ALGEBRA QUALIFYING EXAM PROBLEMS
RING THEORY

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RING THEORY

General Ring Theory

1. Give an example of each of the following.
   (a) An irreducible polynomial of degree 3 in $\mathbb{Z}_3[x]$.
   (b) A polynomial in $\mathbb{Z}[x]$ that is not irreducible in $\mathbb{Z}[x]$ but is irreducible in $\mathbb{Q}[x]$.
   (c) A non-commutative ring of characteristic $p$, $p$ a prime.
   (d) A ring with exactly 6 invertible elements.
   (e) An infinite non-commutative ring with only finitely many ideals.
   (f) An infinite non-commutative ring with non-zero characteristic.
   (g) An integral domain which is not a unique factorization domain.
   (h) A unique factorization domain that is not a principal ideal domain.
   (i) A principal ideal domain that is not a Euclidean domain.
   (j) A Euclidean domain other than the ring of integers or a field.
   (k) A finite non-commutative ring.
   (l) A commutative ring with a sequence $\{P_n\}_{n=1}^\infty$ of prime ideals such that $P_n$ is properly contained in $P_{n+1}$ for all $n$.
   (m) A non-zero prime ideal of a commutative ring that is not a maximal ideal.
   (n) An irreducible element of a commutative ring that is not a prime element.
   (o) An irreducible element of an integral domain that is not a prime element.
   (p) A commutative ring that has exactly one maximal ideal and is not a field.
   (q) A non-commutative ring with exactly two maximal ideals.

2. (a) How many units does the ring $\mathbb{Z}/60\mathbb{Z}$ have? Explain your answer.
   (b) How many ideals does the ring $\mathbb{Z}/60\mathbb{Z}$ have? Explain your answer.

3. How many ideals does the ring $\mathbb{Z}/90\mathbb{Z}$ have? Explain your answer.

4. Denote the set of invertible elements of the ring $\mathbb{Z}_n$ by $U_n$.
   Answer the following for $n = 18, n = 20, n = 24$.
   (a) List all the elements of $U_n$.
   (b) Is $U_n$ a cyclic group under multiplication? Justify your answer.

5. Find all positive integers $n$ having the property that the group of units of $\mathbb{Z}/n\mathbb{Z}$ is an elementary abelian 2-group.

6. Let $U(R)$ denote the group of units of a ring $R$. Prove that if $m$ divides $n$, then the natural ring homomorphism $\mathbb{Z}_n \rightarrow \mathbb{Z}_m$ maps $U(\mathbb{Z}_n)$ onto $U(\mathbb{Z}_m)$.
   Give an example that shows that $U(R)$ does not have to map onto $U(S)$ under a surjective ring homomorphism $R \rightarrow S$.

7. If $p$ is a prime satisfying $p \equiv 1 \pmod{4}$, then $p$ is a sum of two squares.

8. If $(\cdot)$ denotes the Legendre symbol, prove Euler’s Criterion: if $p$ is a prime and $a$ is any integer relatively prime to $p$, then $a^{(p-1)/2} \equiv \left( \frac{a}{p} \right) \pmod{p}$.
9. Let $R_1$ and $R_2$ be commutative rings with identities and let $R = R_1 \times R_2$. Show that every ideal $I$ of $R$ is of the form $I = I_1 \times I_2$ with $I_i$ an ideal of $R_i$ for $i = 1, 2$.

10. Show that a non-zero ring $R$ in which $x^2 = x$ for all $x \in R$ is of characteristic 2 and is commutative.

11. Let $R$ be a finite commutative ring with more than one element and no zero-divisors. Show that $R$ is a field.

12. Determine for which integers $n$ the ring $\mathbb{Z}/n\mathbb{Z}$ is a direct sum of fields. Prove your answer.

13. Let $R$ be a subring of a field $F$ such that for each $x$ in $F$ either $x \in R$ or $x^{-1} \in R$. Prove that if $I$ and $J$ are two ideals of $R$, then either $I \subseteq J$ or $J \subseteq I$.

14. The Jacobson Radical $J(R)$ of a ring $R$ is defined to be the intersection of all maximal ideals of $R$.

Let $R$ be a commutative ring with 1 and let $x \in R$. Show that $x \in J(R)$ if and only if $1 - xy$ is a unit for all $y$ in $R$.

15. Let $R$ be any ring with identity, and $n$ any positive integer. If $M_n(R)$ denotes the ring of $n \times n$ matrices with entries in $R$, prove that $M_n(I)$ is an ideal of $M_n(R)$ whenever $I$ is an ideal of $R$, and that every ideal of $M_n(R)$ has this form.

16. Let $m, n$ be positive integers such that $m$ divides $n$. Then the natural map $\varphi : \mathbb{Z}_m \to \mathbb{Z}_m$ given by $a + (n) \mapsto a + (m)$ is a surjective ring homomorphism. If $U_n, U_m$ are the units of $\mathbb{Z}_n$ and $\mathbb{Z}_m$, respectively, show that $\varphi : U_n \to U_m$ is a surjective group homomorphism.

