Character Degrees of Extensions of $\mathrm{PSL}_2(q)$

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Abstract

Denote by S the 2-dimensional projective special linear group $\mathrm{PSL}_2(q)$ over the field of q elements. We determine, for all values of $q > 3$, the degrees of the irreducible complex characters of every group H such that $S \leqslant H \leqslant \mathrm{Aut}(S)$. Explicit knowledge of the character tables of $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$ is used along with standard Clifford theory to obtain the degrees.

1 Introduction

In a series of recent articles, M. L. Lewis and the author have studied various properties of the set of degrees of irreducible complex characters of nonsolvable groups. This always requires detailed information on the character degrees of finite simple groups and, in order to extend results to general nonsolvable groups, often requires information on the degrees of almost simple groups; that is, groups H such that $S \leqslant H \leqslant \mathrm{Aut}(S)$ for some simple group S. The most interesting cases tend to be groups involving the smaller simple groups with few character degrees, in particular the 2-dimensional projective special linear groups $\mathrm{PSL}_2(q)$.

Several of these studies required increasingly detailed information about the character degrees of groups H with $\mathrm{PSL}_2(q) \leqslant H \leqslant \mathrm{Aut}(\mathrm{PSL}_2(q))$ (see [5, 6, 7, 8]). Of course, the character table of $\mathrm{PSL}_2(q)$ is well-known, as is the automorphism group, and so the character degrees of H are known in principle. Since $\mathrm{PSL}_2(q)$ is a normal subgroup of such a group H, each character degree of H will be $\chi(1) \cdot j$ for some irreducible character χ of $\mathrm{PSL}_2(q)$ and some divisor j of $|H : \mathrm{PSL}_2(q)|$. Determining the values of j for which $\chi(1) \cdot j$ is a character degree of H for a specific group H and character χ of $\mathrm{PSL}_2(q)$ is not theoretically difficult. However, given the more detailed information required in recent work and the number of different possibilities for H, χ, and q, it has become much more convenient to have a complete answer covering all cases than to derive the necessary information on a case-by-case basis. The result is the present paper, and in Theorem A, we give the list of character degrees of H in all cases.

Let $S = \mathrm{PSL}_2(q)$, where $q = p^f$ for some prime p. The outer automorphism group of S is generated by a field automorphism φ of order f and, if p is odd, a diagonal automorphism δ of order 2. If $p = 2$, then δ is an inner automorphism. We have $S\langle\delta\rangle = \mathrm{PGL}_2(q)$, and the character table of $\mathrm{PGL}_2(q)$ is also known. In §3, we describe explicitly the actions of the automorphisms on the conjugacy classes of $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$, and in §4, we describe the actions of the automorphisms on the irreducible characters.

If $q = p^f > 5$ is odd, the character degree set of $\mathrm{PSL}_2(q)$ is

$$\mathrm{cd}(\mathrm{PSL}_2(q)) = \{1, q, (q + \varepsilon)/2, (q - 1), (q + 1)\},$$

where $\varepsilon = (-1)^{(q-1)/2}$, and the character degree set of $\mathrm{PSL}_2(q)$ for even q or $\mathrm{PGL}_2(q)$ for odd q is

$$\mathrm{cd}(\mathrm{PSL}_2(q)) = \{1, q, (q-1), (q+1)\}.$$

The characters of degrees 1 and q are invariant in $\mathrm{Aut}(S)$ and in fact extend to irreducible characters of H for any $S \leqslant H \leqslant \mathrm{Aut}(S)$. The two characters of degree $(q+\varepsilon)/2$ are invariant under φ and are interchanged by δ, so are easily handled. The characters of degrees $q-1$ and $q+1$ belong to parametrized families, and their stabilizers in $\mathrm{Aut}(S)$ depend on the parameters. In §5, we determine the subgroups of $\mathrm{Aut}(S)$ that are stabilizers of characters of degree $q-1$ or $q+1$ of S or $\mathrm{PGL}_2(q)$.

In §6, we show that for odd q, if H is any subgroup of $\mathrm{Aut}(S)$ containing S but not containing $\mathrm{PGL}_2(q)$, then H/S is cyclic. Hence, in any case, if $S \leqslant H \leqslant \mathrm{Aut}(S)$, then either H/S is cyclic or $H/\mathrm{PGL}_2(q)$ is cyclic. A character of $\mathrm{PGL}_2(q)$ or, if $\mathrm{PGL}_2(q) \nleqslant H$, of S will therefore extend to its stabilizer in H and then the extensions will induce irreducibly to H by Clifford's Theorem. We are then able to determine the character degrees of H using our knowledge of which subgroups of H appear as stabilizers.

2 Notation and Main Theorem

If G is any finite group, $\mathrm{Irr}(G)$ will denote the set of irreducible complex characters of G. We denote by $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ the set of character degrees of G.

We set $q = p^f$, where p is a prime number and f is a positive integer, and denote $S = \mathrm{PSL}_2(q)$ and $A = \mathrm{Aut}(S)$. We will always assume $p^f > 3$ because the character degrees of the automorphism groups of the non-simple groups $\mathrm{PSL}_2(2) \cong S_3$ and $\mathrm{PSL}_2(3) \cong A_4$ are well-known and the general results we prove do not always apply in these cases.

It will be most convenient to work with the conjugacy classes and characters of $\mathrm{SL}_2(q)$ and $\mathrm{GL}_2(q)$, which are described in [3] and [9], respectively. We will use the notation of those sources for the classes and characters.

The outer automorphism group of S is of order $(p-1,2)\cdot f$, and is generated by a field automorphism φ of order f and, if p is odd, a diagonal automorphism δ of order 2 (see [1] or [2]). Observe that $\mathrm{PGL}_2(q) = S\langle\delta\rangle$. If $p = 2$, then δ is an inner automorphism and the center of S is trivial, so that $S = \mathrm{PSL}_2(q) \cong \mathrm{PGL}_2(q) \cong \mathrm{SL}_2(q)$.

We will denote by H a subgroup of A satisfying $S \leqslant H \leqslant A$. The following theorem is our main result, describing the set of character degrees of H for any such subgroup of A for any q. Theorem A follows directly from Corollary 6.3 and Theorems 6.4, 6.5, 6.6, and 6.7.

Theorem A. *Let $S = \mathrm{PSL}_2(q)$, where $q = p^f > 3$ for a prime p, $A = \mathrm{Aut}(S)$, and let $S \leqslant H \leqslant A$. Set $G = \mathrm{PGL}_2(q)$ if $\delta \in H$ and $G = S$ if $\delta \notin H$, and let $|H : G| = d = 2^a m$, m odd. If p is odd, let $\varepsilon = (-1)^{(q-1)/2}$. The set of irreducible character degrees of H is*

$$\mathrm{cd}(H) = \{1, q, (q+\varepsilon)/2\} \cup \{(q-1)2^a\ell : \ell \mid m\} \cup \{(q+1)j : j \mid d\},$$

with the following exceptions:

i. If p is odd with $H \nleqslant S\langle\varphi\rangle$ or if $p = 2$, then $(q+\varepsilon)/2$ is not a degree of H.

ii. If f is odd, $p = 3$, and $H = S\langle\varphi\rangle$, then $\ell \neq 1$.

iii. If f is odd, $p = 3$, and $H = A$, then $j \neq 1$.

iv. If f is odd, $p = 2$, 3, or 5, and $H = S\langle\varphi\rangle$, then $j \neq 1$.

v. If $f \equiv 2 \pmod 4$, $p = 2$ or 3, and $H = S\langle\varphi\rangle$ or $H = S\langle\delta\varphi\rangle$, then $j \neq 2$.

3 Conjugacy Classes and Automorphisms

Let $q = p^f$, where p is a prime. We are considering characters of extensions of $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$. However, the conjugacy classes, characters, and automorphisms are more easily described for $\mathrm{SL}_2(q)$ and $\mathrm{GL}_2(q)$. For $\mathrm{SL}_2(q)$, we will use the notation and character tables of [3, §38], and for $\mathrm{GL}_2(q)$ we will use the character table of [9].

Let ν be a generator of $\mathbb{F}_q^*$, the multiplicative group of the field $\mathbb{F}_q$ of q elements, τ a generator of $\mathbb{F}_{q^2}^*$, and $\gamma = \tau^{q-1}$. We denote

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ z = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\ d = \begin{bmatrix} 1 & 0 \\ \nu & 1 \end{bmatrix},\ a = \begin{bmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{bmatrix},\ b = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{bmatrix}.$$

If q is odd, then every element of $\mathrm{SL}_2(q)$ is conjugate to one of 1, z, c, cz, d, dz, a^l for $1 \leqslant l \leqslant (q-3)/2$, or b^m for $1 \leqslant m \leqslant (q-1)/2$. If q is even, then every element of $\mathrm{SL}_2(q) = \mathrm{PSL}_2(q)$ is conjugate to one of 1, c, a^l for $1 \leqslant l \leqslant (q-2)/2$, or b^m for $1 \leqslant m \leqslant q/2$. Observe that in either case, $1 \leqslant l \leqslant [(q-2)/2]$ and $1 \leqslant m \leqslant [q/2]$, where $[x]$ denotes the greatest integer less than or equal to x.

