Lecture 4.2, MATH-57091 Probability and Statistics for High-School Teachers.

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A small Remark

Use axioms to prove that

- $P(A \cup B) \geq \max\{P(A), P(B)\}$,
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$$\mathbb{E}(X * X) \neq (\mathbb{E}X)(\mathbb{E}X).$$

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Examples of $Var(X)$

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$$Var(X+X) = Var(2X) = 4Var(X) \neq 2Var(X) = Var(X) + Var(X).$$

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When X and Y are discrete random variables the above condition of independence reduces to

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$$\begin{aligned} P(X \le a, Y \le b) &= \sum_{x \le a} \sum_{y \le b} p_{X,Y}(x,y) \\ &= \sum_{x \le a} \sum_{y \le b} p_X(x)p_Y(y) \\ &= \sum_{x \le a} p_X(x) \sum_{y \le b} p_Y(y) \\ &= P(X \le a)P(Y \le b). \end{aligned}$$

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Proof:

$$\begin{aligned}
\mathbb{E}[f(X)g(Y)] &= \sum_x \sum_y f(x)g(y)p_{X,Y}(x,y) \\
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Independent Random Variables: Sum & variance.

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$$= \mathbb{E}\left[X^2 + Y^2 + (\mathbb{E}X)^2 + (\mathbb{E}Y)^2 + 2XY - 2X\mathbb{E}X - 2X\mathbb{E}Y - 2Y\mathbb{E}X - 2Y\mathbb{E}Y + 2\mathbb{E}X\mathbb{E}Y\right]$$

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If X and Y are independent desecrate random variables, then

$$Var(X+Y) = Var(X) + Var(Y)$$

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Now we may use that X and Y are independent to get $\mathbb{E}(XY) = \mathbb{E}X\mathbb{E}Y$ and thus

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Variance of Binomial Random Variable

The Binomial Random Variable

Suppose that n independent trial, each of which results in a "success" with probability p and "failure" with probability $1-p$ are to be performed. If X represents the number of successes that occur in the n trials, then X is said to be a **Binomial Random Variable** with parameters (n,p). We note that

$$p_X(i) = P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, \text{ for } i = 0, 1, \ldots, n,$$

where $\binom{n}{i} = \frac{n!}{(n-i)!i!}$.

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$$Var(X) = Var(X_1) + Var(X_2) + \cdots + Var(X_n) = n(p - p^2).$$