Lecture 1.2,
MATH-57091 Probability and Statistics for High-School Teachers.

Artem Zvavitch

Department of Mathematical Sciences, Kent State University

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Note that for Game 1 it is enough to consider just a total number of heads. For Game 2 we must consider much more complicated sample space.

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from where we get that $P(\emptyset) = 0$.

In this model the Sample Space consists of a finite number of possible outcomes (as for example in the previous example) and the probability law is specified by assigning probability of each element, i.e. $\Omega = \{s_1, \dots, s_n\}$ and $P(s_i) = p_i$ where each $p_i \in [0,1]$ and $\sum_{i=1}^{n} p_i = 1$.

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$$P(A) = \frac{\text{Number of elements in } A}{n},$$

for each event $A \subset \Omega$.

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Be careful with "each outcome above have the same probability" note

$$P(\text{one of dice is 3 and another is 4}) = \frac{2}{36}$$

but

$$P(\text{one of dice is 3 and another is also 3}) = \frac{1}{36}$$

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$$S \cup T = \{\text{sum of dices is an even number, or a number which is greater then 7 or both}\}\\ = \{(1,1),(1,3),(1,5),(2,2),(2,4),(2,6),(3,1),(3,3),(3,5),(3,6),(4,2),\\ (4,4),(4,6),(5,1),(5,3),(5,4),(5,5),(5,6),(6,2),(6,3),(6,4),(6,5),(6,6)\}.$$

and $P(S \cup T) = \frac{23}{36} \neq P(S) + P(T)$ (but this is perfectly fine: $S \cap T \neq \emptyset$).

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- $P(A \cup B) \leq P(A) + P(B)$.
- $P(A^c) = 1 - P(A)$.

Continuous Model

Romeo and Juliet has a meeting at given time but each arrive a "bit" late. Say each of them may be up to one hour late. We assume that all pairs of delays are equally likely. The first who arrive will wait for 15 minutes and then leave, if the other one is not at the place. What is the probability that they will meet?

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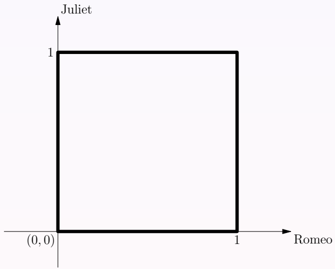
We may model this problem using geometry:



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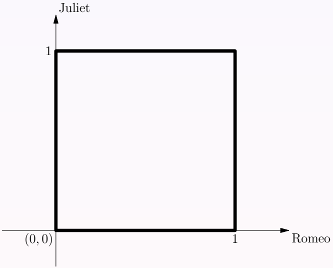

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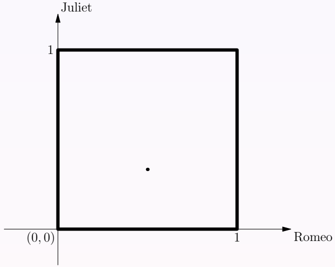

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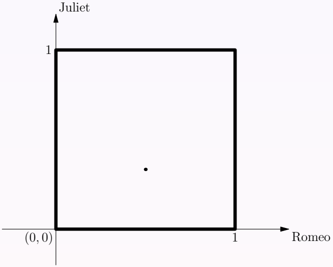

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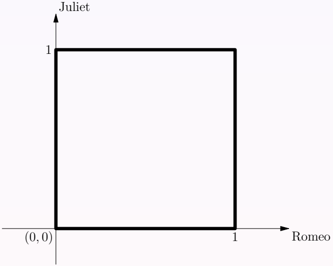

Where on x-axis we have Romeo's chances to be late and on y-axis Juliet's chances to be late. What would be the chances that Romeo is late by **at most** 30 min and Juliet is late by at most 20 min?

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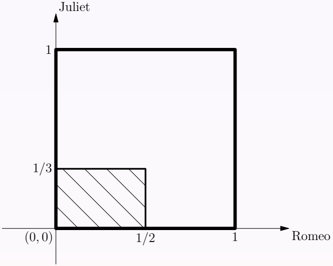

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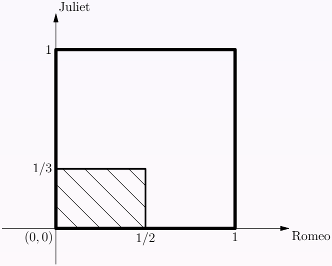

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$$P(\text{Romeo is late by } \textbf{at most } 30 \text{ min and Juliet is late by at most } 20 \text{ min}) = \frac{1}{6}.$$

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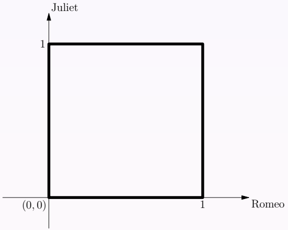

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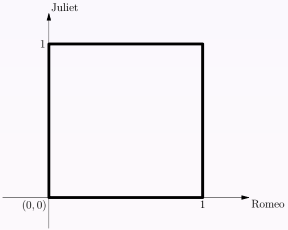

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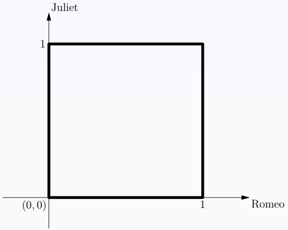

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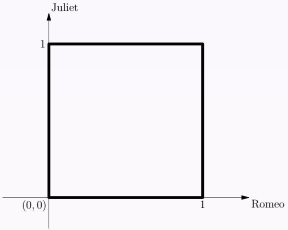

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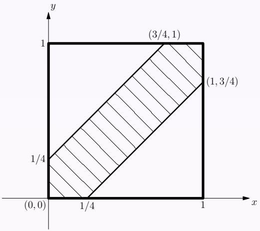

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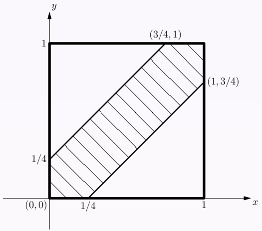

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Finally

$$P(\text{Romeo and Juliet arrive within 15 min of each other}) = 1 - 2\frac{1}{2}\left(\frac{3}{4}\right)^2 = \frac{7}{16}.$$