s - g) + D^j g$, where g is the unique interpolant of f in Sp$(4m+1, 2m+1, \Delta)$ in the sense of (3.6). Because of the inequalities of (3.10) of Theorem 3.5, it suffices to suitably bound $D^j(g - s)$ and $D^j g$ to establish the inequalities of (4.5). Now, since s is also the unique interpolant of g in $\mathrm{Sp}(L, \Delta, z)$ in the sense of (4.2), then applying (4.3),

$$K(\bar{\Delta})^{k+1-j+(1/q)-(1/2)} \| g \|_{W_p^{k+1}[a,b]} \geqslant \begin{cases} \| D^j(g-s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \quad p \leqslant q \leqslant \infty, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m-1, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{4.9}$$

Next, since $\| D^j g \|_{L_p[a,b]} \leqslant \| D^j(f-g)\|_{L_p[a,b]} + \| D^j f \|_{L_p[a,b]}$ for $0 \leqslant j \leqslant k$, the bounds of (3.10) directly give that $\| D^j g \|_{L_p[a,b]} \leqslant K(\bar{\Delta})^{k-j} \omega_p(D^k f, \bar{\Delta}) + \| D^j f \|_{L_p[a,b]}$ for $0 \leqslant j \leqslant k$, and that $\| D^{k+1} g \|_{L_p[a,b]} \leqslant K(\bar{\Delta})^{-1} \omega_p(D^k f, \bar{\Delta})$. Hence, upon adding, $\| g \|_{W_p^{k+1}[a,b]} \leqslant K\{(\bar{\Delta})^{-1} \omega_p(D^k f, \bar{\Delta}) + \| f \|_{W_p^k[a,b]}\}$. Thus, substituting this bound in (4.9) then gives the desired inequalities of (4.5) for the case $0 < k < 2m$. Q.E.D.

It is worth noting that if $f \in W_p^k[a, b]$ with $0 \leqslant k < 2m$ and $2 \leqslant p \leqslant \infty$, and if $\{\Delta_i\}_{i=1}^{\infty} \in \mathscr{P}_\sigma(a, b)$ with $\lim_{i\to\infty} \bar{\Delta}_i = 0$, the above result of (4.5) of Theorem 4.2 does not necessarily imply that $\lim_{i\to\infty} \| D^k(f - s_i)\|_{L_p[a,b]} = 0$, where s_i is the unique interpolant of f in $\mathrm{Sp}(L, \Delta_i, z^{(i)})$, in the sense of (4.4). However, if $p = q = 2$ or if $\omega_p(D^k f, \delta) \leqslant K\delta^{(1/2)-(1/p)} F(\delta)$ where $F(\delta) \to 0$ as $\delta \to 0$, then $\lim_{i\to\infty} \| D^k(f - s_i)\|_{L_p[a,b]} = 0$. However, we shall later show in Section 6 that, for smooth *polynomial* splines over *uniform* meshes, this limit is zero without further restrictions.

Another case in which $\lim_{i\to\infty} \| D^k(f - s_i)\|_{L_p[a,b]} = 0$ for $f \in W_p^k[a, b]$ is that of *Hermite L-splines* (cf. Swartz and Varga [11, Section 6]), i.e., for the L-spline space $\mathrm{Sp}(L, \Delta, \hat{z})$ for which

$$\hat{z} = (\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_N) \qquad \text{with} \quad \hat{z}_i = m, \quad 0 \leqslant i \leqslant N. \tag{4.10}$$

The following result, derived in [11], but stated in a slightly weaker form in [11, Corollary 6.2], is the starting point.

LEMMA 4.3. *Given $f \in W_p^{k+1}[a, b]$ with $0 \leqslant k < 2m$ and $1 \leqslant p \leqslant \infty$, and given $\Delta \in \mathscr{P}_\sigma(a, b)$, let s be the unique interpolant of f in* (cf. (4.10)) $\mathrm{Sp}(L, \Delta, \hat{z})$ *in the sense of*

$$\begin{aligned} D^j(f-s)(x_i) &= 0, \qquad 0 \leqslant j \leqslant \min(k, m-1), \quad 0 \leqslant i \leqslant N, \\ D^j s(x_i) &= 0, \qquad \text{if} \quad k < j \leqslant m-1, \quad 0 \leqslant i \leqslant N. \end{aligned} \tag{4.11}$$

Then,

$$K(\bar{\Delta})^{k+1-j+(1/q)-(1/p)} \| f \|_{W_p^{k+1}[a,b]} \geqslant \begin{cases} \| D^j(f-s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \quad p \leqslant q \leqslant \infty, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m-1, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{4.12}$$

For polynomial splines, i.e., $L = D^m$, the term $\| f \|_{W_p^{k+1}[a,b]}$ can be replaced in (4.12) by $\| D^{k+1} f \|_{L_p[a,b]}$.

