Lecture 2.1, MATH-57091 Probability and Statistics for High-School Teachers.

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But S is not exactly what we are looking for in our question! We need S if we know T happened! i.e. we are actually looking for elements in S which are elements of T, which is exactly $S \cap T$. So (see Lecture 1.2). Denote by $\sharp T$ the number of elements in T, then the probability of an event S, given that we only work "inside" T is

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$$\frac{\sharp(S \cap T)}{\sharp T} = \frac{\sharp(S \cap T)/\sharp\Omega}{\sharp T/\sharp\Omega} = \frac{P(S \cap T)}{P(T)} = \frac{2/36}{11/36} = \frac{2}{11}.$$

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Conditional Probability:

Assume we are looking for probability of S occur given that the event T occurred, such probability is denoted by $P(S|T)$ and is called conditional probability of S given T. The general formula is

$$P(S|T) = \frac{P(S \cap T)}{P(T)}.$$

We note that the above definition is only well defined when $P(T) > 0$ and hence $P(S|T)$ is only defined when $P(T) > 0$.

Examples:

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$$P(S|T) = \frac{P(S \cap T)}{P(T)} = \frac{1/20}{10/20} = \frac{1}{10},$$

note that in above calculation we used that $S \cap T = S$ (indeed, the intersection of events that the number is 6 and that the number is less then 10 is just that the number is 6).

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$$P(S|T) = \frac{P(S \cap T)}{P(T)} = \frac{P(\{(b,b)\})}{P(\{(b,b);(b,g);(g.b)\}} = \frac{1/4}{3/4} = \frac{1}{3}.$$

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$$P(S) = P(S \cap T) = P(T)P(S|T) = \frac{9}{16} \times \frac{8}{15} = \frac{3}{10}.$$

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Putting the all above formulas together we get:

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So our goal, for now, to find $P(H_1 \cup H_2 \cup H_3)$. We will use the formula from the previous slide. We first note that $P(H_i) = \frac{1}{3}$, for $i = 1, 2, 3$. Indeed there are 3 hats and the only one is correct for a give person. Next we would like to find $P(H_1 \cap H_2)$. We note that it is easy to see that

$$P(H_2 | H_1) = \frac{1}{2}.$$

Indeed, if we know that the first man selected a correct hat, then there is only two hats left and one is the hat of the second man.

Problem: Suppose that each of three men at a party throws his hat into the center of the room. The hats are mixed up and the each man randomly selects a hat. What is the probability that none of the man selects his own hat?

Solution: We will first calculate complementary probability, i.e. the probability of event H^c, where H is the event we are looking for. Note that then H^c is the event that at least one man selects his own hat. Let us denote by H_i, $i = 1, 2, 3$ the event that the ith man selects his own hat. Then

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$$P(H_2|H_1) = \frac{1}{2}.$$

Indeed, if we know that the first man selected a correct hat, then there is only two hats left and one is the hat of the second man. So

$$P(H_1 \cap H_2) = P(H_2)P(H_1|H_2) = \frac{1}{6}.$$

Next we use a very similar logic to claim that

$$P(H_1 \cap H_2 \cap H_3) = P(H_1 \cap H_2)P(H_3 | H_1 \cap H_2) = \frac{1}{6} \times 1.$$

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Now we use the formula for the union of three events:

$$\begin{aligned} P(H_1 \cup H_2 \cup H_3) =& P(H_1) + P(H_2) + P(H_3) \\ & - P(H_1 \cap H_2) - P(H_1 \cap H_3) - P(H_2 \cap H_3) \\ & + P(H_1 \cap H_2 \cap H_3) \\ =& 3\frac{1}{3} - 3\frac{1}{6} + \frac{1}{6} = \frac{2}{3}. \end{aligned}$$

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Finally the probability that none of the men elects his own hat is

$$1 - \frac{2}{3} = \frac{1}{3}.$$