ALGEBRA QUALIFYING EXAM PROBLEMS
LINEAR ALGEBRA

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**LINEAR ALGEBRA AND MODULES**

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LINEAR ALGEBRA

General Matrix Theory

1. Let $m > n$ be positive integers. Show that there do not exist matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$ such that $AB = I_m$, where $I_m$ is the $m \times m$ identity matrix.

2. Let $A$ and $B$ be nonsingular $n \times n$ matrices over $\mathbb{C}$.
   (a) Show that if $A^{-1}B^{-1}AB = cI$, $c \in \mathbb{C}$, then $c^n = 1$.
   (b) Show that if $AB - BA = cI$, $c \in \mathbb{C}$, then $c = 0$.

3. Let $A$ be a strictly upper triangular $n \times n$ matrix with real entries, and let $I$ be the $n \times n$ identity matrix. Show that $I - A$ is invertible and express the inverse of $I - A$ as a function of $A$.

4. Let $A$ and $B$ be complex $2 \times 2$ matrices so that $A(AB - BA) = (AB - BA)A$. Prove that the matrix $AB - BA$ is nilpotent.

5. (a) Give an example of a complex $2 \times 2$ matrix that does not have a square root.
   (b) Show that every complex non-singular $n \times n$ matrix has a square root.
       [Hint: Show first that a Jordan block with non-zero eigenvalue has a square root.]

6. Does there exist a $2023 \times 2023$ real matrix $A$ such that $A^2 = -I$, where $I$ is the identity matrix?

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $T(v) = Av$, where

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
2 & 0 & 3 \\
0 & 1 & 0 \\
3 & 4 & 2
\end{bmatrix}
\]

   (a) Find the dimension of the null space of $T$.
   (b) Find a basis for the range space of $T$.

8. (a) Let

\[
J = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}
\]

Show that $J$ is similar to $J^t$ by a symmetric transforming matrix.
[Recall: Matrices $X$ and $Y$ are similar if there is a matrix $P$ so that $P^{-1}XP = Y$, and $P$ is called a transforming matrix.]

(b) Show that if $A$ is an $n \times n$ matrix, then $A$ is similar to $A^t$ by a symmetric transforming matrix.
9. (a) Let
\[ J = \begin{bmatrix}
\lambda & 1 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \lambda & 1 \\
0 & \cdots & 0 & \lambda
\end{bmatrix} \]
be an \( n \times n \) Jordan block. Show that \( J \) is similar to its transpose \( J^T \).
(b) Show that if \( A \) is an \( n \times n \) matrix with entries in \( \mathbb{C} \), then \( A \) is similar to \( A^T \).

Canonical Forms, Diagonalization, and Characteristic and Minimal Polynomials

10. State and prove the Cayley-Hamilton Theorem.

11. Let \( A \) be a complex \( n \times n \) matrix with characteristic polynomial \( f(x) \) and minimal polynomial \( g(x) \). By the Cayley-Hamilton Theorem, we know that \( f(A) = 0 \). Prove that \( g(x) \) divides \( f(x) \) and that \( f(x) \) divides some power of \( g(x) \).

12. Show that if \( A \) is an \( n \times n \) matrix, then \( A^n \) can be written as a linear combination of the matrices \( I, A, A^2, \ldots, A^{n-1} \) (that is, \( A^n = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_{n-1} A^{n-1} \) for some scalars \( \alpha_0, \ldots, \alpha_{n-1} \)).

13. Let \( A \) be an \( n \times n \) Jordan block. Show that any matrix that commutes with \( A \) is a polynomial in \( A \).

14. Let \( A \) be a square matrix whose minimal polynomial is equal to its characteristic polynomial. Show that if \( B \) is any matrix that commutes with \( A \), then \( B \) is a polynomial in \( A \).

15. Let \( A \) be an \( n \times n \) matrix, \( \mathbf{v} \) a column vector, and suppose \( \{ \mathbf{v}, A\mathbf{v}, \ldots, A^{n-1}\mathbf{v} \} \) is linearly independent. Prove that if \( B \) is any matrix that commutes with \( A \), then \( B \) is a polynomial in \( A \).

16. Prove that an \( n \times n \) complex matrix \( A \) is diagonalizable if and only if the minimal polynomial of \( A \) has distinct roots.

