Lecture 5.1, MATH-57091 Probability and Statistics for High-School Teachers.

Artem Zvavitch

Department of Mathematical Sciences, Kent State University

September, 22 - 26, 2014.

We say that X is continuous random variable if there exists a nonnegative function $f(x)$, defined for all real numbers x, having property that for any set of real numbers B:

$$P(X \in B) = \int_B f(x)dx.$$

The function $f(x)$ is called the **probability density function** of the random variable X.

We say that X is continuous random variable if there exists a nonnegative function $f(x)$, defined for all real numbers x, having property that for any set of real numbers B:

$$P(X \in B) = \int_B f(x)dx.$$

The function $f(x)$ is called the **probability density function** of the random variable X.

Yes! This may and should remind you some "physics": to find the mass of an object we need to integrate the density of the object. To find a probability of random variable X to be in set B we need to integrate the probability density function over the set B.

We say that X is continuous random variable if there exists a nonnegative function $f(x)$, defined for all real numbers x, having property that for any set of real numbers B:

$$P(X \in B) = \int_B f(x)dx.$$

The function $f(x)$ is called the **probability density function** of the random variable X.

Yes! This may and should remind you some "physics": to find the mass of an object we need to integrate the density of the object. To find a probability of random variable X to be in set B we need to integrate the probability density function over the set B. This definition immediately give an essential property of $f(x)$:

$$\int_{-\infty}^{\infty} f(x)dx = P(X \in (-\infty, \infty)) = 1.$$

We say that X is continuous random variable if there exists a nonnegative function $f(x)$, defined for all real numbers x, having property that for any set of real numbers B:

$$P(X \in B) = \int_B f(x)dx.$$

The function $f(x)$ is called the **probability density function** of the random variable X.

Yes! This may and should remind you some "physics": to find the mass of an object we need to integrate the density of the object. To find a probability of random variable X to be in set B we need to integrate the probability density function over the set B. This definition immediately give an essential property of $f(x)$:

$$\int_{-\infty}^{\infty} f(x)dx = P(X \in (-\infty, \infty)) = 1.$$

We also note that if B is just interval $[a, b]$ then we get

$$P(a \le X \le b) = \int_a^b f(x)dx,$$

We say that X is continuous random variable if there exists a nonnegative function $f(x)$, defined for all real numbers x, having property that for any set of real numbers B:

$$P(X \in B) = \int_B f(x)dx.$$

The function $f(x)$ is called the **probability density function** of the random variable X.

Yes! This may and should remind you some "physics": to find the mass of an object we need to integrate the density of the object. To find a probability of random variable X to be in set B we need to integrate the probability density function over the set B. This definition immediately give an essential property of $f(x)$:

$$\int_{-\infty}^{\infty} f(x)dx = P(X \in (-\infty, \infty)) = 1.$$

We also note that if B is just interval $[a, b]$ then we get

$$P(a \leq X \leq b) = \int_a^b f(x)dx,$$

and thus letting $a = b$ we get $P(X = a) = \int_a^a f(x)dx = 0$.

We remind the definition of

Cumulative distribution function $F_X(\cdot)$:

$$F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx.$$

Continuous Random Variables X with the density $f(x)$

We remind the definition of

Cumulative distribution function $F_X(\cdot)$:

$$F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx.$$

Taking the derivative of the both sides we get

$$\frac{d}{da}F_X(a) = f(a).$$

We remind the definition of

Cumulative distribution function $F_X(\cdot)$:

$$F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx.$$

Taking the derivative of the both sides we get

$$\frac{d}{da} F_X(a) = f(a).$$

Expected Value of Continuous Random Variable:

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx.$$

Continuous Random Variables X with the density $f(x)$

We remind the definition of

Cumulative distribution function $F_X(\cdot)$:

$$F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx.$$

Taking the derivative of the both sides we get

$$\frac{d}{da} F_X(a) = f(a).$$

Expected Value of Continuous Random Variable:

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx.$$

Variance of Continuous Random Variable:

$$Var(X) = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2.$$

A random variable is said to be uniformly distributed over the interval $(0, 1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

A random variable is said to be uniformly distributed over the interval $(0,1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

We first note that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 1dx = 1$.