We need to consider $\mathrm{GL}_2(q)$ and $\mathrm{PGL}_2(q)$ only when q is odd. We denote

$$A_1(l) = \begin{bmatrix} \nu^l & 0 \\ 0 & \nu^l \end{bmatrix},\ A_2(l) = \begin{bmatrix} \nu^l & 0 \\ 1 & \nu^l \end{bmatrix},\ A_3(l_1, l_2) = \begin{bmatrix} \nu^{l_1} & 0 \\ 0 & \nu^{l_2} \end{bmatrix},\ B_1(l) = \begin{bmatrix} \tau^l & 0 \\ 0 & \tau^{ql} \end{bmatrix}.$$

Every element of $\mathrm{GL}_2(q)$ is conjugate to one of $A_1(l)$ or $A_2(l)$ for $1 \leqslant l \leqslant q-1$, $A_3(l_1, l_2)$ for $1 \leqslant l_1 \leqslant q-1$, $1 \leqslant l_1 \leqslant q-1$, and $l_1 \neq l_2$, or $B_1(l)$ for $1 \leqslant l \leqslant q^2 - 1$ and $(q+1) \nmid l$.

The outer automorphism group of $\mathrm{PSL}_2(q)$, $q = p^f$, is of order df, where $d = (2, q-1)$. It is generated by a diagonal automorphism δ and a field automorphism φ. These automorphisms act on elements of $\mathrm{SL}_2(q)$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\delta = \begin{bmatrix} a & \nu^{-1}b \\ \nu c & d \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^\varphi = \begin{bmatrix} a^p & b^p \\ c^p & d^p \end{bmatrix}.$$

As the center Z of $\mathrm{SL}_2(q)$ is invariant under both δ and φ, these maps induce automorphisms $\bar{\delta}$ and $\bar{\varphi}$ on $\mathrm{PSL}_2(q) = \mathrm{SL}_2(q)/Z$ by $(gZ)^{\bar{\delta}} = g^\delta Z$ and $(gZ)^{\bar{\varphi}} = g^\varphi Z$, as usual. Denoting the induced automorphisms on $\mathrm{PSL}_2(q)$ by δ, φ as well, we have $\mathrm{PSL}_2(q)\langle\delta\rangle \cong \mathrm{PGL}_2(q)$ and $\mathrm{Aut}(\mathrm{PSL}_2(q)) = \mathrm{PSL}_2(q)\langle\delta, \varphi\rangle \cong \mathrm{PGL}_2(q)\langle\varphi\rangle$.

If q is even, then δ is an inner automorphism and $\mathrm{Aut}(\mathrm{PSL}_2(q)) = \mathrm{PSL}_2(q)\langle\varphi\rangle$, and if q is odd, then δ is an outer automorphism but δ^2 is inner. Hence δ is of order $d = (2, q-1)$ modulo inner automorphisms. Since entries in elements of $\mathrm{SL}_2(q)$ are from the field of $q = p^f$ elements, φ is of order f. Moreover, if q is odd, then δ and φ commute modulo inner automorphisms, so that $\mathrm{Aut}(\mathrm{PSL}_2(q))/\mathrm{PSL}_2(q) \cong \langle\delta\rangle \times \langle\varphi\rangle$, hence their actions on conjugacy classes or characters will commute.

We now describe the actions of δ and φ on conjugacy classes. These lemmas follow from straightforward calculations.

Lemma 3.1. *Let q be odd and assume notation as above. In $\mathrm{SL}_2(q)$, the diagonal automorphism δ interchanges the conjugacy classes of c and d, interchanges the conjugacy classes of cz and dz, and fixes all other conjugacy classes.*

Lemma 3.2. *Assume notation as above and let $1 \leqslant k < f$. In $\mathrm{SL}_2(q)$, the automorphism φ^k sends*

i. the conjugacy class of a^l to the class of a^r, where $1 \leqslant r \leqslant [(q-2)/2]$ and $r \equiv \pm lp^k \pmod{q-1}$,

ii. the conjugacy class of b^m to the class of b^s, where $1 \leqslant s \leqslant [q/2]$ and $s \equiv \pm mp^k \pmod{q+1}$, *and fixes all other conjugacy classes.*

The field automorphism φ is defined similarly on $\mathrm{GL}_2(q)$ and as before induces an automorphism on the quotient $\mathrm{PGL}_2(q)$ of $\mathrm{GL}_2(q)$ modulo its center.

Lemma 3.3. *Assume notation as above with q odd and let $1 \leqslant k < f$. In $\mathrm{GL}_2(q)$, the automorphism φ^k sends*

i. the conjugacy class of $A_1(l)$ to the class of $A_1(r)$, where $1 \leqslant r \leqslant q-1$ and $r \equiv lp^k (\mathrm{mod}\ q-1)$,

ii. the conjugacy class of $A_2(l)$ to the class of $A_2(r)$, where $1 \leqslant r \leqslant q-1$ and $r \equiv lp^k (\mathrm{mod}\ q-1)$,

iii. the conjugacy class of $A_3(l_1, l_2)$ to the class of $A_3(r_1, r_2)$, where $1 \leqslant r_1 \leqslant q-1$, $1 \leqslant r_2 \leqslant q-1$, $r_1 \equiv l_1 p^k (\mathrm{mod}\ q-1)$, and $r_2 \equiv l_2 p^k (\mathrm{mod}\ q-1)$, and

iv. the conjugacy class of $B_1(l)$ to the class of $B_1(t)$, where $1 \leqslant t \leqslant q^2-1$, $q+1 \nmid t$, and $t \equiv lp^k (\mathrm{mod}\ q^2-1)$.

4 Characters and Automorphisms

In order to determine which subgroups of $\mathrm{Aut}(\mathrm{PSL}_2(q))$ appear as stabilizers of irreducible characters of $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}_2(q)$, we first determine conditions under which these characters are invariant under the action of δ or powers of φ.

As noted previously, it will be more convenient to work with characters and conjugacy class of $\mathrm{SL}_2(q)$ or $\mathrm{GL}_2(q)$. Let G be either $\mathrm{SL}_2(q)$ or $\mathrm{GL}_2(q)$ and let $Z = Z(G)$, so that G/Z is $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}_2(q)$, respectively. An automorphism σ of G induces an automorphism $\bar{\sigma}$ on G/Z defined by $(gZ)^{\bar{\sigma}} = g^{\sigma}Z$. Similarly, the irreducible characters of G/Z are precisely those defined by $\bar{\chi}(gZ) = \chi(g)$, where $\chi \in \mathrm{Irr}(G)$ and $Z \leqslant \ker \chi$. It is straightforward to check that $\bar{\chi}^{\bar{\sigma}} = \bar{\chi}$ if and only if $\chi^{\sigma} = \chi$. Hence the irreducible characters of G/Z invariant under a particular automorphism σ are those characters of G invariant under σ with kernel containing Z.

We first determine the characters of $\mathrm{SL}_2(q)$ and $\mathrm{GL}_2(q)$ whose kernels contain the center. Of course, if q is even, then the center of $\mathrm{SL}_2(q)$ is trivial and the diagonal automorphism is an inner automorphism, and so $\mathrm{SL}_2(q) \cong \mathrm{PSL}_2(q) \cong \mathrm{PGL}_2(q)$. In the notation of [3, §38], when q is even, $\mathrm{SL}_2(q)$ has irreducible characters

i. 1_G of degree 1,

ii. ψ of degree q (= St, the Steinberg character),

iii. χ_i, $1 \leqslant i \leqslant (q-2)/2$, of degree $q+1$, and

iv. θ_j, $1 \leqslant j \leqslant q/2$, of degree $q-1$.

We will assume q is odd in the following. For $\mathrm{SL}_2(q)$, we use the notation and character table in [3, §38] and for $\mathrm{GL}_2(q)$, we use [9, §2]. The following results are easily obtained from the respective character tables.