With Lemma 4.3 and Theorem 3.5, we then establish the following:

THEOREM 4.4. *Given $f \in W_p^k[a, b]$ with $0 \leqslant k < 2m$ and $1 \leqslant p \leqslant \infty$, and given $\Delta \in \mathscr{P}_\sigma(a, b)$, let s be the unique interpolant of f in* $\mathrm{Sp}(L, \Delta, \hat{z})$ *in the sense of* (4.4) *with $\hat{z}_i = m$, $0 \leqslant i \leqslant N$. Then, with $h = \bar{\Delta}$,*

$$K(\bar{\Delta})^{k-j+(1/q)-(1/p)} \{\omega_p(D^k f, \bar{\Delta}) + (\bar{\Delta})^{2m-k} \| f \|_{W_p^k[a,b]}\}$$
$$\geqslant \begin{cases} \| D^j(f-s) \|_{L_q[a,b]}, & 0 \leqslant j \leqslant k-1 \quad \text{if} \quad k > 0, \quad p \leqslant q \leqslant \infty, \\ \| D^k(f-s) \|_{L_p[a,b]}, & j = k, \quad p = q, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m-1, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{4.13}$$

For polynomial splines, i.e., $L = D^m$, the term $(\bar{\Delta})^{2m-k} \| f \|_{W_p^k[a,b]}$ can be deleted in (4.13).

Proof. Assume $k > 0$, and let $g \in \mathrm{Sp}(4m+1, 2m+1, \Delta)$ be the unique interpolant of f in the sense of (3.6). Writing $D^j(f-s) = D^j(f-g) + D^j(g-s)$, it suffices from (3.10) to suitably bound $\| D^j(g-s) \|_{L_q[a,b]}$ for $0 \leqslant j \leqslant k-1$ and $p \leqslant q \leqslant \infty$. Next, s is by definition also the unique interpolant of g in $\mathrm{Sp}(L, \Delta, \hat{z})$, both in the sense of (4.11), as well as in the sense of

$$D^j(g-s)(x_i) = 0, \qquad 0 \leqslant j \leqslant m-1, \quad 0 \leqslant i \leqslant N.$$

As such, it follows from Swartz and Varga [11, Eq. (6.4)] that, for $x \in [x_i, x_{i+1}]$ and for $0 \leqslant j \leqslant 2m-1$,

$$| D^j(g-s)(x) | \leqslant K h_i^{2m-j} \{\| D^{2m} g \|_{L_\infty[x_i, x_{i+1}]} + \| D^{2m} s \|_{L_\infty[x_i, x_{i+1}]}\},$$

where $h_i \equiv x_{i+1} - x_i$. Thus,

$$\| D^j(g-s) \|_{L_q[x_i, x_{i+1}]} \leqslant K h_i^{2m-j+1/q} \{\| D^{2m} g \|_{L_\infty[x_i, x_{i+1}]} + \| D^{2m} s \|_{L_\infty[x_i, x_{i+1}]}\}. \tag{4.14}$$

Since $L * Ls(x) = 0$ in (x_i, x_{i+1}) and $c_m(x) \geqslant \delta > 0$ for $x \in [a, b]$, we have, as in [11], that

$$\| D^{2m} s \|_{L_\infty[x_i, x_{i+1}]} \leqslant K \sum_{l=0}^{2m-1} \| D^l s \|_{L_\infty[x_i, x_{i+1}]},$$

which, with the triangle inequality, yields

$$\| D^{2m} s \|_{L_\infty[x_i, x_{i+1}]} \leqslant K \sum_{l=0}^{2m-1} \{\| D^l(s-g) \|_{L_\infty[x_i, x_{i+1}]} + \| D^l g \|_{L_\infty[x_i, x_{i+1}]}\}. \tag{4.15}$$

Because of the local character of Hermite L-spline interpolation, we can apply (4.12) of Lemma 4.3 with $q = \infty$, $k = 2m - 1$, and $[a, b] = [x_i, x_{i+1}]$ to bound the first term on the right of (4.15), i.e., $\| D^l(s - g)\|_{L_\infty[x_i,x_{i+1}]} \leqslant K(\bar{\Delta})^{2m-l-(1/p)} \| g \|_{W_p^{2m}[x_i,x_{i+1}]}$ for $0 \leqslant l \leqslant 2m - 1$. The fact that g is a polynomial on $[x_i, x_{i+1}]$ similarly allows us to bound the last term of (4.15) by (cf. Swartz [10]) $\| D^l g \|_{L_\infty[x_i,x_{i+1}]} \leqslant K(\underline{\Delta})^{-(1/p)} \| D^l g \|_{L_p[x_i,x_{i+1}]}$. With these bounds, (4.15) becomes

$$\| D^{2m} s \|_{L_\infty[x_i,x_{i+1}]} \leqslant K(\bar{\Delta})^{1-(1/p)} \| g \|_{W_p^{2m}[x_i,x_{i+1}]} + (\underline{\Delta})^{-(1/p)} \| g \|_{W_p^{2m-1}[x_i,x_{i+1}]} .$$

With the above bound, (4.14) becomes

$$\| D^j(g - s)\|_{L_q[x_i,x_{i+1}]} \leqslant K(\bar{\Delta})^{2m-j+(1/q)-(1/p)} \| g \|_{W_p^{2m}[x_i,x_{i+1}]} . \qquad (4.16)$$

Summing now on i and applying Jensen's inequality yields for $0 \leqslant j \leqslant 2m - 1$ and $p \leqslant q$,

$$\| D^j(g - s)\|_{L_q[a,b]} \leqslant K(\bar{\Delta})^{2m-j+(1/q)-(1/p)} \| g \|_{W_p^{2m}[a,b]} . \qquad (4.17)$$

To complete the proof, write

$$\| g \|_{W_p^{2m}[a,b]} = \sum_{j=0}^{2m} \| D^j g \|_{L_p[a,b]}$$

$$\leqslant \sum_{j=0}^{k} \{\| D^j(g - f)\|_{L_p[a,b]} + \| D^j f \|_{L_p[a,b]}\} + \sum_{j=k+1}^{2m} \| D^j g \|_{L_p[a,b]} .$$