A random variable is said to be uniformly distributed over the interval $(0,1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

We first note that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 1dx = 1$.

We also notice that if $a \leq 0$ then $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_{-\infty}^{a} 0dx = 0$.

A random variable is said to be uniformly distributed over the interval $(0,1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

We first note that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 1dx = 1$.

We also notice that if $a \leq 0$ then $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_{-\infty}^{a} 0dx = 0$.

And if $a \in (0,1)$ then $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_0^a 1dx = a$, A random variable is said to be uniformly distributed over the interval $(0, 1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

We first note that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 1dx = 1.$

We also notice that if $a \leq 0$ then $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_{-\infty}^{a} 0dx = 0.$

And if $a \in (0,1)$ then $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_0^a 1dx = a,$

finally if $a \geq 1$: $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_0^1 1dx = 1$

A random variable is said to be uniformly distributed over the interval $(0,1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

We first note that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 1dx = 1$.

We also notice that if $a \leq 0$ then $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_{-\infty}^{a} 0dx = 0$.

And if $a \in (0,1)$ then $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_0^a 1dx = a$,

finally if $a \geq 1$: $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_0^1 1dx = 1$

thus

$$F_X(a) = \begin{cases} 0, & a \leq 0 \\ a, & 0 \leq a \leq 1 \\ 1, & a \geq 1. \end{cases}$$

A random variable is said to be uniformly distributed over the interval $(0,1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

We first note that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 1dx = 1$.

We also notice that if $a \leq 0$ then $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_{-\infty}^{a} 0dx = 0$.

And if $a \in (0,1)$ then $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_0^a 1dx = a$,

finally if $a \geq 1$: $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_0^1 1dx = 1$

thus

$$F_X(a) = \begin{cases} 0, & a \leq 0 \\ a, & 0 \leq a \leq 1 \\ 1, & a \geq 1. \end{cases}$$

Moreover if we take $0 < a < b < 1$ then

$$P(a \leq X \leq b) = \int_a^b f(x)dx = \int_a^b 1dx = b - a.$$

A random variable is said to be uniformly distributed over the interval $(0,1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1), \\ 0, & \text{otherwise.} \end{cases}$$

We first note that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 1dx = 1.$

We also notice that if $a \leq 0$ then $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_{-\infty}^{a} 0dx = 0.$

And if $a \in (0,1)$ then $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_0^a 1dx = a,$

finally if $a \geq 1$: $F_X(a) = P(X < a) = \int_{-\infty}^{a} f(x)dx = \int_0^1 1dx = 1$

thus

$$F_X(a) = \begin{cases} 0, & a \leq 0 \\ a, & 0 \leq a \leq 1 \\ 1, & a \geq 1. \end{cases}$$

Moreover if we take $0 < a < b < 1$ then

$$P(a \leq X \leq b) = \int_a^b f(x)dx = \int_a^b 1dx = b - a.$$

In other words, the probability that X is in any particular subinterval of $(0,1)$ is just equal to the length of that subinterval!

The Uniform Random Variable

A random variable is said to be uniformly distributed over the interval $(0,1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

A random variable is said to be uniformly distributed over the interval $(0,1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

We shall also compute the expected value:

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x * 1dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

A random variable is said to be uniformly distributed over the interval $(0,1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

We shall also compute the expected value:

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x * 1dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

Moreover

$$\mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 * 1dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

A random variable is said to be uniformly distributed over the interval $(0,1)$ if its probability density function is given by

$$f(x) = \begin{cases} 1, & \text{if } x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

We shall also compute the expected value:

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x * 1dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

Moreover

$$\mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 * 1dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$$

and

$$Var(X) = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

The Uniform Random Variable (General Case)

We may also consider a uniform random variable on the interval (α, β).