Lemma 4.1. *Let $G = \mathrm{SL}_2(q)$ with q odd. The irreducible characters of G with kernel containing $Z(G)$ are as follows:*

i. 1_G of degree 1;

ii. ψ of degree q (= St, the Steinberg character);

iii. χ_i, for $1 \leqslant i \leqslant (q-3)/2$ and i even, of degree $q+1$;

iv. θ_j *for* $1 \leqslant j \leqslant (q-1)/2$ *and* j *even, of degree* $q-1$*;*

v. ξ_1 *and* ξ_2 *of degree* $(q+1)/2$*, if* $q \equiv 1(\mathrm{mod}\ 4)$*;*

vi. η_1 *and* η_2 *of degree* $(q-1)/2$*, if* $q \equiv -1(\mathrm{mod}\ 4)$.

Lemma 4.2. *Let* $G = \mathrm{GL}_2(q)$ *with* q *odd. The irreducible characters of* G *with kernel containing* $Z(G)$ *are as follows:*

i. $\chi_1^{(n)}$ *and* $\chi_q^{(n)}$ *for* $n = (q-1)/2$ *and* $n = q-1$,

ii. $\chi_{q+1}^{(m,n)}$ *for* $1 \leqslant n \leqslant (q-3)/2$ *and* $m = (q-1)-n$,

iii. $\chi_{q-1}^{((q-1)n)}$ *for* $1 \leqslant n \leqslant (q-1)/2$.

Notation 4.3.
We will write $\chi_{q+1}^{(n)}$ *for the character* $\chi_{q+1}^{(m,n)}$ *with* $1 \leqslant n \leqslant (q-3)/2$ *and* $m = (q-1)-n$.
We will write $\theta_{q-1}^{(n)}$ *for the character* $\chi_{q-1}^{((q-1)n)}$ *with* $1 \leqslant n \leqslant (q-1)/2$.

We first consider the action of δ on the irreducible characters of $\mathrm{PSL}_2(q)$ for odd q and relate the characters of $\mathrm{PSL}_2(q)$ to those of $\mathrm{PGL}_2(q)$. For the following lemma, let $\varepsilon = (-1)^{(q-1)/2}$ and denote by μ_1, μ_2 the irreducible characters of $\mathrm{SL}_2(q)$ of degree $(q+\varepsilon)/2$. Thus $\mu_i = \xi_i$ if $q \equiv 1(\mathrm{mod}\ 4)$ and $\mu_i = \eta_i$ if $q \equiv -1(\mathrm{mod}\ 4)$, so that $\mu_1, \mu_2 \in \mathrm{Irr}(\mathrm{PSL}_2(q))$.

Lemma 4.4. *Let* q *be odd. All characters of* $\mathrm{PSL}_2(q)$ *of degrees* 1, q, $q+1$, *and* $q-1$ *are invariant under* δ *and each extends to two distinct irreducible characters of* $\mathrm{PGL}_2(q)$.
The characters μ_1 *and* μ_2 *of* $\mathrm{PSL}_2(q)$ *of degree* $(q+\varepsilon)/2$ *form a single orbit under the action of* δ *and induce to a single irreducible character of* $\mathrm{PGL}_2(q)$ *of degree* $q+\varepsilon$.

Proof. Observe first that $|\,\mathrm{PGL}_2(q) : \mathrm{PSL}_2(q)| = 2$. Therefore, by Gallagher's Theorem ([4, 6.17]), an invariant character of $\mathrm{PSL}_2(q)$ extends to two distinct irreducible characters of $\mathrm{PGL}_2(q)$. An orbit of non-invariant characters consists of exactly two characters that induce to the same irreducible character of $\mathrm{PGL}_2(q)$.

The characters of degrees 1, q, $q+1$, and $q-1$ have the same value on the classes of c and d, and so also on the classes of cz and dz. Hence these characters are invariant under δ by Lemma 3.1.

The characters μ_1, μ_2 of degree $(q+\varepsilon)/2$ are equal on all classes except the classes of c, d, cz, and dz. We have $\mu_1(c) = \mu_2(d)$, $\mu_1(d) = \mu_2(c)$, $\mu_1(cz) = \mu_2(dz)$, and $\mu_1(dz) = \mu_2(cz)$. Therefore, $\mu_1^{\delta} = \mu_2$ and $\mu_1^{\delta} = \mu_2$ by Lemma 3.1. □

It follows that $\mathrm{PGL}_2(q)$ has 2 irreducible characters of each of the degrees 1 and q. Observe that if $q \equiv 1(\mathrm{mod}\ 4)$, then $(q-1)/2$ is even and $\mathrm{PSL}_2(q)$ has $(q-1)/4$ irreducible characters of degree $q-1$. In this case, $(q-3)/2$ is odd and $\mathrm{PSL}_2(q)$ has $(q-5)/4$ irreducible characters of degree $q+1$ and 2 irreducible characters of degree $(q+1)/2$. Similarly, if $q \equiv 3(\mathrm{mod}\ 4)$, then $\mathrm{PSL}_2(q)$ has $(q-3)/4$ irreducible characters of degree $q-1$ and 2 of degree $(q-1)/2$, and has $(q-3)/4$ of degree $q+1$. Hence the lemma implies that $\mathrm{PGL}_2(q)$ has exactly $(q-1)/2$ irreducible characters of degree $q-1$ and $(q-3)/2$ of degree $q+1$ in any case. This is confirmed in the character table of $\mathrm{PGL}_2(q)$ in [9].

We now consider the action of the field automorphism φ on the irreducible characters of $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$. Unless stated otherwise, q may be either even or odd.

Lemma 4.5. *If* χ *is an irreducible character of either* $\mathrm{PSL}_2(q)$ *or* $\mathrm{PGL}_2(q)$ *of degree* 1, q, *or* $(q+\varepsilon)/2$, *then* χ *is invariant under the action of* φ.

Proof. It is clear from the character table that the characters of $\mathrm{PSL}_2(q)$ of degree 1 and q are invariant under φ, as are the principal character and Steinberg character of $\mathrm{PGL}_2(q)$. For odd q, the remaining characters of degree 1, q of $\mathrm{PGL}_2(q)$ have value $(-1)^{k+l}$ on the class of $A_3(k,l)$, and either $(-1)^l$ or $-(-1)^l$ on the class of $B_1(l)$. By Lemma 3.3, φ sends $A_3(k,l)$ to $A_3(kp,lp)$ and $B_1(l)$ to $B_1(lp)$. Since p is odd in this case, $(-1)^{k+l} = (-1)^{kp+lp}$ and $(-1)^l = (-1)^{lp}$, and so these characters are also invariant under φ.

The characters μ_1, μ_2 of degree $(q+\varepsilon)/2$ occur only for $\mathrm{PSL}_2(q)$ with q odd. By Lemma 3.2, the only conjugacy classes moved by φ are those of a^l and b^l. The values of the μ_i on these classes are 0, $(-1)^l$ or $-(-1)^l$. The class of a^l, b^l is sent to $a^{\pm lp}$, $b^{\pm lp}$, respectively. Again, since p is odd, $(-1)^l = (-1)^{\pm lp}$ and these characters are invariant under φ. □

We next determine conditions under which a given character of degree $q+1$ or $q-1$ is invariant under a power of φ. The following general result will be very useful.

Lemma 4.6. *If ϵ is a complex kth root of unity and i, j are integers, then $\epsilon^i + \epsilon^{-i} = \epsilon^j + \epsilon^{-j}$ if and only if $i \equiv \pm j \pmod k$.*

Proof. Observe that $\epsilon^i + \epsilon^{-i} = \epsilon^j + \epsilon^{-j}$ if and only if

$$\epsilon^i - \epsilon^j = \epsilon^{-j} - \epsilon^{-i} = \frac{\epsilon^i - \epsilon^j}{\epsilon^{i+j}}.$$

This holds if and only if either $\epsilon^i = \epsilon^j$, in which case $i \equiv j \pmod k$, or $\epsilon^{i+j} = 1$, in which case $i \equiv -j \pmod k$. □

Lemma 4.7. *Let $q = p^f$ for a prime p and $f \geqslant 2$, and let $1 \leqslant k \leqslant f$.*

i. The character χ_n of $\mathrm{PSL}_2(q)$ or $\chi_{q+1}^{(n)}$ of $\mathrm{PGL}_2(q)$ of degree $q+1$ is invariant under φ^k if and only if

$$p^f - 1 \mid (p^k-1)n \text{ or } p^f - 1 \mid (p^k+1)n.$$

ii. The character θ_n of $\mathrm{PSL}_2(q)$ or $\theta_{q-1}^{(n)}$ of $\mathrm{PGL}_2(q)$ of degree $q-1$ is invariant under φ^k if and only if

$$p^f + 1 \mid (p^k-1)n \text{ or } p^f + 1 \mid (p^k+1)n.$$

Proof. Distinct characters of $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}_2(q)$ of degree $q+1$ differ only on the classes of a^l of $\mathrm{SL}_2(q)$ or $A_3(l_1,l_2)$ of $\mathrm{GL}_2(q)$. Similarly, distinct characters of degree $q-1$ differ only on the classes of b^m of $\mathrm{SL}_2(q)$ or $B_1(l)$ of $\mathrm{GL}_2(q)$.