17. Let \( G = GL_n(\mathbb{C}) \) be the multiplicative group of invertible \( n \times n \) matrices with complex entries and let \( g \) be an element of \( G \) of finite order. Show that \( g \) is diagonalizable.

18. Let \( V \) be a vector space and let \( T : V \to V \) be a linear transformation.
   (a) Show that \( T \) is invertible if and only if the minimal polynomial of \( T \) has non-zero constant term.
   (b) Show that if \( T \) is invertible then \( T^{-1} \) can be expressed as a polynomial in \( T \).

19. Let \( A \) and \( B \) be complex \( 3 \times 3 \) matrices having the same eigenvectors. Suppose the minimal polynomial of \( A \) is \( (x - 1)^2 \) and the characteristic polynomial of \( B \) is \( x^3 \). Show that the minimal polynomial of \( B \) is \( x^2 \).

20. Let \( A \) and \( B \) be \( 5 \times 5 \) complex matrices and suppose that \( A \) and \( B \) have the same eigenvectors. Show that if the minimal polynomial of \( A \) is \( (x + 1)^2 \) and the characteristic polynomial of \( B \) is \( x^5 \), then \( B^3 = 0 \).
21. A square matrix $A$ over $\mathbb{C}$ is Hermitian if $\bar{A}^t = A$. Prove that the eigenvalues of a Hermitian matrix are all real.

22. (a) Prove that a $2 \times 2$ scalar matrix $A$ over a field $F$ has a square root (i.e., a matrix $B$ satisfying $B^2 = A$).

(b) Prove that a real symmetric matrix having the property that every negative eigenvalue occurs with even multiplicity has a square root. [Hint: Use (a).]

23. Let $A$ and $B$ be complex $n \times n$ matrices. Prove that if $AB = BA$, then $A$ and $B$ share a common eigenvector.

24. Let $A$ be a $5 \times 5$ matrix with complex entries such that $A^3 = 0$. Find all possible Jordan canonical forms for $A$.

25. The characteristic polynomial of a certain $4 \times 4$ matrix $A$ has the two distinct roots 2 and 3, with the multiplicity of the root 3 less than or equal to the multiplicity of the root 2. List all possible Jordan canonical forms of $A$, up to rearrangements of the Jordan blocks.

26. Let $A$ be an $n \times n$ matrix and let $I$ be the $n \times n$ identity matrix. Show that if $A^2 = I$ and $A \neq I$, then $\lambda = -1$ is an eigenvalue of $A$.

27. (a) Show that two $3 \times 3$ complex matrices are similar if and only if they have the same characteristic and minimal polynomials.

(b) Is the conclusion of part (a) true for larger matrices? Prove or give a counter-example.

28. (a) Find all possible Jordan canonical forms of a $5 \times 5$ complex matrix with minimal polynomial $(x - 2)^2(x - 1)$.

(b) Find all possible Jordan canonical forms of a complex matrix with characteristic polynomial $(x - 3)^3(x - 5)^2$.

29. Find all possible Jordan canonical forms for the following. EXPLAIN your answers.

(a) A linear operator $T$ with characteristic polynomial $\Delta(x) = (x - 2)^4(x - 3)^2$ and minimal polynomial $m(x) = (x - 2)^2(x - 3)^2$.

(b) A linear operator $T$ with characteristic polynomial $\Delta(x) = (x - 4)^5$ and such that $\dim \ker(T - 4I) = 3$.

30. A matrix $A$ has characteristic polynomial $\Delta(x) = (x - 3)^5$ and minimal polynomial $m(x) = (x - 3)^3$.

(a) List all possible Jordan canonical forms for $A$.

(b) Determine the Jordan canonical form of the matrix

$$A = \begin{bmatrix}
3 & -1 & 2 & 0 & 0 \\
2 & 3 & 0 & -2 & 0 \\
1 & 0 & 3 & -1 & 0 \\
0 & -1 & 2 & 3 & 0 \\
0 & 2 & -3 & 0 & 3
\end{bmatrix}$$

which has the given characteristic and minimal polynomials.
31. Let \( T : V \rightarrow V \) be a linear transformation defined on the finite dimensional vector space \( V \). Let \( \lambda \) be an eigenvalue of \( T \), and set \( W_\lambda = \{ v \in V \mid (T - \lambda I)^i(v) = 0 \} \). If \( m \) is the multiplicity of \( \lambda \) as a root of the characteristic polynomial of \( T \), prove that \( W_m = W_{m+1} \).

32. Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \) be a matrix over the field \( F \), where \( F \) is either the field of rational numbers or the field of \( p \) elements for some prime \( p \).

(a) Find a basis of eigenvectors for \( A \) over those fields for which such a basis exists.

(b) What is the Jordan canonical form of \( A \) over the fields not included in part (a)?

33. Let \( A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \), let \( B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 3 \end{bmatrix} \), and let \( C = \begin{bmatrix} 2 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \).

(a) Find the characteristic polynomial of \( A, B, \) and \( C \).

(b) Find the minimal polynomial of \( A, B, \) and \( C \).

(c) Find the eigenvalues of \( A, B, \) and \( C \).

(d) Find the dimensions of all eigenspaces of \( A, B, \) and \( C \).

(e) Find the Jordan canonical form of \( A, B, \) and \( C \).

34. Let \( A \) be the following matrix:

\[
A = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 2 \\
0 & 2 & 1 & 1 & 2 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 2
\end{bmatrix}
\]

Find the characteristic polynomial, minimal polynomial, eigenvalues, eigenvectors, dimensions of eigenspaces, and the Jordan canonical form of this matrix.

35. Let \( A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{bmatrix} \).

(a) Determine conditions on \( a, b, c, \) and \( d \) so that there is only one Jordan block for each eigenvalue of \( A \) in the Jordan canonical form of \( A \).

(b) Suppose now \( a = c = d = 2 \) and \( b = -2 \). Find the Jordan canonical form of \( A \).

36. Let \( A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & a & -1 \end{bmatrix} \).

(a) Determine conditions on \( a, b, c, \) and \( d \) so that there is only one Jordan block for each eigenvalue of \( A \) in the Jordan canonical form of \( A \).

(b) Suppose now \( a = b = c = d = 2 \). Find the Jordan canonical form of \( A \).
37. Let $A$ be a square complex matrix with a single eigenvalue $\lambda$. Show that the number of blocks in the Jordan form of $A$ is the dimension of the $\lambda$-eigenspace.

38. Let $T : V \to V$ be a linear transformation satisfying $T^2 = 0$. Prove that the Jordan canonical form of $T$ consists of $\dim(\ker T)$ Jordan blocks, $\dim(\text{Im } T)$ of which are $2 \times 2$ blocks.

39. Let $A$ be an $n \times n$ nilpotent matrix such that $A^{n-1} \neq 0$. Show that $A$ has exactly one Jordan block.

40. Let $A = \begin{bmatrix} -1 & 4 & -2 \\ -2 & 5 & -2 \\ -1 & 2 & 0 \end{bmatrix}$ with characteristic polynomial $\Delta(x) = (x - 1)^2(x - 2)$.

(a) For each eigenvalue $\lambda$ of $A$, find a basis for the eigenspace $E_{\lambda}$.

(b) Determine if $A$ is diagonalizable. If so, give matrices $P, B$ such that $P^{-1}AP = B$ and $B$ is diagonal. If not, explain carefully why $A$ is not diagonalizable.

41. Let $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$.

(a) Verify that the characteristic polynomial of $A$ is $\Delta(x) = x(x - 1)^2$.

(b) For each eigenvalue $\lambda$ of $A$, find a basis for the eigenspace $E_{\lambda}$.

(c) Determine if $A$ is diagonalizable. If so, give matrices $P, B$ such that $P^{-1}AP = B$ and $B$ is diagonal. If not, explain carefully why $A$ is not diagonalizable.

42. Let $A = \begin{bmatrix} 5 & 0 & 6 \\ 2 & 2 & 4 \\ -2 & 0 & -2 \end{bmatrix}$.

(a) Verify that the characteristic polynomial of $A$ is $\Delta(x) = (x - 1)(x - 2)^2$.

(b) For each eigenvalue $\lambda$ of $A$, find a basis for the eigenspace $E_{\lambda}$.

(c) Determine if $A$ is diagonalizable. If so, give matrices $P, B$ such that $P^{-1}AP = B$ and $B$ is diagonal. If not, explain carefully why $A$ is not diagonalizable.

43. Let $A$ be a matrix of the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ c_1 & c_2 & c_3 & \cdots & c_n \end{bmatrix}$$

Show that the minimal polynomial and characteristic polynomial of $A$ are equal.