17. Let $R$ be a ring with ideals $A$ and $B$. Let $R/A \times R/B$ be the ring with coordinate-wise addition and multiplication. Show the following.
   (a) The map $R \to R/A \times R/B$ given by $r \mapsto (r + A, r + B)$ is a ring homomorphism.
   (b) The homomorphism in part (a) is surjective if and only if $A + B = R$.

18. Let $m$ and $n$ be relatively prime integers.
   (a) Show that if $c$ and $d$ are any integers, then there is an integer $x$ such that $x \equiv c \pmod{m}$ and $x \equiv d \pmod{n}$.
   (b) Show that $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m \times \mathbb{Z}_n$ are isomorphic as rings.

19. Let $R$ be a commutative ring with 1 and let $I$ and $J$ be ideals of $R$ such that $I + J = R$. Show that $I \cdot J = I \cap J$.

20. [NEW]
    Give an example of a commutative ring $R$ and ideals $I$ and $J$ in which $I \cdot J \neq I \cap J$.
    Also, prove that if $I + J = R$ then necessarily $I \cdot J = I \cap J$.

21. Let $R$ be a commutative ring with 1 and let $I$ and $J$ be ideals of $R$ such that $I + J = R$. Show that $R/(I \cap J) \cong R/I \oplus R/J$.

22. Let $R$ be a commutative ring with identity and let $I_1, I_2, \ldots, I_n$ be pairwise co-maximal ideals of $R$ (i.e., $I_i + I_j = R$ if $i \neq j$). Show that $I_i + \bigcap_{j \neq i} I_j = R$ for all $i$. 

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23. Let $R$ be a commutative ring, not necessarily with identity, and assume there is some fixed positive integer $n$ such that $nr = 0$ for all $r \in R$. Prove that $R$ embeds in a ring $S$ with identity so that $R$ is an ideal of $S$ and $S/R \cong \mathbb{Z}/n\mathbb{Z}$.

24. Let $R$ be a ring with identity 1 and $a, b \in R$ such that $ab = 1$. Denote $X = \{x \in R \mid ax = 1\}$.

(a) If $x \in X$, then $b + (1 - xa) \in X$.

(b) If $\varphi : X \to X$ is the mapping given by $\varphi(x) = b + (1 - xa)$, then $\varphi$ is one-to-one.

(c) If $X$ has more than one element, then $X$ is an infinite set.

25. Let $R$ be a commutative ring with identity and define $U_2(R) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in R \right\}$.

Prove that every $R$-automorphism of $U_2(R)$ is inner.

26. Let $R$ be the field of real numbers and let $F$ be the set of all $2 \times 2$ matrices of the form $\begin{bmatrix} a & b \\ -3b & a \end{bmatrix}$, where $a, b \in \mathbb{R}$. Show that $F$ is a field under the usual matrix operations.

27. Let $R$ be the ring of all $2 \times 2$ matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where $a$ and $b$ are real numbers.

Prove that $R$ is isomorphic to $\mathbb{C}$, the field of complex numbers.

28. Let $p$ be a prime and let $R$ be the ring of all $2 \times 2$ matrices of the form $\begin{bmatrix} a & b \\ pb & a \end{bmatrix}$, where $a, b \in \mathbb{Z}$. Prove that $R$ is isomorphic to $\mathbb{Z}[\sqrt{p}]$.

29. Let $p$ be a prime and $F_p$ the set of all $2 \times 2$ matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, where $a, b \in \mathbb{Z}_p$.

(a) Show that $F_p$ is a commutative ring with identity.

(b) Show that $F_7$ is a field.

(c) Show that $F_{13}$ is not a field.

30. Let $I \subseteq J$ be right ideals of a ring $R$ such that $J/I \cong R$ as right $R$-modules. Prove that there exists a right ideal $K$ such that $I \cap K = (0)$ and $I + K = J$.

31. A ring $R$ is called simple if $R^2 \neq 0$ and 0 and $R$ are its only ideals. Show that the center of a simple ring is 0 or a field.

32. Give an example of a field $F$ and a one-to-one ring homomorphism $\varphi : F \to F$ which is not onto. Verify your example.

33. Let $D$ be an integral domain and let $D[x_1, x_2, \ldots, x_n]$ be the polynomial ring over $D$ in the $n$ indeterminates $x_1, x_2, \ldots, x_n$. Let

$$V = \begin{bmatrix} x_1^{n-1} & \cdots & x_1^2 & x_1 & 1 \\
 x_2^{n-1} & \cdots & x_2^2 & x_2 & 1 \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 x_n^{n-1} & \cdots & x_n^2 & x_n & 1 \end{bmatrix}.$$  

Prove that the determinant of $V$ is $\prod_{1 \leq i < j \leq n} (x_i - x_j)$. 