Applying (3.10) of Theorem 3.5, then gives that

$$\| g \|_{W_p^{2m}[a,b]} \leqslant K(\bar{\Delta})^{k-2m}\, \omega_p(D^k f, \bar{\Delta}) + \| f \|_{W_p^{k}[a,b]} .$$

The above inequality, when combined with (4.16) and (3.10) of Theorem 3.5, yields the desired first two inequalities of (4.13) for the case $k > 0$. To obtain the third inequality of (4.13) for the case $k > 0$, it suffices to write $D^j s = D^j(s - g) + D^j g$, and to apply the same analysis. The case $k = 0$ can be similarly established with obvious modifications in the above analysis. Q.E.D.

As an immediate consequence of Theorem 4.4, we have the following:

COROLLARY 4.5. *With the assumptions of Theorem* 4.4, *let* $\{\Delta_i\}_{i=1}^{\infty} \in \mathscr{P}_\sigma(a, b)$ *with* $\lim_{i\to\infty} \bar{\Delta}_i = 0$, *and let* s_i *be the unique interpolant of* f *in the Hermite* L*-spline space* $\mathrm{SP}(L, \Delta_i, \hat{z}^{(i)})$ *in the sense of* (4.11). *Then, with the additional hypothesis* (cf. (2.3)) *that* $D^k f \in C^0[a, b]$ *if* $p = \infty$,

$$\lim_{i\to\infty} \| D^k(f - s_i)\|_{L_p[a,b]} = 0. \qquad (4.18)$$

It is interesting to remark that if $f \in W_p{}^k[a, b]$ with $m \leqslant k < 2m$, then the unique interpolant s of f in $\mathrm{Sp}(L, \Delta, \mathfrak{z})$ in the sense of (4.4) is *independent* of f_h, and thus, this interpolant s is identical with the one considered in Swartz and Varga [11] in their Theorem 6.1 and Corollary 6.2. Comparing these results, we see that the special case $p = \infty$ of Theorem 4.4 above essentially reduces to Theorem 6.1 of [11], the only change being that the hypothesis that $f \in C^k[a, b]$ in [11, Theorem 6.1] is weakened to $f \in W_\infty{}^k[a, b]$ in Theorem 4.4. Moreover, when $m \leqslant k < 2m$, Theorem 4.4 above sharpens the corresponding result of Lemma 4.3 (cf. [11, Corollary 6.2]), with $k + 1$ replaced by k, in the following ways: (i) an upper bound for $\| D^k(f - s)\|_{L_p[a,b]}$ is determined in Theorem 4.4 which is not provided by Lemma 4.3, and (ii) the quantity $\|f\|_{W_p{}^k[a,b]}$ in (4.12) of Lemma 4.3 is replaced in (4.13) of Theorem 4.4 by the smaller quantity (when $\bar{\Delta} \to 0$): $\{\omega_p(D^k f, \bar{\Delta}) + (\bar{\Delta})^{2m-k} \|f\|_{W_p{}^k[a,b]}\}$. Finally, if $f \in W_p^{2m}[a, b]$, a case already covered by Lemma 4.3, we remark that the use of (2.4) in conjunction with the case $k = 2m - 1$ of (4.13) of Theorem 4.4 gives the same upper bounds as in (4.12), i.e.,

$$K(\bar{\Delta})^{2m-j+(1/q)-(1/p)} \|f\|_{W_p^{2m}[a,b]} \geqslant \| D^j(f - s)\|_{L_q[a,b]}, \qquad 0 \leqslant j \leqslant 2m - 1, \quad p \leqslant q \leqslant \infty. \tag{4.19}$$

We remark that the exponents of $\bar{\Delta}$, as given in (4.13), cannot in general be improved. This can be seen from counterexamples in Schultz and Varga [8], Birkhoff, Schultz, and Varga [4], and Subbotin [12]. We further remark that upper bounds for interpolation errors in terms of norms of Besov spaces, as described in Hedstrom and Varga [5], could also be readily carried out here, but such extensions will not be considered further here.

5. Stability of L-Spline Interpolation

As discussed in Swartz and Varga [11], one can suitably perturb the data defining an L-spline interpolant without affecting the nature of the original error bounds for this interpolation. Such results are referred to as *stability results* for L-spline interpolation (cf. [11]). We now give such a stability result for the Hermite L-spline interpolation of Theorem 4.4 (which covers the case of Lagrange interpolation, as discussed in [11]). Its proof is based on the following slightly improved result of [11, Corollary 6.4].