The Uniform Random Variable (General Case)

We may also consider a uniform random variable on the interval (α, β). Remember that the idea is that density of this random variable must be zero outside of (α, β) must be (positive) constant (say c) on (α, β), but the integral must be 1:

The Uniform Random Variable (General Case)

We may also consider a uniform random variable on the interval (α, β). Remember that the idea is that density of this random variable must be zero outside of (α, β) must be (positive) constant (say c) on (α, β), but the integral must be 1:

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{\alpha}^{\beta} cdx = c * (\beta - \alpha).$$

thus $c = 1/(\beta - \alpha)$:

We may also consider a uniform random variable on the interval (α, β). Remember that the idea is that density of this random variable must be zero outside of (α, β) must be (positive) constant (say c) on (α, β), but the integral must be 1:

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{\alpha}^{\beta} cdx = c * (\beta - \alpha).$$

thus $c = 1/(\beta - \alpha)$:

A random variable is said to be uniformly distributed over the interval (α, β) if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \text{if } x \in (\alpha, \beta) \\ 0, & \text{otherwise.} \end{cases}$$

We may also consider a uniform random variable on the interval (α, β). Remember that the idea is that density of this random variable must be zero outside of (α, β) must be (positive) constant (say c) on (α, β), but the integral must be 1:

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{\alpha}^{\beta} cdx = c * (\beta - \alpha).$$

thus $c = 1/(\beta - \alpha)$:

A random variable is said to be uniformly distributed over the interval (α, β) if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha}, & \text{if } x \in (\alpha, \beta) \\ 0, & \text{otherwise.} \end{cases}$$

You can easily recalculate $F_X(a)$, $\mathbb{E}X$ and $Var(X)$ directly from the $(0, 1)$ example.

Let X be uniformly distributed over $(-1, 3)$.

The Uniform Random Variable (General Case)

Let X be uniformly distributed over $(-1,3)$.

Then density $f(x)$ and and cumulative distribution function $F_X(\cdot)$:

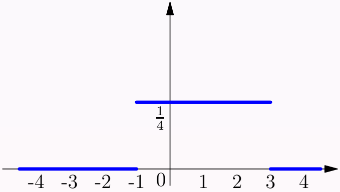
$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } x \in (-1,3) \\ 0, & \text{otherwise.} \end{cases} \qquad F_X(x) = P(X < x) = \begin{cases} 0, & \text{if } x \leq -1 \\ \frac{1}{4}(x+1), & x \in (-1,3) \\ 1 & x \geq 3. \end{cases}$$

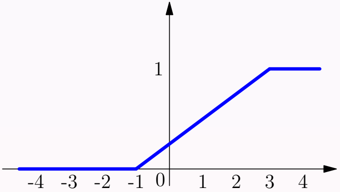
The Uniform Random Variable (General Case)

Let X be uniformly distributed over $(-1, 3)$.

Then density $f(x)$ and and cumulative distribution function $F_X(\cdot)$:

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } x \in (-1,3) \\ 0, & \text{otherwise.} \end{cases} \qquad F_X(x) = P(X < x) = \begin{cases} 0, & \text{if } x \leq -1 \\ \frac{1}{4}(x+1), & x \in (-1,3) \\ 1 & x \geq 3. \end{cases}$$





The Uniform Random Variable (General Case)

If X is uniformly distributed over $(-2,5)$, calculate probability that $X<0$; $X>2$; $X \in (-1,8)$. Also find $\mathbb{E}X$ and $Var(X)$.

If X is uniformly distributed over $(-2,5)$, calculate probability that $X<0$; $X>2$; $X\in(-1,8)$. Also find $\mathbb{E}X$ and $Var(X)$.

Solution:

$$f(x)=\begin{cases}\frac{1}{7}, & \text{if } x\in(-2,5)\\ 0, & \text{otherwise.}\end{cases}$$

If X is uniformly distributed over $(-2,5)$, calculate probability that $X<0$; $X>2$; $X\in(-1,8)$. Also find $\mathbb{E}X$ and $Var(X)$.

Solution:

$$f(x)=\begin{cases}\frac{1}{7}, & \text{if } x\in(-2,5)\\ 0, & \text{otherwise.}\end{cases}$$

Thus

$$P(X<0)=\int_{-\infty}^{0} f(x)dx=\int_{-2}^{0}\frac{1}{7}dx=\frac{2}{7}.$$

If X is uniformly distributed over $(-2, 5)$, calculate probability that $X < 0$; $X > 2$; $X \in (-1, 8)$. Also find $\mathbb{E}X$ and $Var(X)$.