Let ρ be a complex primitive $(q-1)$th root of unity, so that $\chi_n(a^l) = \rho^{nl} + \rho^{-nl}$ for the character χ_n of $\mathrm{PSL}_2(q)$ of degree $q+1$. The character χ_n is then invariant under φ^k if and only if

$$\chi_n(a^l) = \chi_n((a^l)^{\varphi^k}) = \chi_n(a^{lp^k})$$

for all l, $1 \leqslant l \leqslant [(q-2)/2]$, by Lemma 3.2, hence if and only if

$$\rho^{nl} + \rho^{-nl} = \rho^{nlp^k} + \rho^{-nlp^k}$$

for all l. By Lemma 4.6, this holds if and only if $nlp^k \equiv \pm nl \pmod{q-1}$ for all l. Observing that $q = p^f$ and that if the congruence holds for $l = 1$ then it holds for all l, we see that χ_n is invariant under φ^k if and only if $np^k \equiv \pm n \pmod{p^f - 1}$ as claimed.

Similarly, for the character $\chi_{q+1}^{(n)}$ of $\mathrm{PGL}_2(q)$, we have

$$\chi_{q+1}^{(n)}(A_3(l_1,l_2)) = \rho^{n(l_2-l_1)} + \rho^{-n(l_2-l_1)}$$

and, by Lemma 3.3, $\chi_{q+1}^{(n)}$ is invariant under φ^k if and only if

$$\rho^{n(l_2-l_1)} + \rho^{-n(l_2-l_1)} = \rho^{n(l_2-l_1)p^k} + \rho^{-n(l_2-l_1)p^k}$$

for all $l_1 \neq l_2$, $1 \leqslant l_1 \leqslant q-1$, $1 \leqslant l_2 \leqslant q-1$. As before, this is equivalent to

$$n(l_2 - l_1)p^k \equiv \pm n(l_2 - l_1)(\mathrm{mod}\ q-1)$$

for all l_1, l_2. Since $q \geqslant 4$, this must hold in particular for $l_1 = 1$, $l_2 = 2$, and so holds for all l_1, l_2 if and only if $np^k \equiv \pm n(\mathrm{mod}\ p^f - 1)$ as claimed.

Let σ be a complex primitive $(q+1)$th root of unity, so that $\theta_n(b^m) = -(\sigma^{nm} + \sigma^{-nm})$ for the character θ_n of $\mathrm{PSL}_2(q)$ of degree $q-1$. The character θ_n is then invariant under φ^k if and only if

$$\theta_n(b^m) = \theta_n((b^m)^{\varphi^k}) = \theta_n(b^{mp^k})$$

for all m, $1 \leqslant m \leqslant [q/2]$, by Lemma 3.2, hence if and only if

$$\sigma^{nm} + \sigma^{-nm} = \sigma^{nmp^k} + \sigma^{-nmp^k}$$

for all m. By Lemma 4.6, this holds if and only if $nmp^k \equiv \pm nm(\mathrm{mod}\ q+1)$ for all m. Again, $q = p^f$ and if the congruence holds for $m = 1$ then it holds for all m, hence θ_n is invariant under φ^k if and only if $np^k \equiv \pm n(\mathrm{mod}\ p^f + 1)$ as claimed.

For the character $\theta_{q-1}^{(n)}$ of $\mathrm{PGL}_2(q)$, we have

$$\theta_{q-1}^{(n)}(B_1(l)) = -(\sigma^{nl} + \sigma^{nlq}) = -(\sigma^{nl} + \sigma^{-nl}),$$

since $\sigma^q = \sigma^{-1}$. By Lemma 3.3, $\theta_{q-1}^{(n)}$ is invariant under φ^k if and only if

$$\theta^{nl} + \theta^{-nl} = \theta^{nlp^k} + \theta^{-nlp^k}$$

for all l, $1 \leqslant l \leqslant q^2 - 1$ and $q + 1 \nmid l$. As before, this is equivalent to

$$nlp^k \equiv \pm nl(\mathrm{mod}\ q+1),$$

which holds for all l if and only if $np^k \equiv \pm n(\mathrm{mod}\ p^f + 1)$. □

The following number-theoretic result will be helpful in applying Lemma 4.7.

Lemma 4.8. *If p is a prime and f, k are positive integers such that $k \mid f$, then*

i. $(p^f - 1, p^k - 1) = p^k - 1$,

ii. $(p^f - 1, p^k + 1) = \begin{cases} (p-1, 2) & \text{if } f/k \text{ is odd} \\ p^k + 1 & \text{if } f/k \text{ is even,} \end{cases}$

iii. $(p^f + 1, p^k - 1) = (p - 1, 2)$,

iv. $(p^f + 1, p^k + 1) = \begin{cases} p^k + 1 & \text{if } f/k \text{ is odd} \\ (p-1, 2) & \text{if } f/k \text{ is even.} \end{cases}$

Proof. Observe first that $p^k \equiv 1(\mathrm{mod}\ p^k - 1)$ and $p^k \equiv -1(\mathrm{mod}\ p^k + 1)$, hence

$$p^f \equiv (p^k)^{f/k} \equiv (1)^{f/k} \equiv 1(\mathrm{mod}\ p^k - 1)$$

and

$$p^f \equiv (p^k)^{f/k} \equiv (-1)^{f/k}(\mathrm{mod}\ p^k + 1).$$

It follows that $p^k - 1 \mid p^f - 1$, so that (i) holds. Similarly, if f/k is even, then $p^k + 1 \mid p^f - 1$ and $(p^f - 1, p^k + 1) = p^k + 1$, whereas if f/k is odd, then $p^k + 1 \mid p^f + 1$ and $(p^f + 1, p^k + 1) = p^k + 1$.

Since $p^k - 1 \mid p^f - 1$, we have that $(p^f + 1, p^k - 1)$ must divide 2. Similarly, if f/k is odd, then $p^k + 1 \mid p^f + 1$ and so $(p^f - 1, p^k + 1)$ divides 2, while if f/k is even, then $p^k + 1 \mid p^f - 1$ and $(p^f + 1, p^k + 1)$ divides 2. If $p = 2$, then all of $p^k \pm 1$ and $p^f \pm 1$ are odd, but if p is odd then all of these integers are even. Hence these greatest common divisors are 1 when $p = 2$ and 2 when p is odd, hence are equal to $(p - 1, 2)$ as claimed. □

5 Stabilizers of Characters of Degree $q - 1$ or $q + 1$

We now determine the subgroups of $\mathrm{Aut}(\mathrm{PSL}_2(q))$ that occur as stabilizers of characters of $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}_2(q)$ of degree $q-1$ or $q+1$. Throughout this section, we denote $A = \mathrm{Aut}(\mathrm{PSL}_2(q))$, where $q = p^f$ for some prime p and integer $f \geqslant 2$. Also, recall that if q is even, then δ is an inner automorphism and $\mathrm{PGL}_2(q) = \mathrm{PSL}_2(q)$. We first restrict the structure of subgroups of A that we need to consider.

Lemma 5.1. *If a subgroup K of A is the stabilizer in A of an irreducible character of $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}_2(q)$ of degree $q - 1$ or $q + 1$, then $K = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$, where $k \mid f$.*

Proof. By Lemma 4.4, every character of $\mathrm{PSL}_2(q)$ of degree $q - 1$ or $q + 1$ is invariant under δ, hence $\mathrm{PGL}_2(q) \leqslant K$ in any case. Therefore $K = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$ for some positive integer k, and we may assume $k \mid f$ as $|\langle\varphi\rangle| = f$. □

For the remainder of this section, K will denote a subgroup of A of the form $K = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$ for some positive divisor k of f, again keeping in mind that $\mathrm{PGL}_2(q) = \mathrm{PSL}_2(q)$ when q is even. We first determine when K is a stabilizer of a character of degree $q - 1$.