Lemma 5.1. *Given $f \in W_p^{k+1}[a, b]$ with $0 \leqslant k < 2m$ and $1 \leqslant p \leqslant \infty$, and given $\Delta \in \mathscr{P}_\sigma(a, b)$, let s be the unique interpolant of f in the Hermite L-spline space* $\mathrm{Sp}(L, \Delta, \mathfrak{z})$ *such that*

$$D^j s(x_i) = \alpha_{i,j}, \qquad 0 \leqslant j \leqslant m - 1, \quad 0 \leqslant i \leqslant N, \tag{5.1}$$

where it is assumed that functions $F_i(f, \bar{\Delta})$, $0 \leqslant i \leqslant N$, *exist such that*

$$K(\bar{\Delta})^{k+1-j} F_i(f, \bar{\Delta}) \geqslant \begin{cases} | D^j f(x_i) - \alpha_{i,j} |, & 0 \leqslant j \leqslant \min(k, m-1), \quad 0 \leqslant i \leqslant N, \\ | \alpha_{i,j} |, & \text{if} \quad k < j \leqslant m-1, \quad 0 \leqslant i \leqslant N. \end{cases} \tag{5.2}$$

With

$$\| F \|_r \equiv \begin{cases} \left(\bar{\Delta} \sum_{i=0}^{N} F_i^r(f, \bar{\Delta})\right)^{1/r}, & 1 \leqslant r < \infty, \\ \max(F_i(f, \bar{\Delta}): 0 \leqslant i \leqslant N), & r = \infty, \end{cases} \tag{5.3}$$

then

$$K(\bar{\Delta})^{k+1-j+(1/q)-(1/p)} \{\| f \|_{W_p^{k+1}[a,b]} + \| F \|_p\} \geqslant \begin{cases} \| D^j(f-s) \|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \quad p \leqslant q \leqslant \infty, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m-1, \mathrm{M}p \leqslant q \leqslant \infty. \end{cases} \tag{5.4}$$

For polynomial splines, i.e., $L = D^m$, $\| f \|_{W_p^{k+1}[a,b]}$ can be replaced in (5.4) by $\| D^{k+1} f \|_{L_p[a,b]}$.

The following application of Theorem 3.5 is then an improvement of the above result.

THEOREM 5.2. *Given* $f \in W_p^k[a, b]$ *with* $0 \leqslant k < 2m$ *and* $1 \leqslant p \leqslant \infty$, *and given* $\Delta \in \mathscr{P}_\sigma(a, b)$, *let s be the unique interpolant in the Hermite L-spline space* $\mathrm{Sp}(L, \Delta, \mathfrak{z})$ *in the sense of* (5.1), *where it is assumed that functions* $F_i(f, \bar{\Delta})$, $0 \leqslant i \leqslant N$, *exist such that*

$$K(\bar{\Delta})^{k-j} F_i(f, \bar{\Delta}) \geqslant \begin{cases} | D^j f(x_i) - \alpha_{i,j} |, & 0 \leqslant j \leqslant \min(k-1, m-1) \ \text{if} \ k > 0, \ 0 \leqslant i \leqslant N, \\ | D^k f_h(x_i) - \alpha_{i,k} |, & j = k, \ \text{if} \ k \leqslant m-1, \quad 0 \leqslant i \leqslant N, \\ | \alpha_{i,j} |, & \text{if} \quad k < j \leqslant m-1, \quad 0 \leqslant i \leqslant N. \end{cases} \tag{5.5}$$

Then, with $h = \bar{\Delta}$,

$$K(\bar{\Delta})^{k-j+(1/q)-(1/p)} \{(\omega_p(D^k f, \bar{\Delta}) + \| F \|_p + \bar{\Delta} \cdot \| f \|_{W_p^k[a,b]})\} \geqslant \begin{cases} \| D^j(f-s) \|_{L_q[a,b]}, & 0 \leqslant j \leqslant k-1 \quad \text{if} \quad k > 0, \quad p \leqslant q \leqslant \infty, \\ \| D^k(f-s) \|_{L_p[a,b]}, & j = k, \quad p = q, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m-1, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{5.6}$$

For polynomial splines, i.e., $L = D^m$, the term $\| f \|_{W_p^k[a,b]}$ can be replaced by $\| D^k f \|_{L_p[a,b]}$.

Proof. Because of the similarity with past proofs, it is necessary only to outline the basic idea of this proof. First, let u be the unique element in $\mathrm{Sp}(4m+1, 2m+1, \Delta)$ such that

$$\begin{aligned} D^j u(x_i) &= D^j f(x_i) - \alpha_{i,j}, \qquad 0 \leqslant j \leqslant \min(k-1, m-1) \quad \text{if} \quad k > 0, \\ &\qquad\qquad\qquad\qquad\qquad 0 \leqslant i \leqslant N, \\ &= D^k f_h(x_i) - \alpha_{i,k}, \qquad j = k \quad \text{if} \quad k \leqslant m-1, \quad 0 \leqslant i \leqslant N, \\ &= -\alpha_{i,j}, \qquad \text{if} \quad k < j \leqslant m-1, \quad 0 \leqslant i \leqslant N, \\ &= 0, \qquad m \leqslant j \leqslant 2m, \quad 0 \leqslant i \leqslant N, \end{aligned} \tag{5.7}$$

and let t be the unique interpolant of f in $\mathrm{Sp}(L, \Delta, \hat{z})$ in the sense of (4.4) with $\hat{z}_i = m$, $0 \leqslant i \leqslant N$. Thus, we can write that

$$f - s = (f - t) + u + [(t - s) - u], \tag{5.8}$$

and we see by definition that $t - s$ is the unique interpolant of u in $\mathrm{Sp}(L, \Delta, \hat{z})$ in the sense of (4.4). As such, we can directly apply the result of (4.12) of Theorem 4.4 to $(f - t)$, and the result of (5.4) of Lemma 5.1 to $[u - (t - s)]$. Because this last mentioned bound for $[u - (t - s)]$ depends, from (5.4), on $\| u \|_{W_p^{k+1}[a,b]}$, it is necessary to estimate $\| u \|_{W_p^{k+1}[a,b]}$. However, from the hypotheses of (5.5), Lemma 4.3 of [11], due to Swartz [10], shows that $\| u \|_{W_p^{k+1}[a,b]} \leqslant K(\bar{\Delta})^{-1} \| F \|_p$, from which (5.6) then follows. Q.E.D.