Solution:

$$f(x) = \begin{cases} \frac{1}{7}, & \text{if } x \in (-2,5) \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$P(X < 0) = \int_{-\infty}^{0} f(x)dx = \int_{-2}^{0} \frac{1}{7}dx = \frac{2}{7}.$$

$$P(X > 2) = \int_{2}^{\infty} f(x)dx = \int_{2}^{5} \frac{1}{7}dx = \frac{3}{7}.$$

The Uniform Random Variable (General Case)

If X is uniformly distributed over $(-2,5)$, calculate probability that $X<0$; $X>2$; $X \in (-1,8)$. Also find $\mathbb{E}X$ and $Var(X)$.

Solution:

$$f(x) = \begin{cases} \frac{1}{7}, & \text{if } x \in (-2,5) \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$P(X<0) = \int_{-\infty}^{0} f(x)dx = \int_{-2}^{0} \frac{1}{7}dx = \frac{2}{7}.$$

$$P(X>2) = \int_{2}^{\infty} f(x)dx = \int_{2}^{5} \frac{1}{7}dx = \frac{3}{7}.$$

$$P(-1<X<8) = \int_{-1}^{8} f(x)dx = \int_{-1}^{5} \frac{1}{7}dx = \frac{6}{7}.$$

The Uniform Random Variable (General Case)

If X is uniformly distributed over $(-2, 5)$, calculate probability that $X < 0$; $X > 2$; $X \in (-1, 8)$. Also find $\mathbb{E}X$ and $Var(X)$.

Solution:

$$f(x) = \begin{cases} \frac{1}{7}, & \text{if } x \in (-2, 5) \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$P(X < 0) = \int_{-\infty}^{0} f(x)dx = \int_{-2}^{0} \frac{1}{7}dx = \frac{2}{7}.$$

$$P(X > 2) = \int_{2}^{\infty} f(x)dx = \int_{2}^{5} \frac{1}{7}dx = \frac{3}{7}.$$

$$P(-1 < X < 8) = \int_{-1}^{8} f(x)dx = \int_{-1}^{5} \frac{1}{7}dx = \frac{6}{7}.$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx = \int_{-2}^{5} x * \frac{1}{7}dx = \frac{1}{7}\frac{x^2}{2}\Big|_{-2}^{5} = \frac{3}{2}.$$

If X is uniformly distributed over $(-2,5)$, calculate probability that $X<0$; $X>2$; $X\in(-1,8)$. Also find $\mathbb{E}X$ and $Var(X)$.

Solution:

$$f(x)=\begin{cases}\frac{1}{7}, & \text{if } x\in(-2,5)\\ 0, & \text{otherwise.}\end{cases}$$

Thus

$$P(X<0)=\int_{-\infty}^{0} f(x)dx=\int_{-2}^{0}\frac{1}{7}dx=\frac{2}{7}.$$

$$P(X>2)=\int_{2}^{\infty} f(x)dx=\int_{2}^{5}\frac{1}{7}dx=\frac{3}{7}.$$

$$P(-1<X<8)=\int_{-1}^{8} f(x)dx=\int_{-1}^{5}\frac{1}{7}dx=\frac{6}{7}.$$

$$\mathbb{E}X=\int_{-\infty}^{\infty} xf(x)dx=\int_{-2}^{5} x*\frac{1}{7}dx=\frac{1}{7}\frac{x^2}{2}\Big|_{-2}^{5}=\frac{3}{2}.$$

$$\mathbb{E}X^2=\int_{-\infty}^{\infty} x^2f(x)dx=\int_{-2}^{5} x^2*\frac{1}{7}dx=\frac{1}{7}\frac{x^3}{3}\Big|_{-2}^{5}=\frac{125-4}{21}.$$

If X is uniformly distributed over $(-2,5)$, calculate probability that $X<0$; $X>2$; $X\in(-1,8)$. Also find $\mathbb{E}X$ and $Var(X)$.