34. Let $R = C[0,1]$ be the set of all continuous real-valued functions on $[0, 1]$. Define addition and multiplication on $R$ as follows. For $f, g \in R$ and $x \in [0, 1]$,

$$(f + g)(x) = f(x) + g(x) \text{ and } (fg)(x) = f(x)g(x).$$

(a) Show that $R$ with these operations is a commutative ring with identity.
(b) Find the units of $R$.
(c) If $f \in R$ and $f^2 = f$, then $f = 0_R$ or $f = 1_R$.
(d) If $n$ is a positive integer and $f \in R$ is such that $f^n = 0_R$, then $f = 0_R$.

35. Let $S$ be the ring of all bounded, continuous functions $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Let $I$ be the set of functions $f$ in $S$ such that $f(t) \to 0$ as $|t| \to \infty$.

(a) Show that $I$ is an ideal of $S$.
(b) Suppose $x \in S$ is such that there is an $i \in I$ with $ix = x$. Show that $x(t) = 0$ for all sufficiently large $|t|$.

36. Let $\mathbb{Q}$ be the field of rational numbers and $D = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$.

(a) Show that $D$ is a subring of the field of real numbers.
(b) Show that $D$ is a principal ideal domain.
(c) Show that $\sqrt{3}$ is not an element of $D$.

37. Show that if $p$ is a prime such that $p \equiv 1 \pmod{4}$, then $x^2 + 1$ is not irreducible in $\mathbb{Z}_p[x]$.

38. Show that if $p$ is a prime such that $p \equiv 3 \pmod{4}$, then $x^2 + 1$ is irreducible in $\mathbb{Z}_p[x]$.

39. Show that if $p$ is a prime such that $p \equiv 1 \pmod{6}$, then $x^3 + 1$ splits in $\mathbb{Z}_p[x]$.

**Prime, Maximal, and Primary Ideals**

40. Let $R$ be a non-zero commutative ring with 1. Show that an ideal $M$ of $R$ is maximal if and only if $R/M$ is a field.

41. Let $R$ be a commutative ring with 1. Show that an ideal $P$ of $R$ is prime if and only if $R/P$ is an integral domain.

42. (a) Let $R$ be a commutative ring with 1. Show that if $M$ is a maximal ideal of $R$ then $M$ is a prime ideal of $R$.
   
   (b) Give an example of a non-zero prime ideal in a ring $R$ that is not a maximal ideal.

43. Let $R$ be a non-zero ring with identity. Show that every proper ideal of $R$ is contained in a maximal ideal.

44. Let $R$ be a commutative ring with 1 and $P$ a prime ideal of $R$. Show that if $I$ and $J$ are ideals of $R$ such that $I \cap J \subseteq P$ and $J \not\subseteq P$, then $I \subseteq P$.

45. Let $M_1 \neq M_2$ be two maximal ideals in the commutative ring $R$ and let $I = M_1 \cap M_2$. Prove that $R/I$ is isomorphic to the direct sum of two fields.
46. Let $R$ be a non-zero commutative ring with $1$. Show that if $I$ is an ideal of $R$ such that $1 + a$ is a unit in $R$ for all $a \in I$, then $I$ is contained in every maximal ideal of $R$.

47. Let $R$ be a commutative ring with identity. Suppose $R$ contains an idempotent element $a$ other than $0$ or $1$. Show that every prime ideal in $R$ contains an idempotent element other than $0$ or $1$. (An element $a \in R$ is idempotent if $a^2 = a$.)

48. Let $R$ be a commutative ring with $1$.
   (a) Prove that $(x)$ is a prime ideal in $R[x]$ if and only if $R$ is an integral domain.
   (b) Prove that $(x)$ is a maximal ideal in $R[x]$ if and only if $R$ is a field.

49. Find all values of $a$ in $\mathbb{Z}_3$ such that the quotient ring

\[ \mathbb{Z}_3[x]/(x^3 + x^2 + ax + 1) \]

is a field. Justify your answer.

50. Find all values of $a$ in $\mathbb{Z}_5$ such that the quotient ring

\[ \mathbb{Z}_5[x]/(x^3 + 2x^2 + ax + 3) \]

is a field. Justify your answer.

51. Let $R$ be a commutative ring with identity and let $U$ be maximal among non-finitely generated ideals of $R$. Prove $U$ is a prime ideal.

52. Let $R$ be a commutative ring with identity and let $U$ be maximal among non-principal ideals of $R$. Prove $U$ is a prime ideal.

53. Let $R$ be a non-zero commutative ring with $1$ and $S$ a multiplicative subset of $R$ not containing $0$. Show that if $P$ is maximal in the set of ideals of $R$ not intersecting $S$, then $P$ is a prime ideal.

54. Prove that the set of nilpotent elements of a commutative ring $R$ is contained in the intersection of all prime ideals of $R$.

55. [NEW]
   Let $R$ be a ring in which there are no non-zero nilpotent elements. Prove that every idempotent is central.

56. Let $R$ be a non-zero commutative ring with $1$.
   (a) Let $S$ be a multiplicative subset of $R$ not containing $0$ and let $P$ be maximal in the set of ideals of $R$ not intersecting $S$. Show that $P$ is a prime ideal.
   (b) Show that the set of nilpotent elements of $R$ is the intersection of all prime ideals.