44. Find the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -c_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}$$
Linear Transformations

45. (Fitting’s Lemma for vector spaces) Let $\varphi: V \to V$ be a linear transformation of a finite dimensional vector space to itself. Prove that there exists a decomposition of $V$ as $V = U \oplus W$, where each summand is $\varphi$-invariant, $\varphi|_U$ is nilpotent, and $\varphi|_W$ is nonsingular.

46. Let $V$ be a vector space and $T: V \to V$ a linear transformation such that $T^2 = T$. Show that $V = \ker T \oplus \text{Im } T$.

47. Let $F$ be a field and $V$ a finite dimensional vector space over $F$ with $\dim V > 1$. Suppose $f: V \to V$ and $g: V \to V$ are distinct nilpotent linear transformations satisfying $f^2 = g^2 = 0$ and that the only subspaces of $V$ that are both $f$-invariant and $g$-invariant are $V$ and $\{0\}$. Prove the following:
   (a) The image of $f$ equals the null space of $f$ and the image of $g$ equals the null space of $g$.
   (b) $V$ is the direct sum of the null spaces of $f$ and $g$.
   (c) $\dim V$ is even.

48. Let $V$ be a vector space over a field $F$. A linear transformation $T: V \to V$ is said to be idempotent if $T^2 = T$. Prove that if $T$ is idempotent then $V = V_0 \oplus V_1$, where $T(v_0) = 0$ for all $v_0 \in V_0$ and $T(v_1) = v_1$ for all $v_1 \in V_1$.

49. Let $U$, $V$, and $W$ be finite dimensional vector spaces with $U$ a subspace of $V$. Show that if $T: V \to W$ is a linear transformation having the same rank as $T|_U: U \to W$, then $U$ is complemented in $V$ by a subspace $K$ satisfying $T(x) = 0$ for all $x \in K$.

50. Let $V$ and $W$ be finite dimensional vector spaces and let $T: V \to W$ be a linear transformation. Show that $\dim(\ker T) + \dim(\text{Im } T) = \dim(V)$.

51. Let $V$ be a finite dimensional vector space and $T: V \to V$ a non-zero linear operator. Show that if $\ker T = \text{Im } T$, then $\dim V$ is an even integer and the minimal polynomial of $T$ is $m(x) = x^2$.

52. Let $V$ be a finite dimensional vector space over a field $F$ and let $T: V \to V$ be a nilpotent linear transformation. Show that the trace of $T$ is 0.

53. Let $V$ be the vector space of $2 \times 2$ matrices over a field $F$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ and let $T: V \to V$ be the linear transformation defined by $T(X) = AX$. Compute $\det(T)$.

54. Let $T: V \to W$ be a surjective linear transformation of finite dimensional vector spaces over a field $F$ (acting on the left). Show that there is a linear transformation $S: W \to V$ such that $T \circ S$ is the identity map on $W$.

55. A linear transformation $T: V \to W$ is said to be independence preserving if $T(I) \subseteq W$ is linearly independent whenever $I \subseteq V$ is a linearly independent set. Show that $T$ is independence preserving if and only if $T$ is one-to-one.

56. Let $V$ and $W$ be vector spaces and let $T: V \to W$ be a surjective linear transformation. Assume for all subsets $S \subseteq V$ that if $T(S)$ spans $W$, then $S$ spans $V$. Prove that $T$ is one-to-one.
57. Let $T : V \to W$ be a linear transformation of vector spaces over a field $F$.
   (a) Show that $T$ is injective if and only if $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent in $W$ whenever $\{v_1, \ldots, v_n\}$ is linearly independent in $V$.
   (b) Show that $T$ is surjective if and only if $\{T(x) \mid x \in X\}$ is a spanning set for $W$ whenever $X$ is a spanning set for $V$.

58. Let $A$ be a complex $n \times n$ matrix, and let $L : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ be the linear transformation given by $L(M) = AM$ for $M \in \mathbb{C}^{n \times n}$. Express $\det L$ in terms of $\det A$ and prove your formula is correct.

59. Let $A$ be a complex $n \times n$ matrix, and let $L : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ be the linear transformation given by $L(X) = AX + XA$ for $X \in \mathbb{C}^{n \times n}$. Prove that if $A$ is a nilpotent matrix, then $L$ is a nilpotent operator. [Note: The result is also true with $\mathbb{C}$ replaced by an arbitrary field.]