Solution:

$$f(x)=\begin{cases}\frac{1}{7}, & \text{if } x\in(-2,5)\\ 0, & \text{otherwise.}\end{cases}$$

Thus

$$P(X<0)=\int_{-\infty}^{0} f(x)dx=\int_{-2}^{0}\frac{1}{7}dx=\frac{2}{7}.$$

$$P(X>2)=\int_{2}^{\infty} f(x)dx=\int_{2}^{5}\frac{1}{7}dx=\frac{3}{7}.$$

$$P(-1<X<8)=\int_{-1}^{8} f(x)dx=\int_{-1}^{5}\frac{1}{7}dx=\frac{6}{7}.$$

$$\mathbb{E}X=\int_{-\infty}^{\infty} xf(x)dx=\int_{-2}^{5} x*\frac{1}{7}dx=\frac{1}{7}\frac{x^2}{2}\Big|_{-2}^{5}=\frac{3}{2}.$$

$$\mathbb{E}X^2=\int_{-\infty}^{\infty} x^2f(x)dx=\int_{-2}^{5} x^2*\frac{1}{7}dx=\frac{1}{7}\frac{x^3}{3}\Big|_{-2}^{5}=\frac{125-4}{21}.$$

and

$$Var(X)=\frac{121}{21}-\frac{9}{4}.$$

Exponential Random Variable

Fix parameter $\lambda > 0$. A continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

is said to be exponential random variable with parameter λ.

Exponential Random Variable

Fix parameter $\lambda > 0$. A continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

is said to be exponential random variable with parameter λ.

Note that for $a > 0$

$$F_X(a) = \int_{-\infty}^{a} f(x)dx = \int_0^a \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^a = 1 - e^{-\lambda a}.$$

Exponential Random Variable

Fix parameter $\lambda > 0$. A continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

is said to be exponential random variable with parameter λ.

Note that for $a > 0$

$$F_X(a) = \int_{-\infty}^{a} f(x)dx = \int_{0}^{a} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^a = 1 - e^{-\lambda a}.$$

and thus

$$F_X(\infty) = \lim_{a \to \infty} 1 - e^{-\lambda a} = 1.$$

A bit of old story...

Consider two functions $f(x)$ and $g(x)$ differentiable and integrable on the interval (a, b), then we all know the product rule of differentiation:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

A bit of old story...

Consider two functions $f(x)$ and $g(x)$ differentiable and integrable on the interval (a, b), then we all know the product rule of differentiation:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Now integrate the above equality over the interval (a, b):

$$\int_a^b (f(x)g(x))' dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx.$$

A bit of old story...

Consider two functions $f(x)$ and $g(x)$ differentiable and integrable on the interval (a, b), then we all know the product rule of differentiation:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Now integrate the above equality over the interval (a, b):

$$\int_a^b (f(x)g(x))' dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx.$$

Applying the fundamental theorem of calculus to the right hand side we get

Integration by parts formula.

Consider two functions $f(x)$ and $g(x)$ differentiable and integrable on the interval (a, b), then

$$f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx.$$

A bit of old story...

Consider two functions $f(x)$ and $g(x)$ differentiable and integrable on the interval (a, b), then we all know the product rule of differentiation:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Now integrate the above equality over the interval (a, b):

$$\int_a^b (f(x)g(x))' dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx.$$

Applying the fundamental theorem of calculus to the right hand side we get

Integration by parts formula.

Consider two functions $f(x)$ and $g(x)$ differentiable and integrable on the interval (a, b), then

$$f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx.$$

Example: Assume X is exponential random variable with parameter λ then

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} x\lambda e^{-\lambda x} dx = \int_0^{\infty} x(-e^{-\lambda x})' dx$$

A bit of old story...

Consider two functions $f(x)$ and $g(x)$ differentiable and integrable on the interval (a, b), then we all know the product rule of differentiation:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Now integrate the above equality over the interval (a, b):

$$\int_a^b (f(x)g(x))' dx = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx.$$

Applying the fundamental theorem of calculus to the right hand side we get

Integration by parts formula.

Consider two functions $f(x)$ and $g(x)$ differentiable and integrable on the interval (a, b), then

$$f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx.$$

Example: Assume X is exponential random variable with parameter λ then

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} x\lambda e^{-\lambda x} dx = \int_0^{\infty} x(-e^{-\lambda x})' dx$$

$$x(-e^{-\lambda x})\Big|_0^{\infty} - \int_0^{\infty} (x)'(-e^{-\lambda x})dx = \int_0^{\infty} e^{-\lambda x} dx = \frac{-1}{\lambda} e^{-\lambda x}\Big|_0^{\infty} = \frac{1}{\lambda}.$$