57. Let $R$ be a commutative ring with identity and let $x \in R$ be a non-nilpotent element. Prove that there exists a prime ideal $P$ of $R$ such that $x \not\in P$.

58. Let $R$ be a commutative ring with identity and let $S$ be the set of all elements of $R$ that are not zero-divisors. Show that there is a prime ideal $P$ such that $P \cap S$ is empty. (Hint: Use Zorn’s Lemma.)
59. Let $R$ be a commutative ring with identity and let $C$ be a chain of prime ideals of $R$. Show that $\bigcup_{P \in C} P$ and $\bigcap_{P \in C} P$ are prime ideals of $R$.

60. Let $R$ be a commutative ring and $P$ a prime ideal of $R$. Show that there is a prime ideal $P_0 \subseteq P$ that does not properly contain any prime ideal.

61. Let $R$ be a commutative ring with 1 such that for every $x$ in $R$ there is an integer $n > 1$ (depending on $x$) such that $x^n = x$. Show that every prime ideal of $R$ is maximal.

62. Let $R$ be a commutative ring with 1 in which every ideal is a prime ideal. Prove that $R$ is a field. (Hint: For $a \neq 0$ consider the ideals $(a)$ and $(a^2)$.)

63. Let $D$ be a principal ideal domain. Prove that every nonzero prime ideal of $D$ is a maximal ideal.

64. Show that if $R$ is a finite commutative ring with identity, then every prime ideal of $R$ is a maximal ideal.

65. Let $R = C[0, 1]$ be the ring of all continuous real-valued functions on $[0, 1]$, with addition and multiplication defined as follows. For $f, g \in R$ and $x \in [0, 1]$,

\[
(f + g)(x) = f(x) + g(x) \\
(fg)(x) = f(x)g(x).
\]

Prove that if $M$ is a maximal ideal of $R$, then there is a real number $x_0 \in [0, 1]$ such that $M = \{f \in R \mid f(x_0) = 0\}$.

66. Let $R$ be a commutative ring with identity, and let $P \subset Q$ be prime ideals of $R$. Prove that there exist prime ideals $P^*, Q^*$ satisfying $P \subseteq P^* \subset Q^* \subseteq Q$, such that there are no prime ideals strictly between $P^*$ and $Q^*$. HINT: Fix $x \in Q - P$ and show that there exists a prime ideal $P^*$ containing $P$, contained in $Q$ and maximal with respect to not containing $x$.

67. Let $R$ be a commutative ring with 1. An ideal $I$ of $R$ is called a primary ideal if $I \neq R$ and for all $x, y \in R$ with $xy \in I$, either $x \in I$ or $y^n \in I$ for some integer $n \geq 1$.

(a) Show that an ideal $I$ of $R$ is primary if and only if $R/I \neq 0$ and every zero-divisor in $R/I$ is nilpotent.

(b) Show that if $I$ is a primary ideal of $R$ then the radical $\text{Rad}(I)$ of $I$ is a prime ideal. (Recall that $\text{Rad}(I) = \{x \in R \mid x^n \in I$ for some $n\}$.)

**Commutative Rings**

68. Let $R$ be a commutative ring with identity. Show that $R$ is an integral domain if and only if $R$ is a subring of a field.

69. Let $R$ be a commutative ring with identity. Show that if $x$ and $y$ are nilpotent elements of $R$ then $x + y$ is nilpotent and the set of all nilpotent elements is an ideal in $R$.

70. Let $R$ be a commutative ring with identity. An ideal $I$ of $R$ is irreducible if it cannot be expressed as the intersection of two ideals of $R$ neither of which is contained in the other. Show the following.

(a) If $P$ is a prime ideal then $P$ is irreducible.

(b) If $x$ is a non-zero element of $R$, then there is an ideal $I_x$, maximal with respect to the property that $x \notin I_x$, and $I_x$ is irreducible.

(c) If every irreducible ideal of $R$ is a prime ideal, then 0 is the only nilpotent element of $R$. 
71. Let \( R \) be a commutative ring with 1 and let \( I \) be an ideal of \( R \) satisfying \( I^2 = \{0\} \). Show that if \( a + I \in R/I \) is an idempotent element of \( R/I \), then the coset \( a + I \) contains an idempotent element of \( R \).

72. Let \( R \) be a commutative ring with identity that has exactly one prime ideal \( P \). Prove the following.
   (a) \( R/P \) is a field.
   (b) \( R \) is isomorphic to \( R_P \), the ring of quotients of \( R \) with respect to the multiplicative set \( R - P = \{s \in R \mid s \notin P\} \).

73. Let \( R \) be a commutative ring with identity and \( \sigma : R \to R \) a ring automorphism.
   (a) Show that \( F = \{r \in R \mid \sigma(r) = r\} \) is a subring of \( R \) and the identity of \( R \) is in \( F \).
   (b) Show that if \( \sigma^2 \) is the identity map on \( R \), then each element of \( R \) is the root of a monic polynomial of degree two in \( F[x] \).