As previously mentioned, the case for the Lagrange interpolation of data, as described in [11], is effectively covered by the above stability result for Hermite L-spline interpolation. Specifically, assume that $f \in W_p^k[a, b]$ with $0 \leqslant k < 2m$, and that $\Delta \in \mathscr{P}_\sigma(a, b)$ with $N \geqslant 2m - 1$ (cf. (2.10)). Extending f to an element in $W_p^k[2a - b, 2b - a]$ via (2.1) and similarly extending the partition Δ to a partition $\tilde{\Delta}$ in $\mathscr{P}_\sigma(2a - b, 2b - a)$, we can associate with each knot x_i of Δ in $[a, b]$, $2m - 1$ consecutive knots of $\tilde{\Delta}$, say $x_{i+1}, x_{i+2}, \ldots, x_{i+2m-1}$, to its right. If $L_{2m-1,i}f$ denotes the Lagrange interpolation of f of degree $2m - 1$ in these consecutive knots, in the sense that

$$\begin{aligned} (L_{2m-1,i}f)(x_j) &= f(x_j), \qquad i \leqslant j \leqslant i + 2m - 1, \quad \text{if} \quad k > 0, \\ (L_{2m-1,i}f)(x_j) &= f_h(x_j), \qquad i \leqslant j \leqslant i + 2m - 1, \quad \text{if} \quad k = 0, \end{aligned} \tag{5.9}$$

then let s be the unique interpolant of f in the Hermite L-spline space $\mathrm{Sp}(L, \Delta, \hat{z})$ such that

$$D^j s(x_i) = D^j (L_{2m-1,i}f)(x_i), \qquad 0 \leqslant j \leqslant m - 1, \quad 0 \leqslant i \leqslant N. \tag{5.10}$$

In other words, s is the interpolant of (5.1) with $\alpha_{i,j} \equiv D^j(L_{2m-1,i}f)(x_i)$. From known error bounds for Lagrange interpolation (cf. [11, Corollary 4.2]), it can be shown that the conditions of (5.5) of Theorem 5.2 are fulfilled, and that $\| F \|_p \leqslant K\omega_p(D^k f, \bar{\Delta})$. This then establishes the following:

COROLLARY 5.3. *Given $f \in W_p^k[a, b]$ with $0 \leqslant k < 2m$ and $1 \leqslant p \leqslant \infty$, and given $\Delta \in \mathscr{P}_\sigma(a, b)$ with $N \geqslant 2m - 1$, let s be the unique Hermite L-spline interpolant of f in the Lagrange sense of* (5.10). *Then, with $h = \bar{\Delta}$ if $k = 0$,*

$$K(\bar{\Delta})^{k-j+(1/q)-(1/p)} \{\omega_p(D^k f, \bar{\Delta}) + \bar{\Delta} \cdot \|f\|_{W_p^k[a,b]}\}$$
$$\geqslant \begin{cases} \| D^j(f-s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k-1 \quad \text{if} \quad k > 0, \quad p \leqslant q \leqslant \infty, \\ \| D^k(f-s)\|_{L_p[a,b]}, & j = k, \quad p = q, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m-1, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{5.11}$$

It is interesting to note from the definition of the Lagrange interpolation in (5.9) that, for $f \in W_p^k[a, b]$ with $k > 0$, the Hermite L-spline interpolant s of (5.10) is *independent* of f_h. In this case, this interpolant s agrees with the Lagrange-type Hermite L-spline interpolant considered in [11, Corollary 6.4], and again, the above result of (5.11) of Corollary 5.3 sharpens the corresponding result of [11, Corollary 6.4].

A result similar to Theorem 5.2 can also be easily deduced for general L-spline interpolation, but for brevity, this is omitted. The following special case of Corollary 5.3, however, is included.

COROLLARY 5.4. *With the assumptions of Corollary 5.3, let $\{\Delta_i\}_{i=1}^{\infty} \in \mathscr{P}_\sigma(a, b)$ with $\lim_{i\to\infty} \bar{\Delta}_i = 0$, and let s_i be the unique interpolant of f in the Hermite L-spline space* $\mathrm{Sp}(L, \Delta_i, \hat{z}^{(i)})$ *in the Lagrange sense of* (5.10). *Then, with the additional hypothesis that $D^k f \in C^0[a, b]$ if $p = \infty$,*

$$\lim_{i\to\infty} \| D^k(f - s_i)\|_{L_p[a,b]} = 0. \tag{5.12}$$

6. POLYNOMIAL SPLINE INTERPOLATION OVER UNIFORM MESHES

As in Section 8 of Swartz and Varga [11], consider any set of $2m$ real point functionals $B = \{B_j\}_{j=0}^{2m-1}$ on $W_2^{2m}[a, b]$, called *boundary conditions*, of the form

$$B_j g = \sum_{i=0}^{2m-1} \{a_{j,i} D^i g(a) + b_{j,i} D^i g(b)\}, \qquad 0 \leqslant j \leqslant 2m-1, \tag{6.1}$$