Lemma 5.2. *Let $K = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$, where $k \mid f$. If K stabilizes an irreducible character of $\mathrm{PSL}_2(q)$ or $\mathrm{PGL}_2(q)$ of degree $q - 1$, then f/k is odd. If f/k is odd, then K is the stabilizer in A of both a character of $\mathrm{PSL}_2(q)$ and a character of $\mathrm{PGL}_2(q)$ of degree $q - 1$, with the exception that if $p = 3$ and $k = 1$, then $K = A$ does not stabilize any character of $\mathrm{PSL}_2(q)$ of degree $q - 1$.*

Proof. All characters of $\mathrm{PSL}_2(q)$ of degree $q - 1$ are θ_n for some $1 \leqslant n \leqslant [q/2]$, with n even if q is odd, and for odd q, characters of $\mathrm{PGL}_2(q)$ of degree $q - 1$ are $\theta_{q-1}^{(n)}$ for $1 \leqslant n \leqslant (q - 1)/2$. In particular, note that

$$1 \leqslant n < (q + 1)/2 = (p^f + 1)/2$$

in any case. By Lemma 4.7, θ_n and $\theta_{q-1}^{(n)}$ are fixed by φ^k if and only if

$$p^f + 1 \mid (p^k - 1)n \text{ or } p^f + 1 \mid (p^k + 1)n.$$

If f/k is even, then

$$(p^f + 1, p^k - 1) = (p^f + 1, p^k + 1) = (2, p - 1)$$

by Lemma 4.8. Thus if φ^k stabilizes θ_n or $\theta_{q-1}^{(n)}$, then $(p^f + 1)/(2, p - 1)$ must divide n, which is impossible as $1 \leqslant n < (p^f + 1)/2$. Hence if f/k is even, φ^k does not stabilize any character of degree $q - 1$, and so we have that f/k is odd.

Suppose now that $p = 3$ and $k = 1$, so that $K = A$ and $f \geqslant 3$ is odd. By Lemma 4.1, the characters of $\mathrm{PSL}_2(3^f)$ of degree $q - 1$ are the θ_n with $1 \leqslant n \leqslant (3^f - 1)/2$ and n even. By Lemma 4.7, θ_n is fixed by φ if and only if $3^f + 1 \mid 2n$ or $3^f + 1 \mid 4n$. Thus if $K = A$ stabilizes θ_n, then $3^f + 1 \mid 4n$. Since f is odd, $3^f + 1 \equiv 4 \pmod 8$, and so $(3^f + 1)/4$ is odd and divides n. Since n is even, this implies $(3^f + 1)/2$ divides n, contradicting $1 \leqslant n \leqslant (3^f - 1)/2$. Therefore, if $p = 3$ and $k = 1$, then $K = A$ does not stabilize any character of $\mathrm{PSL}_2(q)$ of degree $q - 1$.

It remains to show that if $K = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$, where $k \mid f$ and f/k is odd, then K is the stabilizer in A of a character of $\mathrm{PGL}_2(q)$ of degree $q-1$ and, unless $p^k = 3$, K is the stabilizer of a character of $\mathrm{PSL}_2(q)$ of degree $q-1$. As noted above, every character of degree $q-1$ is stabilized by $\mathrm{PGL}_2(q)$.

By Lemma 4.8, since f/k is odd, $p^k+1 \mid p^f+1$. Set $n = (p^f+1)/(p^k+1)$. Since $p^k+1 \geqslant 3$, we have

$$1 \leqslant n = \frac{p^f+1}{p^k+1} \leqslant \frac{p^f+1}{3} < \frac{p^f+1}{2} = \frac{q+1}{2},$$

hence n is an integer and $n < (q+1)/2$. If q is odd, this implies $n \leqslant (q-1)/2$ and if q is even, this implies $n \leqslant q/2$, and so $\theta_{q-1}^{(n)}$ is a character of $\mathrm{PGL}_2(q)$ for odd q and θ_n is a character of $\mathrm{PGL}_2(q) = \mathrm{PSL}_2(q)$ for even q. Moreover, we have $p^f+1 \mid (p^k+1)n$, so that by Lemma 4.7, $\theta_{q-1}^{(n)}$, respectively θ_n, is stabilized by K.

If the stabilizer, T, of $\theta_{q-1}^{(n)}$ or θ_n properly contains K, then $T = \mathrm{PGL}_2(q)\langle\varphi^t\rangle$ for some divisor t of k with $1 \leqslant t < k$. Since φ^t stabilizes $\theta_{q-1}^{(n)}$ or θ_n, we have $p^f+1 \mid (p^t+1)n$ or $p^f+1 \mid (p^t-1)n$. Hence one of $(p^t+1)/(p^k+1)$ or $(p^t-1)/(p^k+1)$ is an integer. In any case, this implies $p^k+1 \leqslant p^t+1$, contradicting $1 \leqslant t < k$. Therefore, K is the stabilizer in A of $\theta_{q-1}^{(n)}$ for odd q and of θ_n for even q.

Finally, assume that $q = p^f$ is odd and $p^k > 3$. Set $n = 2(p^f+1)/(p^k+1)$. Again, since f/k is odd, $p^k+1 \mid p^f+1$, and so n is an even integer. Since $p^k > 3$, we have

$$1 \leqslant n = 2 \cdot \frac{p^f+1}{p^k+1} < 2 \cdot \frac{p^f+1}{4} = \frac{p^f+1}{2} = \frac{q+1}{2},$$

hence n is a positive integer and $n < (q+1)/2$, which implies $n \leqslant (q-1)/2$. Therefore, θ_n is a character of $\mathrm{PSL}_2(q)$ of degree $q-1$, and since $p^f+1 \mid (p^k+1)n$, θ_n is stabilized by φ^k.

As before, if the stabilizer of θ_n properly contains K, then there is a divisor t of k with $1 \leqslant t < k$ such that φ^t stabilizes θ_n. Hence $p^f+1 \mid (p^t+1)n$ or $p^f+1 \mid (p^t-1)n$, and so one of $2(p^t+1)/(p^k+1)$ or $2(p^t-1)/(p^k+1)$ is an integer. In particular, we must have $p^k+1 \leqslant 2(p^t+1)$; that is, $p^k \leqslant 2p^t+1$. But $p \geqslant 3$ and $1 \leqslant t < k$, and so

$$p^k \geqslant p^{t+1} \geqslant 3p^t = 2p^t + p^t > 2p^t + 1,$$

a contradiction. Therefore, K is the stabilizer in A of the character θ_n of $\mathrm{PSL}_2(q)$. □

We next determine when $K = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$ is the stabilizer of a character of degree $q+1$.

Lemma 5.3. *Let q be odd. The subgroup $K = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$ of A is the stabilizer in A of an irreducible character of $\mathrm{PGL}_2(q)$ of degree $q+1$, with the exception that if $p = 3$, $k = 1$, and f is odd, then $K = A$ does not stabilize any character of $\mathrm{PGL}_2(q)$ of degree $q+1$.*

Proof. Characters of $\mathrm{PGL}_2(q)$ of degree $q+1$ are $\chi_{q+1}^{(n)}$ for $1 \leqslant n \leqslant (q-3)/2$. By Lemma 4.7, $\chi_{q+1}^{(n)}$ is fixed by φ^k if and only if

$$p^f - 1 \mid (p^k-1)n \text{ or } p^f - 1 \mid (p^k+1)n.$$

Assume first that $p^k \geqslant 5$ and let $n = (p^f-1)/(p^k-1)$. We then have $p^k - 1 > 2$, so that $n < (p^f-1)/2 = (q-1)/2$. Therefore,

$$n \leqslant \frac{q-1}{2} - 1 \leqslant \frac{q-3}{2},$$

and so $\chi_{q+1}^{(n)}$ is an irreducible character of $\mathrm{PGL}_2(q)$. Moreover, $(p^k-1)n = p^f-1$, hence p^f-1 divides $(p^k-1)n$ and $\chi_{q+1}^{(n)}$ is invariant under $\langle\varphi^k\rangle$. Therefore, K is contained in the stabilizer, T, of $\chi_{q+1}^{(n)}$ in A.

We have $K = \mathrm{PGL}_2(q)\langle\varphi^k\rangle \leqslant \mathrm{PGL}_2(q)\langle\varphi^t\rangle = T$ for some divisor t of k. Since $\chi_{q+1}^{(n)}$ is invariant under φ^t, we have that $p^f - 1$ divides either $(p^t - 1)n$ or $(p^t + 1)n$, that is,

$$p^f - 1 \mid (p^t - 1) \cdot \frac{p^f - 1}{p^k - 1} \text{ or } p^f - 1 \mid (p^t + 1) \cdot \frac{p^f - 1}{p^k - 1}.$$

Hence either $p^k - 1 \mid p^t - 1$ or $p^k - 1 \mid p^t + 1$. Since $t \mid k$, we have $p^t - 1 \mid p^k - 1$, and so if $p^k - 1 \mid p^t + 1$, then $p^t - 1 \mid p^t + 1 = (p^t - 1) + 2$. Therefore, $p^t - 1 \mid 2$ and since p is odd, this means $p^t = 3$. In particular, $p^k - 1 = 3^k - 1$ divides $p^t + 1 = 4$, and hence $k = 1$ and $p^k = 3$, contradicting $p^k \geqslant 5$. Therefore, $p^k - 1 \mid p^t - 1$, so that $k \mid t$, and since $t \mid k$, we have $k = t$ and so $K = T$.