74. Let \( R \) be a commutative ring with identity that has exactly three ideals, \( \{0\} \), \( I \), and \( R \).
   (a) Show that if \( a \notin I \), then \( a \) is a unit of \( R \).
   (b) Show that if \( a, b \in I \) then \( ab = 0 \).

75. Let \( R \) be a commutative ring with 1. Show that if \( u \) is a unit in \( R \) and \( n \) is nilpotent, then \( u + n \) is a unit.

76. Let \( R \) be a commutative ring with identity. Suppose that for every \( a \in R \), either \( a \) or \( 1 - a \) is invertible. Prove that \( N = \{a \in R \mid a \text{ is not invertible}\} \) is an ideal of \( R \).

77. Let \( R \) be a commutative ring in which any two ideals are comparable (that is, either \( I \subseteq J \) or \( J \subseteq I \)). Prove that every finitely generated ideal of \( R \) is principal.

78. Let \( R \) be a commutative ring with 1. Show that the sum of any two principal ideals of \( R \) is principal if and only if every finitely generated ideal of \( R \) is principal.

79. Let \( R \) be an integral domain. Show that if all prime ideals of \( R \) are principal, then \( R \) is a Principal Ideal Domain.

80. Let \( R \) be a commutative ring with identity such that not every ideal is a principal ideal.
   (a) Show that there is an ideal \( I \) maximal with respect to the property that \( I \) is not a principal ideal.
   (b) If \( I \) is the ideal of part (a), show that \( R/I \) is a principal ideal ring.

81. Recall that if \( R \subseteq S \) is an inclusion of commutative rings (with the same identity) then an element \( s \in S \) is \textit{integral over} \( R \) if \( s \) satisfies some monic polynomial with coefficients in \( R \). Prove the equivalence of the following statements.
   (i) \( s \) is integral over \( R \).
   (ii) \( R[s] \) is finitely generated as an \( R \)-module.
   (iii) There exists a faithful \( R[s] \) module which is finitely generated as an \( R \)-module.

82. Recall that if \( R \subseteq S \) is an inclusion of commutative rings (with the same identity) then \( S \) is an \textit{integral} extension of \( R \) if every element of \( S \) satisfies some monic polynomial with coefficients in \( R \). Prove that if \( R \subseteq S \subseteq T \) are commutative rings with the same identity, then \( S \) is integral over \( R \) and \( T \) is integral over \( S \) if and only if \( T \) is integral over \( R \).
83. Let $R \subseteq S$ be commutative domains with the same identity, and assume that $S$ is an integral extension of $R$. Let $I$ be a nonzero ideal of $S$. Prove that $I \cap R$ is a nonzero ideal of $R$.

**Domains**

84. Suppose $R$ is a domain and $I$ and $J$ are ideals of $R$ such that $IJ$ is principal. Show that $I$ (and by symmetry $J$) is finitely generated.

[Hint: If $IJ = (a)$, then $a = \sum_{i=1}^{n} x_i y_i$ for some $x_i \in I$ and $y_i \in J$. Show the $x_i$ generate $I$.]

85. Prove that if $D$ is a Euclidean Domain, then $D$ is a Principal Ideal Domain.

86. Show that if $p$ is a prime such that there is an integer $b$ with $p = b^2 + 4$, then $\mathbb{Z}[\sqrt{p}]$ is not a unique factorization domain.

87. Show that if $p$ is a prime such that $p \equiv 1 \pmod{4}$, then $\mathbb{Z}[\sqrt{p}]$ is not a unique factorization domain.

88. Let $D = \mathbb{Z}(\sqrt{5}) = \{m + n\sqrt{5} \mid m, n \in \mathbb{Z}\}$ — a subring of the field of real numbers and necessarily an integral domain (you need not show this) — and $F = \mathbb{Q}(\sqrt{5})$ its field of fractions. Show the following:

(a) $x^2 + x - 1$ is irreducible in $D[x]$ but not in $F[x]$.

(b) $D$ is not a unique factorization domain.

89. Let $D = \mathbb{Z}(\sqrt{21}) = \{m + n\sqrt{21} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{21})$, the field of fractions of $D$. Show the following:

(a) $x^2 - x - 5$ is irreducible in $D[x]$ but not in $F[x]$.

(b) $D$ is not a unique factorization domain.

90. Let $D = \mathbb{Z}(\sqrt{-11}) = \{m + n\sqrt{-11} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{-11})$ its field of fractions. Show the following:

(a) $x^2 - x + 3$ is irreducible in $D[x]$ but not in $F[x]$.

(b) $D$ is not a unique factorization domain.

91. Let $D = \mathbb{Z}(\sqrt{13}) = \{m + n\sqrt{13} \mid m, n \in \mathbb{Z}\}$ and $F = \mathbb{Q}(\sqrt{13})$ its field of fractions. Show the following:

(a) $x^2 + 3x - 1$ is irreducible in $D[x]$ but not in $F[x]$.

(b) $D$ is not a unique factorization domain.

92. Let $D$ be an integral domain and $F$ a subring of $D$ that is a field. Show that if each element of $D$ is algebraic over $F$, then $D$ is a field.