60. Let $T : V \to V$ be a linear transformation. Let $X = \ker T^{n-2}$, $Y = \ker T^{n-1}$, and $Z = \ker T^n$. Observe that $X \subseteq Y \subseteq Z$ (you need not prove this). Suppose
\[
\{u_1, \ldots, u_r\}, \{u_1, \ldots, u_r, v_1, \ldots, v_s\}, \{u_1, \ldots, u_r, v_1, \ldots, v_s, w_1, \ldots, w_t\}
\]
are bases for $X, Y, Z$, respectively. Show that $S = \{u_1, \ldots, u_r, T(w_1), \ldots, T(w_t)\}$ is contained in $Y$ and is linearly independent.

Vector Spaces

61. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space $V$ over $\mathbb{R}$. Show that if $w$ is any vector in $V$, then for some choice of sign $\pm$, $\{v_1 \pm w, v_2, \ldots, v_n\}$ is a basis for $V$.

62. Let $V$ be a finite dimensional vector space over the field $F$. Let $V^*$ be the dual space of $V$ (that is, $V^*$ is the vector space of linear transformations $T : V \to F$). Show that $V \cong V^*$.

63. Let $V$ be a vector space over the field $F$. Let $V^*$ be the dual space of $V$ and let $V^{**}$ be the dual space of $V^*$. Show that there is an injective linear transformation $\varphi : V \to V^{**}$.

64. Let $V$ be a finite dimensional vector space and let $W$ be a subspace. Show that $\dim V/W = \dim V - \dim W$.

65. Let $V$ be a vector space and let $U$ and $W$ be finite dimensional subspaces of $V$. Show that $\dim(U + W) = \dim U + \dim W - \dim U \cap W$.

66. Let $V$ be a finite-dimensional vector space over a field $F$ and let $U$ be a subspace. Show that there is a subspace $W$ of $V$ such that $V = U \oplus W$.

67. (a) Let $F$ be an algebraically closed field and $n > 1$. Let $M_n(F)$ be the vector space of $n \times n$ matrices over $F$. Prove that if $V$ is a subspace of $M_n(F)$ with $\dim V \geq 2$, then $V$ contains a nonzero singular matrix.
   (b) Show by example that (a) is false if $F$ is not algebraically closed.

68. Let $V$ be the vector space of $n \times n$ matrices over the field $\mathbb{R}$ of real numbers. Let $U$ be the subspace of $V$ consisting of symmetric matrices and $W$ the subspace of $V$ consisting of skew-symmetric matrices. Show that $V = U \oplus W$. 
69. Let $V$ be the vector space over the field $\mathbb{R}$ of real numbers consisting of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let $U$ be the subspace of even functions and $W$ the subspace of odd functions. Show that $V = U \oplus W$.

70. Let $U$, $V$, and $W$ be vector spaces over a field $F$ and let $S : U \rightarrow V$ and $T : V \rightarrow W$ be linear transformations such that $T \circ S = 0$, the zero map. Show that

$$\dim(W/\text{Im } T) - \dim(\ker T/\text{Im } S) + \dim \ker S = \dim W - \dim V + \dim U.$$ 

71. Let $A$, $B$, and $C$ be subspaces of the nonzero vector space $V$ satisfying

$$V = A \oplus B = B \oplus C = A \oplus C.$$ 

Show that there exists a 2-dimensional subspace $W \subseteq V$ such that each of $W \cap A$, $W \cap B$, and $W \cap C$ has dimension 1.

72. Let

$$V_{-1} = 0 \xrightarrow{L_{-1} = 0} V_0 \xrightarrow{L_0} V_1 \xrightarrow{L_1} \cdots \xrightarrow{L_{n-1}} V_n \xrightarrow{L_n = 0} V_{n+1} = 0$$

be a sequence of finite dimensional vector spaces and linear transformations with $L_{i+1} \circ L_i = 0$ for all $i = 0, \ldots, n$. Therefore, the quotients $H_i = \ker(L_i)/\text{im}(L_{i-1})$ are defined for $0 \leq i \leq n$. Prove that

$$\sum_i (-1)^i \dim V_i = \sum_i (-1)^i \dim H_i.$$ 

73. If $V$ is a finite dimensional vector space, let $V^*$ denote the dual of $V$. If $(\cdot, \cdot)$ is a non-degenerate bilinear form on $V$, and $W$ is a subspace of $V$, define $W^\perp = \{ v \in V \mid (v, w) = 0 \text{ for all } w \in W \}$. Show that if $X$ and $Y$ are subspaces of $V$ with $Y \subseteq X$, then $X^\perp \subseteq Y^\perp$ and $Y^\perp/X^\perp \cong (X/Y)^*$. 