Now let $p^k = 3$ so that $p^k - 1 = 2$ and $p^k + 1 = 4$. It follows that if $\varphi^k = \varphi$ stabilizes $\chi_{q+1}^{(n)}$ for some n, then $3^f - 1 \mid 2n$ or $3^f - 1 \mid 4n$. Hence $3^f - 1 \mid 4n$ in any case and if f is odd, then $(3^f - 1)/2$ is odd and divides $2n$. Hence $(q - 1)/2$ divides n, contradicting $1 \leqslant n \leqslant (q - 3)/2$. Therefore, if φ stabilizes $\chi_{q+1}^{(n)}$ for some n, then f is even.

Conversely, if f is even, then $4 \mid 3^f - 1$. Setting $n = (3^f - 1)/4 = (q - 1)/4$, we have $n < (q - 1)/2$, so that $n \leqslant (q - 3)/2$ and $\chi_{q+1}^{(n)} \in \mathrm{Irr}(\mathrm{PGL}_2(q))$. Moreover, $3^f - 1 \mid 4n$ so that $\chi_{q+1}^{(n)}$ is invariant under φ. Hence $K = A$ is the stabilizer of $\chi_{q+1}^{(n)}$. □

Lemma 5.4. *The subgroup $K = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$ of A is the stabilizer in A of an irreducible character of $\mathrm{PSL}_2(q)$ of degree $q + 1$ if and only if either $p^k \in \{2, 3, 2^2, 5, 3^2\}$ with f/k even or $p^k \notin \{2, 3, 2^2, 5, 3^2\}$. If $p = 2$ or 3, $k = 2$, and $f/2$ is odd, then there is an irreducible character of $\mathrm{PSL}_2(q)$ of degree $q + 1$ whose stabilizer is A, hence is invariant under $K = \mathrm{PGL}_2(q)\langle\varphi^2\rangle$.*

Proof. Characters of $\mathrm{PSL}_2(q)$ of degree $q + 1$ are χ_n for $1 \leqslant n \leqslant (q - 2)/2$ if q is even and $1 \leqslant n \leqslant (q - 3)/2$, n even, if q is odd. By Lemma 4.7, χ_n is fixed by φ^k if and only if

$$p^f - 1 \mid (p^k - 1)n \text{ or } p^f - 1 \mid (p^k + 1)n.$$

Assume first that K is the stabilizer of χ_n for some n, and suppose $p^k \in \{2, 3, 2^2, 5, 3^2\}$. We also assume f/k is odd and work for a contradiction.

If $p^k = 2$, 3, or 5, so $k = 1$, and K stabilizes χ_n, then $p^f - 1 \mid (p - 1)n$ or $p^f - 1 \mid (p + 1)n$. Since $f = f/k$ is odd, $(p^f - 1, p + 1) \mid 2$ by Lemma 4.8, and so if $p^f - 1 \mid (p + 1)n$, then $p^f - 1 \mid 2n$, i.e., $q - 1 \mid 2n$. This contradicts the fact that $n < (q - 1)/2$ in all cases. Hence $p^f - 1 \mid (p - 1)n$, and so $(q - 1)/(p - 1) \mid n$. For $p = 2$ or 3, this again contradicts $n < (q - 1)/2$. Finally, for $p = 5$, we have $(5^f - 1)/4 \mid n$ and since f is odd, $(5^f - 1)/4$ is odd. Recalling that n must be even, we then have $2 \cdot (5^f - 1)/4 \mid n$, and so $(q - 1)/2 \mid n$, again contradicting $n < (q - 1)/2$.

Now suppose $p^k = 2^2$ or 3^2, so $k = 2$, and assume χ_n is invariant under K, hence under φ^2, for some n. In this case we have $p^f - 1 \mid (p^2 - 1)n$ or $p^f - 1 \mid (p^2 + 1)n$. Since $f/k = f/2$ is odd, we have $(p^f - 1, p^2 + 1) \mid 2$ by Lemma 4.8, and as before, this implies that if $p^f - 1 \mid (p^2 + 1)n$, then $q - 1 \mid 2n$, a contradiction.

Hence we must have $p^f - 1 \mid (p^2 - 1)n$. If $p = 2$, this means $2^f - 1 \mid 3n$. Hence in fact $2^f - 1 \mid (2^1 + 1)n$, and so χ_n is invariant under φ. If $p = 3$, we have $3^f - 1 \mid 8n$. Since $f/2$ is odd, f is not divisible by 4, and it follows that $(3^f - 1)/8$ is odd and divides n, which is even. Hence $2 \cdot (3^f - 1)/8 \mid n$, and so $3^f - 1 \mid 4n = (3^1 + 1)n$ and again this implies χ_n is invariant under φ. Therefore, in either case $K = \mathrm{PGL}_2(q)\langle\varphi^2\rangle$ is not the full stabilizer of χ_n.

On the other hand, if $k = 2$ and $f/2$ is odd, then both $(2^f - 1)/3$ and $(3^f - 1)/4$ are integers, $(2^f - 1)/3 \leqslant (2^f - 2)/2$ and $(3^f - 1)/4 \leqslant (3^f - 3)/2$, and $(3^f - 1)/4$ is even. Hence, setting $n = (2^f - 1)/3$ when $p = 2$ and $n = (3^f - 1)/4$ when $n = 3$, we have that $\chi_n \in \mathrm{Irr}(\mathrm{PSL}_2(q))$. Since $2^f - 1 = 3n$ and $3^f - 1 = 4n$, we obtain $2^f - 1 \mid (2^1 + 1)n$ and $3^f - 1 \mid (3^1 + 1)n$, and so χ_n is invariant under φ, and hence A, in either case.

We now have that if K is the stabilizer in A of an irreducible character of $\mathrm{PSL}_2(q)$ of degree $q+1$ and $p^k \in \{2,3,2^2,5,3^2\}$, then f/k is even. We next consider the converse, and so let $K = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$ and if $p^k \in \{2,3,2^2,5,3^2\}$, we assume f/k is even. We will find an n in each case such that K is the stabilizer of χ_n.

We first suppose $p^k \in \{2,3,5\}$, so that $k=1$ and f is even. If $p=2$, this implies $3 \mid 2^f-1$ and we set $n=(2^f-1)/3$, so that $n<(2^f-1)/2$, so $n \leqslant (q-2)/2$ and $\chi_n \in \mathrm{Irr}(\mathrm{PSL}_2(q))$. If $p=3$ or 5, then f even implies $8 \mid p^f-1$ and we set $n=(p^f-1)/4$. Thus n is even and strictly less than $(q-1)/2$, hence $n \leqslant (q-3)/2$, and so $\chi_n \in \mathrm{Irr}(\mathrm{PSL}_2(q))$. In all three cases, we have $n=(p^f-1)/(p\pm1)$ and so $p^f-1 \mid (p\pm1)n$, which implies that χ_n is invariant under φ. Hence the stabilizer in A of χ_n is $A=K$.

Next, let $p^k=2^2$ or 3^2, so that $k=2$ and $f/k=f/2$ is even. By Lemma 4.8, $p^2+1 \mid p^f-1$ and we set $n=(p^f-1)/(p^2+1)$. For $p=2$, we have $n=(2^f-1)/5<(2^f-1)/2$, hence $n\leqslant(q-2)/2$ and $\chi_n \in \mathrm{Irr}(\mathrm{PSL}_2(q))$. For $p=3$, we have $n=(3^f-1)/10<(3^f-1)/2$, hence $n \leqslant (q-3)/2$. Also, since f is even, $8 \mid 3^f-1$ and so n is even. Thus again $\chi_n \in \mathrm{Irr}(\mathrm{PSL}_2(q))$. In both cases, $p^f-1 \mid (p^2+1)n$, and so χ_n is stabilized by $\varphi^2=\varphi^k$. Also $p\pm1<p^2+1$, so $(p\pm1)n<p^f-1$ and hence p^f-1 divides neither $(p+1)n$ nor $(p-1)n$. It follows that χ_n is not invariant under φ, and so the stabilizer of χ_n in A is $K=\mathrm{PGL}_2(q)\langle\varphi^2\rangle$, as claimed.

We now suppose $p^k \notin \{2,3,2^2,5,3^2\}$. Set $n=2\cdot(p^f-1)/(p^k-1)$, which is even as $p^k-1 \mid p^f-1$. Since $p^k \geqslant 7$, we have

$$n = 2\cdot\frac{q-1}{p^k-1} \leqslant 2\cdot\frac{q-1}{6} < \frac{q-1}{2}.$$

Therefore, if q is even, then $n \leqslant (q-2)/2$ and if q is odd, then n is even and $n \leqslant (q-3)/2$, so that $\chi_n \in \mathrm{Irr}(\mathrm{PSL}_2(q))$ in any case. Moreover, $p^f-1 \mid (p^k-1)n$, and so χ_n is invariant under φ^k and K is contained in the stabilizer T of χ_n in A.