93. Let $R$ be an integral domain containing the subfield $F$ and assume that $R$ is finite dimensional over $F$ when viewed as a vector space over $F$. Prove that $R$ is a field.

94. Let $D$ be an integral domain.

(a) For $a, b \in D$ define a **greatest common divisor** of $a$ and $b$.

(b) For $x \in D$ denote $(x) = \{dx \mid d \in D\}$. Prove that if $(a) + (b) = (d)$, then $d$ is a greatest common divisor of $a$ and $b$. 

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95. Let $D$ be a principal ideal domain.
   (a) For $a, b \in D$, define a least common multiple of $a$ and $b$.
   (b) Show that $d \in D$ is a least common multiple of $a$ and $b$ if and only if $(a) \cap (b) = (d)$.

96. Let $D$ be a principal ideal domain and let $a, b \in D$.
   (a) Show that there is an element $d \in D$ that satisfies the properties
      i. $d | a$ and $d | b$ and
      ii. if $e | a$ and $e | b$ then $e | d$.
   (b) Show that there is an element $m \in D$ that satisfies the properties
      i. $a | m$ and $b | m$ and
      ii. if $a | e$ and $b | e$ then $m | e$.

97. Let $R$ be a principal ideal domain. Show that if $(a)$ is a nonzero ideal in $R$, then there are only finitely many ideals in $R$ containing $(a)$.

98. Let $D$ be a unique factorization domain and $F$ its field of fractions. Prove that if $d$ is an irreducible element in $D$, then there is no $x \in F$ such that $x^2 = d$.

99. Let $D$ be a Euclidean domain. Prove that every non-zero prime ideal is a maximal ideal.

100. Let $\pi$ be an irreducible element of a principal ideal domain $R$. Prove that $\pi$ is a prime element (that is, $\pi | ab$ implies $\pi | a$ or $\pi | b$).

101. Let $D$ with $\varphi : D - \{0\} \to \mathbb{N}$ be a Euclidean domain. Suppose $\varphi(a + b) \leq \max\{\varphi(a), \varphi(b)\}$ for all $a, b \in D$. Prove that $D$ is either a field or isomorphic to a polynomial ring over a field.

102. Let $D$ be an integral domain and $F$ its field of fractions. Show that if $g$ is an isomorphism of $D$ onto itself, then there is a unique isomorphism $h$ of $F$ onto $F$ such that $h(d) = g(d)$ for all $d \in D$.

103. Let $D$ be a unique factorization domain such that if $p$ and $q$ are irreducible elements of $D$, then $p$ and $q$ are associates. Show that if $A$ and $B$ are ideals of $D$, then either $A \subseteq B$ or $B \subseteq A$.

104. Let $D$ be a unique factorization domain and $p$ a fixed irreducible element of $D$ such that $q$ is any irreducible element of $D$, then $q$ is an associate of $p$. Show the following.
   (a) If $d$ is a nonzero element of $D$, then $d$ is uniquely expressible in the form $up^n$, where $u$ is a unit of $D$ and $n$ is a non-negative integer.
   (b) $D$ is a Euclidean domain.

105. Prove that $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$ is a Euclidean domain.

106. Show that the ring $\mathbb{Z}[i]$ of Gaussian integers is a Euclidean ring and compute the greatest common divisor of $5 + i$ and $13$ using the Euclidean algorithm.

**Polynomial Rings**

107. Show that the polynomial $f(x) = x^4 + 5x^2 + 3x + 2$ is irreducible over the field of rational numbers.
108. Let \( D \) be an integral domain and \( D[x] \) the polynomial ring over \( D \). Suppose \( \varphi : D[x] \rightarrow D[x] \) is an isomorphism such that \( \varphi(d) = d \) for all \( d \in D \). Show that \( \varphi(x) = ax + b \) for some \( a, b \in D \) and that \( a \) is a unit of \( D \).

109. Let \( f(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots + a_n x^n \in \mathbb{Z}[x] \) and \( p \) a prime such that \( p \mid a_i \) for \( i = 1, \ldots, k - 1 \), \( p \nmid a_k \), \( p \nmid a_n \), and \( p^2 \nmid a_0 \). Show that \( f(x) \) has an irreducible factor in \( \mathbb{Z}[x] \) of degree at least \( k \).

110. Let \( D \) be an integral domain and \( D[x] \) the polynomial ring over \( D \) in the indeterminate \( x \). Show that if every nonzero prime ideal of \( D[x] \) is a maximal ideal, then \( D \) is a field.

111. Let \( R \) be a commutative ring with \( 1 \) and let \( f(x) \in R[x] \) be nilpotent. Show that the coefficients of \( f \) are nilpotent.

112. Show that if \( R \) is an integral domain and \( f(x) \) is a unit in the polynomial ring \( R[x] \), then \( f(x) \) is in \( R \).

113. Let \( D \) be a unique factorization domain and \( F \) its field of fractions. Prove that if \( f(x) \) is a monic polynomial in \( D[x] \) and \( \alpha \in F \) is a root of \( f \), then \( \alpha \in D \).

