Lecture 4.1, MATH-57091 Probability and Statistics for High-School Teachers.

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It is usually a quite tricky to find $p_{X,Y}(x, y)$ even if we know p_X and p_Y. The point is that we must know how X and Y are "related" to each other and if events $X = x$ and $Y = y$ are not independent then

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Note that the above formula does not require independence assumption.

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NOTICE: WE DO NOT NEED X AND Y TO BE INDEPENDENT!

Problem: The table below represents the annual incomes (in $ 1000) of 8 men and 8 women who are residents of Kent

Women	Men
43.6	51.3
55.8	44.2
62.2	56.9
77.3	83.2
64.6	85.3
95.8	84.2
74.6	55.3
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Suppose that a woman and a man are randomly chosen. Find the expected value of the sum of their incomes.

Solution: Let X be the man's income and Y the woman's income. Since X is equally likely to be any of the 8 values in the women's column, we see that:

$$\mathbb{E}Y = \frac{1}{8}(43.6 + 55.8 + 62.2 + 77.3 + 64.6 + 95.8 + 74.6 + 85.8) = 69.9625,$$

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Finally

$$\mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y = 139.2875.$$

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The Binomial Random Variable

Suppose that n independent trial, each of which results in a "success" with probability p and "failure" with probability $1-p$ are to be performed. If X represents the number of successes that occur in the n trials, then X is said to be a **Binomial Random Variable** with parameters (n,p). We note that

$$p_X(i) = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}, \text{ for } i = 0, 1, \ldots, n,$$

where $\binom{n}{i} = \frac{n!}{(n-i)!i!}$.

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$$\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 + \cdots + \mathbb{E}X_N = N * \frac{1}{N} = 1.$$

We remark that it is clear that random variables X_i are **"very dependent!"** but we do not care about this when calculating the expected value of the sum!

Problem: At a party N men throw their hats into the center of a room. The hats are mixed up and each man randomly selects one. Find the expected number of men who selected their own hats.

Solution: Let X denote the number of men that selected their own hats. You may try to find Probability Mass Function of X, this will be quite non-trivial!!! The only reason to try it, is to really appreciate the solution we will present! The idea is to compute $\mathbb{E}X$ by representing X as

$$X = X_1 + X_2 + \cdots + X_N,$$

where

$$X_i = \begin{cases} 1, & \text{if the } i\text{th man selected his own hat} \\ 0, & \text{otherwise.} \end{cases}$$

Now, because the ith man is equally likely to select any of the N hats, it follows that

$$P(X_i = 1) = P(i\text{th man selected his own hat}) = \frac{1}{N},$$

moreover

$$\mathbb{E}X_i = 1P(X_i = 1) + 0P(X_i = 0) = \frac{1}{N},$$

finally

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We also remark that the answer to the problem is really cool: it does not meter how many people are at the party, on the average exactly one of the men will select his own hat!

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Thus

$$\mathbb{E}X = \mathbb{E}X_1 + \mathbb{E}X_2 + \cdots + \mathbb{E}X_{25} = 25 * \left[1 - \left(\frac{24}{25}\right)^{10}\right].$$

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and $\mathbb{E}X^2 = .5$ but $(\mathbb{E}X)^2 = .25$ so $\mathbb{E}X^2 \neq (\mathbb{E}X)^2$.