We know that T is of the form $T=\mathrm{PGL}_2(q)\langle\varphi^t\rangle$ for some positive divisor t of k. In particular, observe that $t \mid k$ implies $p^t-1 \mid p^k-1$. Also, since φ^t stabilizes χ_n, we have $p^f-1 \mid (p^t\pm1)n$; that is

$$p^f-1 \mid (p^t\pm1)\cdot 2\cdot\frac{p^f-1}{p^k-1},$$

which implies that either $p^k-1 \mid 2(p^t-1)$ or $p^k-1 \mid 2(p^t+1)$.

First, we show that $p^k-1 \mid 2(p^t+1)$ cannot occur if $p^k \notin \{2,3,2^2,5,3^2\}$. If $p^k-1 \mid 2(p^t+1)$, then $t \mid k$ implies $p^t-1 \mid 2(p^t+1)=2(p^t-1)+4$. Hence $p^t-1 \mid 4$ and $p^t=2$, 3, or 5, so $t=1$ and we have $p^k-1 \mid 2(p+1)$ with $p=2$, 3, or 5. If $p=2$, then $2^k-1 \mid 6$, hence $k=1$ or 2 and $p^k=2$ or 2^2. If $p=3$, then $3^k-1 \mid 8$, hence $k=1$ or 2 and $p^k=3$ or 3^2. If $p=5$, then $5^k-1 \mid 12$, hence $k=1$ and $p^k=5$. Therefore, if $p^k \notin \{2,3,2^2,5,3^2\}$, then $p^k-1 \nmid 2(p^t+1)$.

Finally, suppose $p^k-1 \mid 2(p^t-1)$. If $p=2$, this implies $p^k-1 \mid (p^t-1)$, hence $k \mid t$ and so $t=k$ and $T=K$. If p is odd, this implies $(p^k-1)/(p^t-1)$ divides 2. Suppose $k>t$, so that $(p^k-1)/(p^t-1)=2$ and $p^k-1=2(p^t-1)$. We have $p^k \geqslant p\cdot p^t$, hence

$$2(p^t-1)=p^k-1 \geqslant p\cdot p^t-1 = p(p^t-1)+(p-1) > 2(p^t-1),$$

a contradiction. Hence $t=k$ and $T=K$ in this case as well. Therefore, if $p^k \notin \{2,3,2^2,5,3^2\}$, then K is the stabilizer in A of χ_n. □

6 Subgroups of $\mathrm{Aut}(\mathrm{PSL}_2(q))$ and Their Degrees

Our goal is to determine the character degrees of every group H with $\mathrm{PSL}_2(q) \leqslant H \leqslant \mathrm{Aut}(\mathrm{PSL}_2(q))$. Recall that if q is even, then δ is an inner automorphism, $\mathrm{PGL}_2(q)=\mathrm{PSL}_2(q)$, and $\mathrm{Aut}(\mathrm{PSL}_2(q)) = \mathrm{PSL}_2(q)\langle\varphi\rangle$. Hence, if $\mathrm{PSL}_2(q) < H \leqslant \mathrm{Aut}(\mathrm{PSL}_2(q))$ with q even, then $H=\mathrm{PSL}_2(q)\langle\varphi^k\rangle$ for some $k \mid f$ with $1 \leqslant k < f$, and $H/\mathrm{PSL}_2(q)$ is cyclic.

If q is odd, then $\mathrm{Aut}(\mathrm{PSL}_2(q)) = \mathrm{PSL}_2(q)\langle \delta, \varphi \rangle$ and the outer automorphism group of $\mathrm{PSL}_2(q)$ is $\mathrm{Aut}(\mathrm{PSL}_2(q))/\mathrm{PSL}_2(q) \cong \langle \bar{\delta} \rangle \times \langle \bar{\varphi} \rangle$, where $\bar{\delta}$ is of order 2 and $\bar{\varphi}$ is of order f. The following elementary lemma will be useful in describing the subgroups of $\mathrm{Aut}(\mathrm{PSL}_2(q))$ in this case.

Lemma 6.1. *If $A \cong \langle x \rangle \times \langle y \rangle$, where $|\langle x \rangle| = 2$ and $|\langle y \rangle| = f$, then any subgroup of A that does not contain x is cyclic.*

Proof. If f is odd, then A is cyclic, so we may assume f is even. In this case, A contains exactly three elements of order 2, as does any noncyclic subgroup of A. Thus a subgroup of A not containing the element x of order 2 cannot contain three elements of order 2, hence is cyclic. □

Corollary 6.2. *If $\mathrm{PSL}_2(q) < H \leqslant \mathrm{Aut}(\mathrm{PSL}_2(q))$ with $q = p^f$, p an odd prime, then one of the following occurs:*

i. $\delta \in H$ so that $\mathrm{PGL}_2(q) \leqslant H$ and $H = \mathrm{PGL}_2(q)\langle \varphi^k \rangle$ for some $k \mid f$ with $1 \leqslant k \leqslant f$;

ii. $H = \mathrm{PSL}_2(q)\langle \varphi^k \rangle$ for some $k \mid f$ with $1 \leqslant k < f$;

iii. $H = \mathrm{PSL}_2(q)\langle \delta\varphi^k \rangle$ for some $k \mid f$ with $1 \leqslant k < f$ and f/k even.

Proof. If $\delta \in H$, then H contains $\mathrm{PSL}_2(q)\langle \delta \rangle \cong \mathrm{PGL}_2(q)$. If δ is not in H, then $H/\mathrm{PSL}_2(q)$ is cyclic by the lemma, hence $H = \mathrm{PSL}_2(q)\langle \sigma \rangle$ for some outer automorphism σ. If H is not a subgroup of $\mathrm{PSL}_2(q)\langle \varphi \rangle$, then $\sigma = \delta\varphi^k$ for some $k \mid f$. Finally, f/k is the order of φ^k, and so if f/k is odd, then δ is in H. □

Corollary 6.3. *Let $\mathrm{PSL}_2(q) \leqslant H \leqslant \mathrm{Aut}(\mathrm{PSL}_2(q))$ and set $G = \mathrm{PGL}_2(q)$ if $\delta \in H$ and $G = \mathrm{PSL}_2(q)$ if $\delta \notin H$. If $\hat{\chi} \in \mathrm{Irr}(H)$, then $\hat{\chi}(1) = \chi(1)|H : I_H(\chi)|$, where χ is a constituent of $\hat{\chi}_G$ and $I_H(\chi)$ is the stabilizer of χ in H.*

Proof. We have that $G \trianglelefteq H$ and H/G is cyclic. Hence a character $\chi \in \mathrm{Irr}(G)$ extends to its stabilizer $I_H(\chi)$ in H and each extension induces irreducibly to H by Theorem 6.11 of [4]. Every character of H lying over χ will therefore have degree $\chi(1)|H : I_H(\chi)|$. □

We first determine the stabilizers of characters of $\mathrm{PSL}_2(q)$ of degrees 1, q, and $(q+\varepsilon)/2$, and the degrees of the characters of H lying over these.

Theorem 6.4. *Let $S = \mathrm{PSL}_2(q)$ and let $S \leqslant H \leqslant \mathrm{Aut}(S)$. If $\chi \in \mathrm{Irr}(S)$ has degree 1 or q, then every irreducible character of H lying over χ has degree $\chi(1)$.*

Proof. By Lemmas 4.4 and 4.5, χ is invariant in H. If H does not contain δ, then by Corollary 6.2, H/S is cyclic and χ extends to H. The result then follows from Gallagher's Theorem (see [4, 6.17]).

If $\delta \in H$, then by Lemma 4.4, χ extends to two irreducible characters of $\mathrm{PGL}_2(q)$. By Lemma 4.5, both of these are invariant in H. Since $H/\mathrm{PGL}_2(q)$ is cyclic, these characters extend to H and the result again follows from Gallagher's Theorem. □

Theorem 6.5. *Let $S = \mathrm{PSL}_2(q)$ with q odd and let $S \leqslant H \leqslant \mathrm{Aut}(S)$. Let $\mu \in \mathrm{Irr}(S)$ with $\mu(1) = (q+\varepsilon)/2$.*

i. If $H \leqslant S\langle \varphi \rangle$, then μ is invariant in H and every irreducible character of H lying over μ is of degree $(q+\varepsilon)/2$.

ii. If $H \nleqslant S\langle \varphi \rangle$, then the stabilizer of μ in H is $I_H(\mu) = H \cap (S\langle \varphi \rangle)$ and $|H : I_H(\mu)| = 2$. Every irreducible character of H lying over μ is of degree $q+\varepsilon$.