where $g \in W_2^{2m}[a, b]$. If the $2m \times 4m$ matrix M is defined by

$$M = \begin{bmatrix} a_{0,0} & b_{0,0} & a_{0,1} & b_{0,1} & \cdots & b_{0,2m-1} \\ a_{1,0} & & & & & b_{1,2m-1} \\ \vdots & & & & & \vdots \\ a_{2m-1,0} & & & \cdots & & b_{2m-1,2m-1} \end{bmatrix}, \tag{6.2}$$

we assume that

$$\operatorname{rank} M = 2m, \tag{6.3}$$

i.e., the functionals $\{B_j\}_{j=0}^{2m-1}$ are linearly independent. We further assume that any $g \in W_2^{2m}[a, b]$ with $B_j g = 0$, $0 \leqslant j \leqslant 2m - 1$, satisfies

$$\int_a^b (D^m g(t))^2 \, dt = (-1)^m \int_a^b g(t) \cdot D^{2m} g(t) \, dt. \tag{6.4}$$

Finally, as in [11], there is no loss of generality in assuming that, by means elementary row operations, applied to M, the matrix M is in *lower reduced echelon form*, i.e.,

(i) every leading entry (from the right) of each row is unity;

(ii) every column containing a leading entry (from the right) has all other entries zero; (6.5)

(iii) if the leading entry (from the right) of row i is in column t_i, then $t_1 < t_2 < \cdots < t_{2m}$.

We remark that the elementary row operations which bring M into lower reduced echelon form leave the property of (6.4) invariant. We further remark that a special case of boundary conditions $B = \{B_j\}_{j=0}^{2m-1}$ which do satisfy (6.3)–(6.5), are given by the so-called *Hermite boundary conditions*, defined by

$$B_{2j} g = D^j g(a), \qquad B_{2j+1} g = D^j g(b), \quad 0 \leqslant j \leqslant m - 1. \tag{6.6}$$

Other examples of such boundary conditions satisfying (6.3)–(6.5) are cited in [11].

We now state a particular result of Swartz and Varga [11, Corollary 8.11].

LEMMA 6.1. *Given* $f \in W_p^{k+1}[a, b]$, *with* $0 \leqslant k < 2m$ *and* $2 \leqslant p \leqslant \infty$, *given* $\Delta \in \mathscr{P}_1(a, b)$ *with* $N > m$ (cf. (2.10)), *and given the point functionals* $\{B_j\}_{j=0}^{2m-1}$ *of the form* (6.1) *which satisfy* (6.3)–(6.5), *let s be the unique interpolant of f in* $\mathrm{Sp}(2m - 1, 2m - 1, \Delta)$, *in the following sense*:

$$\begin{aligned} (f - s)(x_i) &= 0, \qquad 1 \leqslant i \leqslant N - 1, \\ B_j s &= \sum_{i=0}^{k} \{a_{j,i} D^i f(a) + b_{j,i} D^i f(b)\}, \qquad 0 \leqslant j \leqslant 2m - 1. \end{aligned} \tag{6.7}$$

Then,

$$K(\bar{\Delta})^{k+1-j+(1/q)-(1/p)} \| D^{k+1} f \|_{L_p[a,b]} \geqslant \begin{cases} \| D^j(f - s) \|_{L_q[a,b]}, & 0 \leqslant j \leqslant k, \quad p \leqslant q \leqslant \infty, \\ \| D^j s \|_{L_q[a,b]}, & \text{if } \; k < j \leqslant 2m - 1, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{6.8}$$

The proof of the following result, based on Lemma 6.1 and Theorem 3.5, is similar to previous proofs given, and is therefore omitted.

THEOREM 6.2. *Given* $f \in W_p^{\,k}[a, b]$, *with* $0 \leqslant k < 2m$ *and* $2 \leqslant p \leqslant \infty$, *given* $\Delta \in \mathscr{P}_1(a, b)$ *with* $N > m$, *and given the point functionals* $\{B_j\}_{j=0}^{2m-1}$ *of the form* (6.1) *which satisfy* (6.3)–(6.5), *let* s *be the unique interpolant of* f *in* $\mathrm{Sp}(2m-1, 2m-1, \Delta)$, *in the following sense*:

$$(f - s)(x_i) = 0, \qquad 1 \leqslant i \leqslant N-1, \quad \text{if} \quad k > 0,$$

$$(f_h - s)(x_i) = 0, \qquad 1 \leqslant i \leqslant N-1, \quad \text{if} \quad k = 0,$$

$$B_j s = \sum_{i=0}^{k-1} \{a_{j,i} D^i f(a) + b_{j,i} D^i f(b)\} + \{a_{j,k} D^k f_h(a) + b_{j,k} D^k f_h(b)\} \tag{6.9}$$

$$0 \leqslant j \leqslant 2m-1, \quad \text{if} \quad k > 0,$$

$$B_j s = a_{j,0} f_h(a) + b_{j,0} f_h(b), \qquad 0 \leqslant j \leqslant 2m-1, \quad \text{if} \quad k = 0.$$

Then, with $h = \bar{\Delta}$,

$$K(\bar{\Delta})^{k-j+(1/q)-(1/p)}\, \omega_p(D^k f, \bar{\Delta})$$

$$\geqslant \begin{cases} \| D^j(f-s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k-1 \quad \text{if} \quad k > 0, \quad p \leqslant q \leqslant \infty, \\ \| D^k(f-s)\|_{L_p[a,b]}, & j = k, \quad p = q, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k < j \leqslant 2m-1, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{6.10}$$