114. Explain why \( F = \mathbb{Z}_3[x]/(x^3 + x^2 + 2) \) is a field and find the multiplicative inverse of \( x^2 + 1 \) in \( F \).

115. (a) Show that \( x^4 + x^3 + x^2 + x + 1 \) is irreducible in \( \mathbb{Z}_3[x] \).
   (b) Show that \( x^4 + 1 \) is not irreducible in \( \mathbb{Z}_3[x] \).

116. Let \( F[x, y] \) be the polynomial ring over a field \( F \) in two indeterminates \( x, y \). Show that the ideal generated by \( \{x, y\} \) is not a principal ideal.

117. Let \( F \) be a field. Prove that the polynomial ring \( F[x] \) is a PID and that \( F[x, y] \) is not a PID.

118. Let \( D \) be an integral domain and let \( c \) be an irreducible element in \( D \). Show that the ideal \( (x, c) \) generated by \( x \) and \( c \) in the polynomial ring \( D[x] \) is not a principal ideal.

119. Show that if \( R \) is a commutative ring with \( 1 \) that is not a field, then \( R[x] \) is not a principal ideal domain.

120. (a) Let \( \mathbb{Z} \left[ \frac{1}{2} \right] = \left\{ \frac{a}{2^n} \mid a, n \in \mathbb{Z}, n \geq 0 \right\} \), the smallest subring of \( \mathbb{Q} \) containing \( \mathbb{Z} \) and \( \frac{1}{2} \).
   Let \( (2x - 1) \) be the ideal of \( \mathbb{Z}[x] \) generated by the polynomial \( 2x - 1 \).
   Show that \( \mathbb{Z}[x]/(2x - 1) \cong \mathbb{Z} \left[ \frac{1}{2} \right] \).
   (b) Find an ideal \( I \) of \( \mathbb{Z}[x] \) such that \( (2x - 1) \not\subseteq I \not\subseteq \mathbb{Z}[x] \).

Non-commutative Rings

121. Let \( R \) be a ring with identity such that the identity map is the only ring automorphism of \( R \). Prove that the set \( N \) of all nilpotent elements of \( R \) is an ideal of \( R \).

122. Let \( p \) be a prime. A ring \( S \) is called a \( p \)-ring if the characteristic of \( S \) is a power of \( p \). Show that if \( R \) is a ring with identity of finite characteristic, then \( R \) is isomorphic to a finite direct product of \( p \)-rings for distinct primes.

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123. If \( R \) is any ring with identity, let \( J(R) \) denote the Jacobson radical of \( R \). Show that if \( e \) is any idempotent of \( R \), then \( J(eRe) = eJ(R)e \).

124. If \( n \) is a positive integer and \( F \) is any field, let \( M_n(F) \) denote the ring of \( n \times n \) matrices with entries in \( F \). Prove that \( M_n(F) \) is a simple ring. Equivalently, \( \text{End}_F(V) \) is a simple ring if \( V \) is a finite dimensional vector space over \( F \).

125. Let \( R \) be a ring.
   (a) Show that there is a unique smallest (with respect to inclusion) ideal \( A \) such that \( R/A \) is a commutative ring.
   (b) Give an example of a ring \( R \) such that for every proper ideal \( I \), \( R/I \) is not commutative. Verify your example.
   (c) For the ring \( R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\} \) with the usual matrix operations, find the ideal \( A \) of part (a).

126. A ring \( R \) is \textit{nilpotent-free} if \( a^n = 0 \) for \( a \in R \) and some positive integer \( n \) implies \( a = 0 \).
   (a) Suppose there is an ideal \( I \) such that \( R/I \) is nilpotent-free. Show there is a unique smallest (with respect to inclusion) ideal \( A \) such that \( R/A \) is nilpotent-free.
   (b) Give an example of a ring \( R \) such that for every proper ideal \( I \), \( R/I \) is not nilpotent-free. Verify your example.
   (c) Show that if \( R \) is a commutative ring with identity, then there is a proper ideal \( I \) of \( R \) such that \( R/I \) is nilpotent-free, and find the ideal \( A \) of part (a).

\textbf{Local Rings, Localization, Rings of Fractions}

127. Let \( R \) be an integral domain. Construct the field of fractions \( F \) of \( R \) by defining the set \( F \) and the two binary operations, and show that the two operations are well-defined. Show that \( F \) has a multiplicative identity element and that every nonzero element of \( F \) has a multiplicative inverse.

128. A \textit{local} ring is a commutative ring with 1 that has a unique maximal ideal. Show that a ring \( R \) is local if and only if the set of non-units in \( R \) is an ideal.

129. Let \( R \) be a commutative ring with 1 \( \neq 0 \) in which the set of nonunits is closed under addition. Prove that \( R \) is local, i.e., has a unique maximal ideal.

130. Let \( D \) be an integral domain and \( F \) its field of fractions. Let \( P \) be a prime ideal in \( D \) and \( D_P = \{ab^{-1} \mid a, b \in D, b \not\in P\} \subseteq F \). Show that \( D_P \) has a unique maximal ideal.