Proof. If $H \leqslant S\langle\varphi\rangle$, then μ is invariant in H by Lemma 4.5. Since H/S is cyclic, μ extends to H and (i) follows from Gallagher's Theorem ([4, 6.17]).

Assume now that $H \nleqslant S\langle\varphi\rangle$, so $\delta\varphi^k \in H$ for some integer k. By Lemma 4.4, μ is not fixed by this automorphism, and so $I_H(\mu) < H$. By Lemma 4.5, μ is invariant in $H \cap (S\langle\varphi\rangle)$ and we have $H \cap (S\langle\varphi\rangle) \leqslant I_H(\mu) < H$. Since δ^2 is an inner automorphism, $|H : H \cap (S\langle\varphi\rangle)| = 2$, and hence $I_H(\mu) = H \cap (S\langle\varphi\rangle)$.

We have that $I_H(\mu) \leqslant S\langle\varphi\rangle$, and so $I_H(\mu)/S$ is cyclic. Thus μ extends to $I_H(\mu)$ and, by Gallagher's Theorem, each extension has degree $(q+\varepsilon)/2$. Finally, by Clifford's Theorem, each extension induces to an irreducible character of H of degree $|H : I_H(\mu)|(q+\varepsilon)/2 = q+\varepsilon$. ☐

Theorem 6.6. *Let $S = \mathrm{PSL}_2(q)$, where $q = p^f > 3$ for a prime p, $A = \mathrm{Aut}(S)$, and let $S \leqslant H \leqslant A$. Let $G = \mathrm{PGL}_2(q)$ if $\delta \in H$ and $G = S$ if $\delta \notin H$, and set $|H : G| = d = 2^a m$, m odd.*

The degrees of the irreducible characters of H lying over characters of G of degree $q-1$ are precisely $(q-1)2^a\ell$, where ℓ is a positive divisor of m, with the exception that if $p = 3$, f is odd, and $H = \mathrm{PSL}_2(q)\langle\varphi\rangle$, then $\ell \neq 1$.

Proof. First suppose $\delta \in H$, so that $H = \mathrm{PGL}_2(q)\langle\varphi^{f/d}\rangle$. (We include here the case where q is even.) By Corollary 6.3, for $j \mid d$, there is a character of H of degree $(q-1)j$ lying over a character θ of $\mathrm{PGL}_2(q)$ of degree $q-1$ if and only if $|H : I_H(\theta)| = j$, hence if and only if $I_H(\theta) = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$, where $k = (f/d)j$. By Lemma 5.2, such a character θ of $\mathrm{PGL}_2(q)$ exists if and only if $f/k = d/j$ is odd, that is, $j = 2^a\ell$ for some $\ell \mid m$.

Suppose now that q is odd and $\delta \notin H$. By Corollary 6.2, $H = \mathrm{PSL}_2(q)\langle\delta^c\varphi^{f/d}\rangle$, where $c \in \{0,1\}$ and if d is odd, then $c = 0$ since $\delta \notin H$. Again by Corollary 6.3, there is a character of H of degree $(q-1)j$ lying over a character θ of $\mathrm{PSL}_2(q)$ of degree $q-1$ if and only if $I_H(\theta) = \mathrm{PSL}_2(q)\langle\delta^{cj}\varphi^k\rangle$, where $k = (f/d)j$. By Lemma 5.2, this implies $f/k = d/j$ must be odd, that is, $j = 2^a\ell$ for some $\ell \mid m$. Conversely, if d/j is odd, then such a character θ of $\mathrm{PSL}_2(q)$ exists except when $p = 3$ and $k = 1$. Since $d \mid f$, $k = (f/d)j = 1$ if and only if $d = f$ (so $f/d = 1$) and $j = 1$. Observe that since $d = f$ is odd in this case, $c = 0$ and $H = \mathrm{PSL}_2(q)\langle\varphi\rangle$. ☐

Theorem 6.7. *Let $S = \mathrm{PSL}_2(q)$, where $q = p^f > 3$ for a prime p, $A = \mathrm{Aut}(S)$, and let $S \leqslant H \leqslant A$. Let $G = \mathrm{PGL}_2(q)$ if $\delta \in H$ and $G = S$ if $\delta \notin H$, and set $|H : G| = d$.*

The degrees of the irreducible characters of H lying over characters of G of degree $q+1$ are precisely $(q+1)j$, where j is a positive divisor of d, with the following exceptions:

i. if f is odd, $p = 3$, and $H = A$, then $j \neq 1$;

ii. if f is odd, $p = 2$, 3, or 5, and $H = S\langle\varphi\rangle$, then $j \neq 1$;

iii. if $f \equiv 2 \pmod 4$, $p = 2$ or 3, and $H = S\langle\varphi\rangle$ or $H = S\langle\delta\varphi\rangle$, then $j \neq 2$.

Proof. First, suppose q is odd and $\delta \in H$, so that $H = \mathrm{PGL}_2(q)\langle\varphi^{f/d}\rangle$. By Corollary 6.3, for $j \mid d$, there is a character of H of degree $(q+1)j$ lying over a character χ of $\mathrm{PGL}_2(q)$ of degree $q+1$ if and only if $|H : I_H(\chi)| = j$, hence if and only if $I_H(\chi) = \mathrm{PGL}_2(q)\langle\varphi^k\rangle$, where $k = (f/d)j$. By Lemma 5.3, such a character χ of $\mathrm{PSL}_2(q)$ exists except when f is odd, $p = 3$, and $k = 1$. Since $d \mid f$, $k = (f/d)j = 1$ if and only if $f = d$ (hence $K = A$) and $j = 1$, which is exception (i) in the statement of the theorem.

Suppose now that either q is odd and $\delta \notin H$ or q is even. By Corollary 6.2, $H = \mathrm{PSL}_2(q)\langle\delta^c\varphi^{f/d}\rangle$, where $c \in \{0,1\}$ and if d is odd, then $c = 0$ since $\delta \notin H$. Again, Corollary 6.3 implies that if $j \mid d$, there is a character of H of degree $(q+1)j$ lying over a character χ of $\mathrm{PSL}_2(q)$ of degree $q+1$ if and only if $I_H(\chi) = \mathrm{PSL}_2(q)\langle\delta^{cj}\varphi^k\rangle$, where $k = (f/d)j$. Lemma 5.4 implies that if $p^k \notin \{2,3,2^2,5,3^2\}$ or if $p^k \in \{2,3,2^2,5,3^2\}$ and f/k is even, then such a character χ of $\mathrm{PSL}_2(q)$ exists and H has a character of degree $(q+1)j$. It therefore remains to consider the cases where $p^k \in \{2,3,2^2,5,3^2\}$ and f/k is odd.

Suppose $p = 2$, 3, or 5, $k = (f/d)j = 1$, and f is odd. Since $d \mid f$, $k = 1$ implies that $j = 1$ and $d = f$, which is odd, so that $H = \mathrm{PSL}_2(q)\langle\varphi\rangle$. In this case, Lemma 5.4 implies there is no character of $\mathrm{PSL}_2(q)$ of degree $q + 1$ stabilized by H, hence H has no character of degree $(q + 1)j$ for $j = 1$, which is exception (ii) in the statement of the theorem.

Finally, suppose $p = 2$ or 3, $k = (f/d)j = 2$, and $f/2$ is odd, that is, $f \equiv 2 (\mathrm{mod}\ 4)$. In this case, $k = (f/d)j = 2$ implies that either $j = 1$ and $d = f/2$, or $j = 2$ and $d = f$.

If $j = 1$ and $d = f/2$, we have $H = \mathrm{PSL}_2(q)\langle\varphi^2\rangle$. By Lemma 5.4, there is a character χ of $\mathrm{PSL}_2(q)$ of degree $q + 1$ invariant under φ, hence under H, and so H has a character of degree $(q + 1)j$ with $j = 1$.

If $j = 2$ and $d = f$, then either $H = \mathrm{PSL}_2(q)\langle\varphi\rangle$ or $H = \mathrm{PSL}_2(q)\langle\delta\varphi\rangle$, and the subgroup of H of index $j = 2$ is $K = \mathrm{PSL}_2(q)\langle\varphi^2\rangle$. However, by Lemma 5.4, K is not the stabilizer in H of any character of $\mathrm{PSL}_2(q)$ of degree $q + 1$ (any such character invariant under φ^2 is also invariant under φ). Hence H does not have a character of degree $(q + 1)j$ for $j = 2$, which is exception (iii) in the statement of the theorem. □

Finally, we observe that Theorem A now follows from Corollary 6.3 and Theorems 6.4, 6.5, 6.6, and 6.7.

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