Lecture 3.1, MATH-57091 Probability and Statistics for High-School Teachers.

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September, 8 - 12, 2014.

A bit different look at functions.

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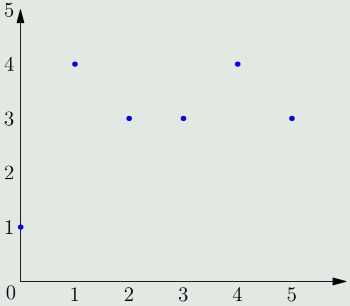

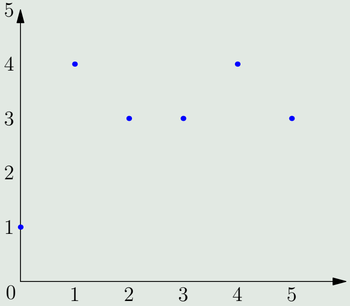

In many cases we do not need to know all information about function (or we simply have no way to know it). We may be interested only to know values of the function and how often those value occurs (and do not care at when/where the value occurs):

$f(x)$	How often
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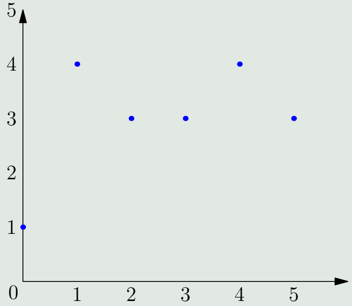
3	3
4	2



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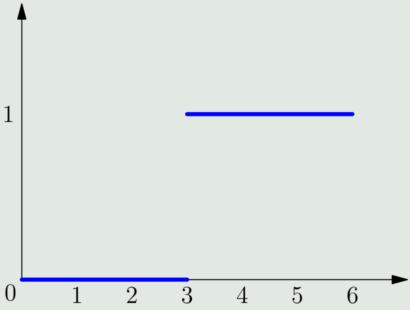

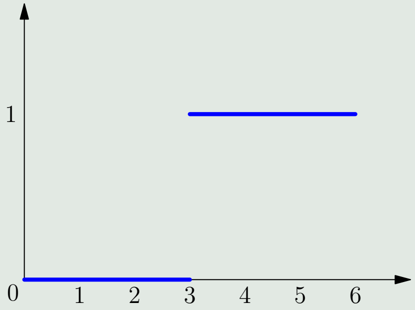

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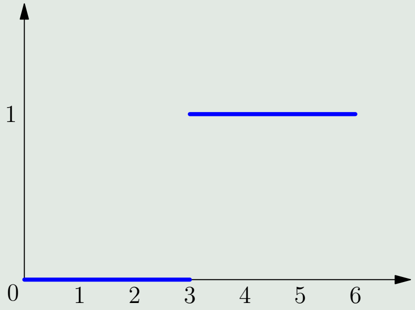

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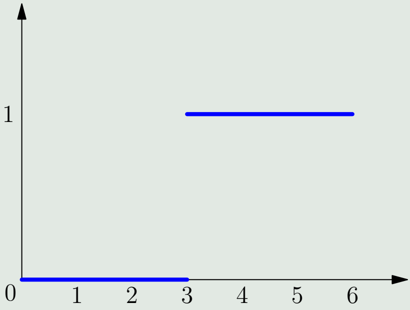

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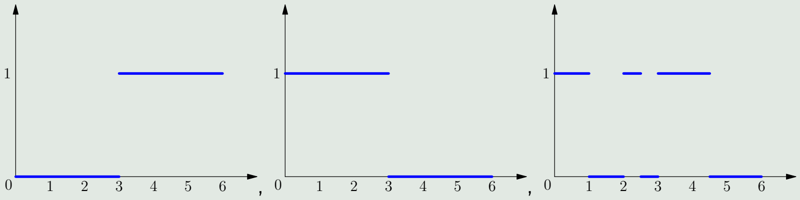

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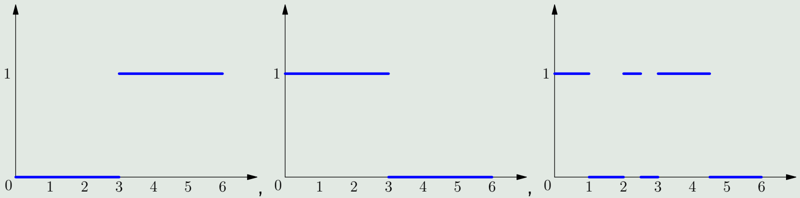

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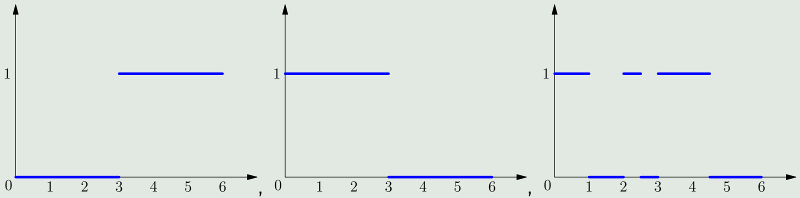

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Random variable

When a probability experiment is performed, often we are not interested in all the details of the experimental result, but rather are interested in the value of some numerical quantity. This gives the definition of **random variable**: a real valued function on the sample space. Since the value of random variable is determined by the outcome of the experiment, we may assign probabilities of its possible values.

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We also may check that

$$\sum_{i=1}^{\infty}(1-p)^{i-1}p=p\sum_{i=1}^{\infty}(1-p)^{i-1}=p\frac{1}{1-(1-p)}=1$$

Distribution function of random variable X: $F_X(t) = P(X \leq t)$

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It easy (simple logic) to check that $F_X(t)$ is not decreasing function (i.e. $F_X(t_1) \leq F_X(t_2)$ for $t_1 \leq t_2$). More over (check!):

$$P(a < X \leq b) = F_X(b) - F_X(a).$$

Discrete Random Variables

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or

$$.4+.1+P(X=3)=1$$

and $P(X=3)=.5$

If X is a discrete Random variable that take one of the possible values $x_1, x_2, \ldots, x_n$ then the **expected value** of X, denoted by $\mathbb{E}X$ is defined by

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Expected Value of Discrete Random Variable

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$$\mathbb{E}Y = \sum_{i=1}^{6} y_i P(Y = y_i) = 1 * \frac{1}{6} + 2 * \frac{1}{6} + 3 * \frac{1}{6} + 4 * \frac{1}{6} + 5 * \frac{1}{6} + 6 * \frac{1}{6} = 3.5$$

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It is interesting to note that $\mathbb{E}X$ is not necessary is one of the possible values of X.

Let X denote the random variable that is defined as the sum of two fair dice, find $\mathbb{E}X$.

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P(X=8) &= \frac{5}{36},\\
P(X=9) &= \frac{4}{36},\\
P(X=10) &= \frac{3}{36},\\
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\end{aligned}$$

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The above computations are direct a correct but there is a much simpler way to do it! Using the properties of expected value. We will do it during next lecture!