Consider $\alpha > 0$, then the gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx.$$

Consider $\alpha > 0$, then the gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx.$$

Note that

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1.$$

Consider $\alpha > 0$, then the gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx.$$

Note that

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1.$$

Also, using integration by parts we get

$$\Gamma(2) = \int_0^\infty x e^{-x} dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} dx = 1$$

Consider $\alpha > 0$, then the gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx.$$

Note that

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1.$$

Also, using integration by parts we get

$$\Gamma(2) = \int_0^\infty x e^{-x} dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} dx = 1$$

and

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx = \int_0^\infty x^{n-1} \left(-e^{-x}\right)' dx = -x^{n-1}e^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x})(x^{n-1})' dx$$

Consider $\alpha > 0$, then the gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha - 1} dx.$$

Note that

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1.$$

Also, using integration by parts we get

$$\Gamma(2) = \int_0^\infty x e^{-x} dx = -x e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} dx = 1$$

and

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx = \int_0^\infty x^{n-1} \left(-e^{-x}\right)' dx = -x^{n-1} e^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x})(x^{n-1})' dx$$

$$(n-1) \int_0^\infty e^{-x} x^{n-2} dx = (n-1)\Gamma(n-1).$$

Consider $\alpha > 0$, then the gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx.$$

Note that

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1.$$

Also, using integration by parts we get

$$\Gamma(2) = \int_0^\infty x e^{-x} dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} dx = 1$$

and

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx = \int_0^\infty x^{n-1} \left(-e^{-x}\right)' dx = -x^{n-1}e^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x})(x^{n-1})' dx$$

$$(n-1)\int_0^\infty e^{-x} x^{n-2} dx = (n-1)\Gamma(n-1).$$

So we proved $\Gamma(n) = (n-1)\Gamma(n-1)$ together with $\Gamma(2) = \Gamma(1) = 1$ it gives us $\Gamma(n) = (n-1)!$.

A continuous random variable whose density is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

for some $\lambda > 0$ and $\alpha > 0$ is said to be the gamma random variable with parameters α and λ.

A continuous random variable whose density is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

for some $\lambda > 0$ and $\alpha > 0$ is said to be the gamma random variable with parameters α and λ.

We note that $f(x) \geq 0$ and

Gamma Random Variable

A continuous random variable whose density is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

for some $\lambda > 0$ and $\alpha > 0$ is said to be the gamma random variable with parameters α and λ.

We note that $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \lambda e^{-\lambda x}(\lambda x)^{\alpha-1}dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-y}(y)^{\alpha-1}dy = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1.$$

A continuous random variable whose density is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

for some $\lambda > 0$ and $\alpha > 0$ is said to be the gamma random variable with parameters α and λ.

We note that $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \lambda e^{-\lambda x}(\lambda x)^{\alpha-1}dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-y}(y)^{\alpha-1}dy = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1.$$

In addition,

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} x\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-\lambda x}(\lambda x)^{\alpha}dx$$

A continuous random variable whose density is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

for some $\lambda > 0$ and $\alpha > 0$ is said to be the gamma random variable with parameters α and λ.

We note that $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \lambda e^{-\lambda x}(\lambda x)^{\alpha-1}dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-y}(y)^{\alpha-1}dy = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1.$$

In addition,

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} x\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-\lambda x}(\lambda x)^{\alpha}dx$$

Make change of variables $y = \lambda x$, $dy = \lambda dx$ and thus $dx = dy/\lambda$:

A continuous random variable whose density is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

for some $\lambda > 0$ and $\alpha > 0$ is said to be the gamma random variable with parameters α and λ.

We note that $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \lambda e^{-\lambda x}(\lambda x)^{\alpha-1}dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-y}(y)^{\alpha-1}dy = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1.$$

In addition,

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} x\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}dx = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-\lambda x}(\lambda x)^{\alpha}dx$$

Make change of variables $y = \lambda x$, $dy = \lambda dx$ and thus $dx = dy/\lambda$:

$$= \frac{1}{\lambda\Gamma(\alpha)} \int_0^{\infty} e^{-y}(y)^{\alpha}dy = \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} = \frac{\alpha\Gamma(\alpha)}{\lambda\Gamma(\alpha)} = \frac{\alpha}{\lambda}.$$