COROLLARY 6.3. *With the assumptions of Theorem* 6.2, *let* $\{\Delta_i\}_{i=1}^{\infty} \in \mathscr{P}_1(a, b)$ *with* $\lim_{i\to\infty} \bar{\Delta}_i = 0$, *and let* s_i *be the unique interpolant of* f *in* $\mathrm{Sp}(2m-1, 2m-1, \Delta_i)$ *in the sense of* (6.9). *Then, with the additional hypothesis* (cf. (2.3)) *that* $D^k f \in C^0[a, b]$ *if* $p = \infty$,

$$\lim_{i\to\infty} \| D^k(f - s_i)\|_{L_p[a,b]} = 0. \tag{6.11}$$

It is interesting to remark that if the point functionals B_j of (6.1) depend only on $D^i g(a)$ and $D^i g(b)$ for $0 \leqslant i \leqslant \tau < 2m-1$, i.e., $a_{j,i} = b_{j,i} = 0$ for all $0 \leqslant j \leqslant 2m-1$, $\tau + 1 \leqslant i \leqslant 2m-1$, and if $f \in W_p^{\,k}[a, b]$ with $\tau < k \leqslant 2m-1$, then the unique interpolant s of f in $\mathrm{Sp}(2m-1, 2m-1, \Delta)$ in the sense of (6.9), is *independent* of f_h. In this case, the interpolants s in $\mathrm{Sp}(2m-1, 2m-1, \Delta)$, as defined by (6.7) and (6.9), are identical, and the error bounds of (6.10) represent a sharpening of the error bounds of (6.8) (with k replaced by $k-1$). If, moreover, $f \in W_p^{2m}[a, b]$ with $2 \leqslant p \leqslant \infty$, a case already covered in Lemma 6.1, we again remark that the use of (2.4) in conjunction with case $k = 2m-1$ of (6.10) of Theorem 6.2 gives the same upper bounds as in (6.8), i.e., for $p \leqslant q \leqslant \infty$,

$$K(\bar{\Delta})^{2m-j+(1/q)-(1/p)} \| D^{2m} f\|_{L_p[a,b]} \geqslant \| D^j(f-s)\|_{L_q[a,b]}, \qquad 0 \leqslant j \leqslant 2m-1. \tag{6.12}$$

We further remark that the inequality of (6.12) in the case $j = 0$ has been shown by Scherer [9, Theorem 9] to hold for more general partitions of $[a, b]$ than uniform partitions. In addition, if $m = 2$ or $m = 3$, i.e., if cubic or quintic splines are considered, it is easy to show that (6.10) is valid for $\Delta \in \mathscr{P}_\sigma(a, b)$ for any $\sigma \geqslant 1$.

7. Even Degree Approximation by Local Integration

Several authors (cf. Anselone and Laurent [3], Scherer [9], and Varga [13]) have considered the approximation of a given smooth function f, defined on $[a, b]$, by even-ordered splines s which, for a given partition Δ of $[a, b]$, interpolates f by means of $\int_{x_i}^{x_{i+1}} (f - s)\, dx = 0$, $0 \leqslant i \leqslant N - 1$, in addition to certain specified boundary interpolation. The object of this section is to derive new error bounds for such interpolation, based on the results of the previous section.

In analogy with Section 6, consider now any set of $2m + 2$ real point functionals $\tilde{B} = \{\tilde{B}_j\}_{j=0}^{2m+1}$ on $W_2^{2m+2}[a, b]$, of the particular form

$$\begin{aligned} &\tilde{B}_0 g = g(a), \qquad \tilde{B}_1 g = g(b), \\ &\tilde{B}_j g = \sum_{i=1}^{2m+1} \{a_{j,i} D^k g(a) + b_{j,i} D^i g(b)\}, \quad 2 \leqslant j \leqslant 2m + 1, \quad g \in W_2^{2m+2}[a, b], \end{aligned} \tag{7.1}$$

where it is assumed that the associated $(2m + 2) \times (4m + 4)$ matrix $\tilde{M}$ satisfies all the hypotheses of (6.3)–(6.5), with m replaced by $m + 1$. Note that since $\tilde{B}_0 g = g(a)$ and $\tilde{B}_1 g = g(b)$ from (7.1), then the assumption of (6.5ii) implies that the sum for $\tilde{B}_j g$ in (7.1), $2 \leqslant j \leqslant 2m + 1$, begins with $i = 1$. For any $\tilde{f} \in W_2^{2m+2}[a, b]$ and for any partition of $[a, b]$ with $N > m + 1$, it follows from the discussion in Section 6 that there is a unique $\tilde{s}$ in $\mathrm{Sp}(2m + 1, 2m + 1, \Delta)$ which interpolates $\tilde{f}$ in the sense that

$$\begin{aligned} (\tilde{f} - \tilde{s})(x_i) &= 0, \qquad 0 \leqslant i \leqslant N, \\ \tilde{B}_j \tilde{s} &= \tilde{B}_j \tilde{f}, \qquad 2 \leqslant j \leqslant 2m + 1. \end{aligned} \tag{7.2}$$