**Inner Product Spaces**

74. Let $(\cdot, \cdot)$ be a positive definite inner product on the finite dimensional real vector space $V$. Let $S = \{ v_1, v_2, \ldots, v_k \}$ be a set of vectors satisfying $(v_i, v_j) < 0$ for all $i \neq j$. Prove that $\dim(\text{span } S) \geq k - 1$.

75. Let $\{ v_1, v_2, \ldots, v_k \}$ be a linearly independent set of vectors in the real inner product space $V$. Show that there exists a unique set $\{ u_1, u_2, \ldots, u_k \}$ of vectors with the property that $(u_i, v_i) > 0$ for all $i$, and $\{ u_1, u_2, \ldots, u_i \}$ is an orthonormal basis for $\text{Span}\{ v_1, v_2, \ldots, v_i \}$ for every $i$.

76. Let $(\cdot, \cdot)$ be a Hermitian inner product defined on the complex vector space $V$. If $\varphi : V \rightarrow V$ is a normal operator ($\varphi \circ \varphi^* = \varphi^* \circ \varphi$, where $\varphi^*$ is the adjoint of $\varphi$), prove that $V$ contains an orthonormal basis of eigenvectors for $\varphi$.

**Modules**

77. Let $M$ be a nonzero $R$-module with the property that every $R$-submodule $N$ is complemented (that is, there exists another $R$-submodule $C$ such that $M = N + C$ and $N \cap C = \{0\}$). Give a direct proof that $M$ contains simple submodules.

78. Let $R = F^{n \times n}$ be the ring of $n \times n$ matrices over a field $F$. Prove that the (right) $R$-module $F^{1 \times n}$, consisting of the row space of $1 \times n$ matrices, is the unique simple $R$-module (up to isomorphism).
79. Let $R \subseteq S$ be an inclusion of rings (sharing the same identity element). Let $S_R$ be the right $R$-module where the module action is right multiplication. Assume $S_R$ is isomorphic to a direct sum of $n$ copies of $R$. Prove that $S$ is isomorphic to a subring of $M_n(R)$, the ring of $n \times n$ matrices over $R$.

80. Let $M$ be a module over a ring $R$ with identity, and assume that $M$ has finite composition length. If $\varphi : M \to M$ is an $R$-endomorphism of $M$, prove that $M$ decomposes as a direct sum of $R$-submodules $M = U \oplus W$ where each summand is $\varphi$-invariant, $\varphi |_U$ is nilpotent, and $\varphi |_W$ is an automorphism.

81. Let $R$ be a ring with identity, and let $I$ be a right ideal of $R$ which is a direct summand of $R$ (i.e., $R = I \oplus J$ for some right ideal $J$). Prove that if $M$ is any $R$-module, and $\varphi : M \to I$ is any surjective $R$-homomorphism, then there exists an $R$-homomorphism $\psi : I \to M$ satisfying $\varphi \circ \psi = 1 |_I$.

82. Let $M$ be an $R$-module and let $N$ be an $R$-submodule of $M$. Prove that $M$ is Noetherian if and only if both $N$ and $M/N$ are Noetherian.

83. Let $M$ be an $R$-module and let $N$ be an $R$-submodule of $M$. Prove that $M$ is Artinian if and only if both $N$ and $M/N$ are Artinian.

84. Let $M$ be an $R$-module, where $R$ is a ring. Prove that the following statements about $M$ are equivalent.

(i) $M$ is a sum (not necessarily direct) of simple submodules.

(ii) $M$ is a direct sum of certain simple submodules.

(iii) For every submodule $N$ of $M$, there exists a complement (i.e., a submodule $C$ such that $M = N + C$ and $N \cap C = 0$).

85. Let $R$ be a ring and let $M$ be a simple $R$-module. Let $D = \text{End}_R(M)$ be the ring of $R$-endomorphisms of $M$ (under composition and pointwise addition). Prove that $D$ is a division ring.

86. Let $M$ be an $R$-module that is generated by finitely many simple submodules. Prove that $M$ is a direct sum of finitely many simple $R$-modules.