131. Let \( R \) be a commutative ring with identity and \( M \) a maximal ideal of \( R \). Let \( R_M \) be the ring of quotients of \( R \) with respect to the multiplicative set \( R - M = \{s \in R \mid s \not\in M\} \). Show the following.
   (a) \( M_M = \{s \mid a \in M, s \not\in M\} \) is the unique maximal ideal of \( R_M \).
   (b) The fields \( R/M \) and \( R_M/M_M \) are isomorphic.

132. Let \( R \) be an integral domain, \( S \) a multiplicative set, and let \( S^{-1}R = \{s^{-1}r \mid r \in R, s \in S\} \) (contained in the field of fractions of \( R \)). Show that if \( P \) is a prime ideal of \( R \), then \( S^{-1}P \) is either a prime ideal of \( S^{-1}R \) or else equals \( S^{-1}R \).
133. Let \( R \) be a commutative ring with identity and \( P \) a prime ideal of \( R \). Let \( R_P \) be the ring of quotients of \( R \) with respect to the set \( R - P = \{ s \in R \mid s \notin P \} \). Show that \( R_P/P_P \) is the field of fractions of the integral domain \( R/P \).

134. Let \( D \) be an integral domain and \( F \) its field of fractions. Denote by \( M \) the set of all maximal ideals of \( D \). For \( M \in M \), let \( D_M = \{ \frac{a}{s} \mid a, s \in D, s \notin M \} \subset F \). Show that \( \bigcap_{M \in M} D_M = D \).

135. Let \( R \) be a commutative ring with 1 and \( D \) a multiplicative subset of \( R \) containing 1. Let \( J \) be an ideal in the ring of fractions \( D^{-1}R \) and let

\[
I = \left\{ a \in R \mid \frac{a}{d} \in J \text{ for some } d \in D \right\}.
\]

Show that \( I \) is an ideal of \( R \).

136. Let \( D \) be a principal ideal domain and let \( P \) be a non-zero prime ideal. Show that \( D_P \), the localization of \( D \) at \( P \), is a principal ideal domain and has a unique irreducible element, up to associates.

Chains and Chain Conditions

137. Let \( R \) be a commutative ring with identity. Prove that any non-empty set of prime ideals of \( R \) contains maximal and minimal elements.

138. Let \( R \) be an integral domain that satisfies the descending chain condition; i.e., whenever \( I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \) is a descending chain of ideals of \( R \), there exists \( m \in \mathbb{N} \) such that \( I_k = I_m \) for all \( k \geq m \). Prove that \( R \) is a field.

139. Let \( R \) be a ring satisfying the descending chain condition on right ideals. Prove that if \( R \) is not a division ring, then \( R \) contains non-trivial zero divisors.

140. Let \( R \) be a commutative ring with 1. We say \( R \) satisfies the ascending chain condition if whenever \( I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \) is an ascending chain of ideals of \( R \), there is an integer \( N \) such that \( I_k = I_N \) for all \( k \geq N \). Show that \( R \) satisfies the ascending chain condition if and only if every ideal of \( R \) is finitely generated.

141. Define Noetherian ring and prove that if \( R \) is Noetherian, then \( R[x] \) is Noetherian.

142. Let \( R \) be a commutative Noetherian ring with identity. Prove that there are only finitely many minimal prime ideals of \( R \).

143. Let \( R \) be a commutative Noetherian ring in which every 2-generated ideal is principal. Prove that \( R \) is a Principal Ideal Domain.

144. Let \( R \) be a commutative Noetherian ring with identity and let \( I \) be an ideal in \( R \). Let \( J = \text{Rad}(I) \). Prove that there exists a positive integer \( n \) such that \( j^n \in I \) for all \( j \in J \).

145. Let \( R \) be a commutative Noetherian domain with identity. Prove that every nonzero ideal of \( R \) contains a product of nonzero prime ideals of \( R \).

146. Let \( R \) be a ring satisfying the descending chain condition on right ideals. If \( J(R) \) denotes the Jacobson radical of \( R \), prove that \( J(R) \) is nilpotent.
147. Show that if $R$ is a commutative Noetherian ring with identity, then the polynomial ring $R[x]$ is also Noetherian.

148. Let $P$ be a nonzero prime ideal of the commutative Noetherian domain $R$. Assume $P$ is principal. Prove that there does not exist a prime ideal $Q$ satisfying $(0) < Q < P$.

149. Let $R$ be a commutative Noetherian ring. Prove that every nonzero ideal $A$ of $R$ contains a product of prime ideals (not necessarily distinct) each of which contains $A$.

150. Let $R$ be a commutative ring with 1 and let $M$ be an $R$-module that is not Artinian (Noetherian, of finite composition length). Let $\mathcal{I}$ be the set of ideals $I$ of $R$ such that there exists an $R$-submodule $N$ of $M$ with the property that $N/NI$ is not Artinian (Noetherian, of finite composition length, respectively). Show that if $A \in \mathcal{I}$ is a maximal element of $\mathcal{I}$, then $A$ is a prime ideal of $R$. 