In particular, if $\tilde{f}(x) = \int_a^x f(t)\, dt$, so that $f \in W_2^{2m+1}[a, b]$, let $\tilde{s}$ be the unique interpolant of $\tilde{f}$ in $\mathrm{Sp}(2m + 1, 2m + 1, \Delta)$ in the above sense, and define $s(x) = D\tilde{s}(x)$. Clearly, $s \in \mathrm{Sp}(2m, 2m, \Delta)$, and it directly follows from (7.1) and (7.2) that

$$\begin{aligned} \int_{x_i}^{x_{i+1}} (f - s)\, dt &= 0, \qquad \leqslant i \leqslant N - 1, \\ B_j s &= B_j f, \qquad 2 \leqslant j \leqslant 2m + 1, \end{aligned} \tag{7.3}$$

where

$$B_j f \equiv \tilde{B}_j \tilde{f} = \sum_{i=0}^{2m} \{a_{j,i+1} D^i f(a) + b_{j,i+1} D^i f(b)\},$$

$$2 \leqslant j \leqslant 2m-1, \quad f \in W_2^{2m+1}[a, b]. \tag{7.4}$$

Conversely, it is readily verified that, for any $f \in W_2^{2m+1}[a, b]$, there is a unique $s \in \mathrm{Sp}(2m, 2m, \Delta)$ which interpolates f in the sense of (7.3).

Since our construction yields $D^j(f-s)(x) \equiv D^{j+1}(\tilde{f}-\tilde{s})(x)$, then the bounds of (6.10) of Theorem 6.2 can be directly used to prove the following result which extends the results of Scherer [9, Theorem 10] and Varga [13].

THEOREM 7.1. *Given $f \in W_p{}^k[a, b]$, with $0 \leqslant k < 2m+1$ and $2 \leqslant p \leqslant \infty$, given $\Delta \in \mathscr{P}_1(a, b)$ with $N > m+1$, and given the point functionals $\{\tilde{B}_j\}_{j=0}^{2m+1}$ of* (7.1) *which satisfy* (6.3)–(6.5), *let s be the unique interpolant of f in* $\mathrm{Sp}(2m, 2m, \Delta)$, *in the following sense*:

$$\int_{x_i}^{x_{i+1}} (f-s)\, dt = 0, \qquad 0 \leqslant i \leqslant N-1,$$

$$B_j s = \sum_{i=0}^{k-1} \{a_{j,i+1} D^i f(a) + b_{j,i+1} D^i f(b)\} + \{a_{j,k+1} D^k f_h(a) + b_{j,k+1} D^k f_h(b)\}, \tag{7.5}$$

$$2 \leqslant j \leqslant 2m+1, \quad \textit{if} \quad k > 0,$$

$$B_j s = a_{j,1} f_h(a) + b_{j,1} f_h(b), \qquad 2 \leqslant j \leqslant 2m+1, \quad \textit{if} \quad k = 0.$$

Then, with $h = \bar{\Delta}$,

$$K(\bar{\Delta})^{k-j+(1/q)-(1/p)}\, \omega_p(D^k f, \bar{\Delta}) \geqslant \begin{cases} \| D^j(f-s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k-1, \quad p \leqslant q \leqslant \infty, \\ \| D^k(f-s)\|_{L_p[a,b]}, & j = k, \quad p = q, \\ \| D^j s \|_{L_q[a,b]}, & \textit{if} \quad k < j \leqslant 2m, \quad p \leqslant q \leqslant \infty. \end{cases} \tag{7.6}$$

COROLLARY 7.2. *With the assumptions of Theorem* 7.1, *let $\{\Delta_i\}_{i=1}^{\infty} \in \mathscr{P}_1(a, b)$ with $\lim_{i\to\infty} \bar{\Delta}_i = 0$, and let s_i be the unique interpolant of f in* $\mathrm{Sp}(2m, 2m, \Delta_i)$ *in the sense of* (7.5). *Then, with the additional assumption that $D^k f \in C^0[a, b]$ if $p = \infty$,*

$$\lim_{i\to\infty} \| D^k(f-s_i)\|_{L_p[a,b]} = 0. \tag{7.7}$$

Making use once more of (2.4), we also have from (7.6) the result of

COROLLARY 7.3. *Given $f \in W_p^{2m+1}[a, b]$, $2 \leqslant p \leqslant \infty$, and the assumptions of Theorem* 7.1, *then*

$$K(\bar{\Delta})^{2m+1-j+(1/q)-(1/p)} \| D^{2m+1} f\|_{L_p[a,b]} \geqslant \| D^j(f-s)\|_{L_q[a,b]},$$

$$0 \leqslant j \leqslant 2m, \quad p \leqslant q \leqslant \infty. \tag{7.8}$$

We remark that this simple idea, viz. that of obtaining error bounds for spline interpolation in $\mathrm{Sp}(2m, 2m, \Delta)$ by considering the derivative of an associated interpolation error in $\mathrm{Sp}(2m+1, 2m+1, \Delta)$, can also be extended to L-spline-like interpolation where the interpolant is defined locally as the solution of an *odd-ordered* ordinary differential equation. It is also clear that a stability analysis for even-ordered splines, i.e., where $s \in \mathrm{Sp}(2m, 2m, \Delta)$ interpolates approximate data for $f \in W_p{}^k[a, b]$, $0 \leqslant k < 2m+1$, can also be easily carried out, in analogy with the results of Section 5. This permits one, as in Scherer [9, Theorem 10], to replace the integrals $\int_{x_i}^{x_{i+1}} f\,dt$ in the first equation in (7.5) with suitable quadratures, with no change in the form of the error bounds of (7.6).

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